6.2 Master Theorem

Note that the cases do not cover all possibilities.

Lemma 5

Let $a \geq 1, b > 1$ and $\epsilon > 0$ denote constants. Consider the recurrence

$$T(n) = aT\left(\frac{n}{b}\right) + f(n) \ .$$

Case 1.

If $f(n) = \mathcal{O}(n^{\log_b(a)-\epsilon})$ then $T(n) = \Theta(n^{\log_b a})$.

Case 2.

If $f(n) = \Theta(n^{\log_b(a)} \log^k n)$ then $T(n) = \Theta(n^{\log_b a} \log^{k+1} n)$, $k \geq 0$.

Case 3.

If $f(n) = \Omega(n^{\log_b(a)+\epsilon})$ and for sufficiently large n $af(\frac{n}{b}) \leq cf(n)$ for some constant $c < 1$ then $T(n) = \Theta(f(n))$.

6.2 Master Theorem

We prove the Master Theorem for the case that n is of the form b^ℓ, and we assume that the non-recursive case occurs for problem size 1 and incurs cost 1.

The Recursion Tree

The running time of a recursive algorithm can be visualized by a recursion tree:

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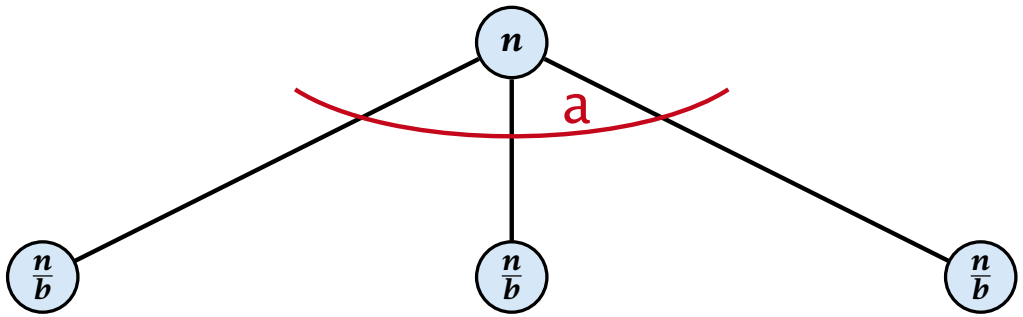

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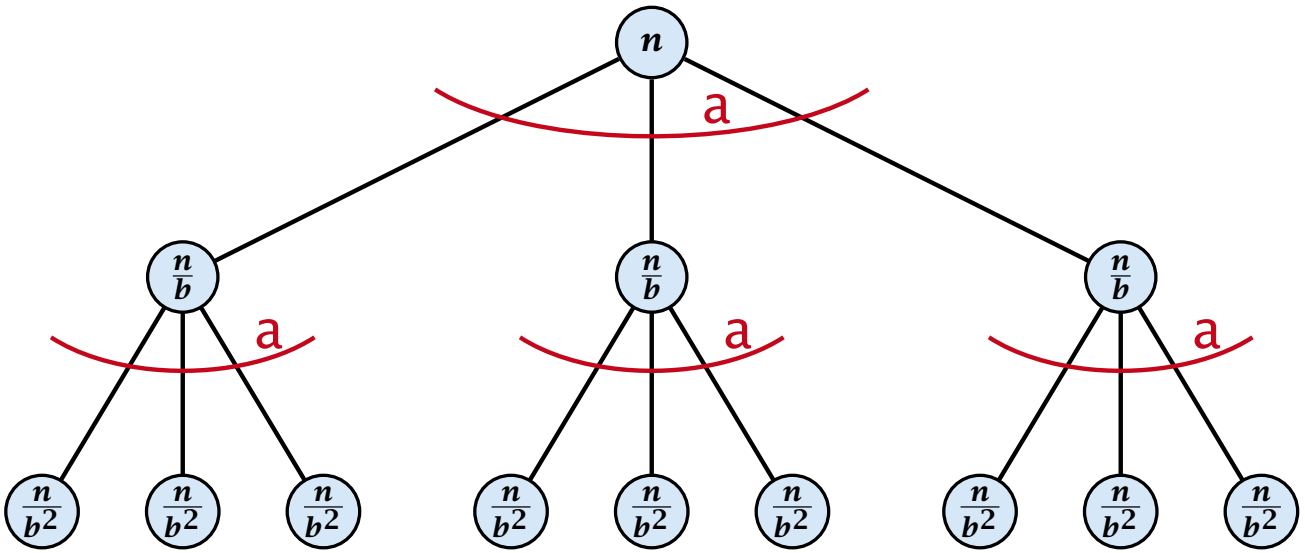

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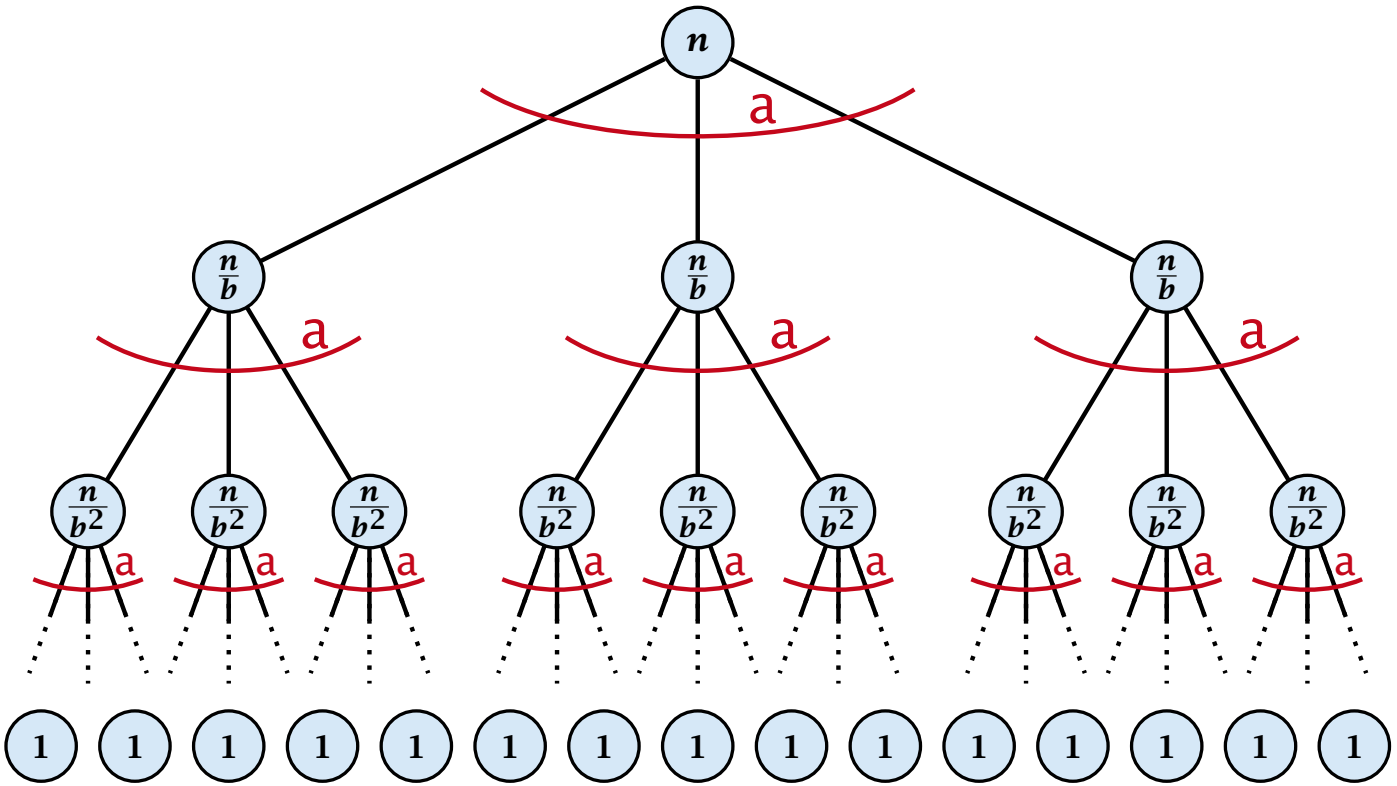

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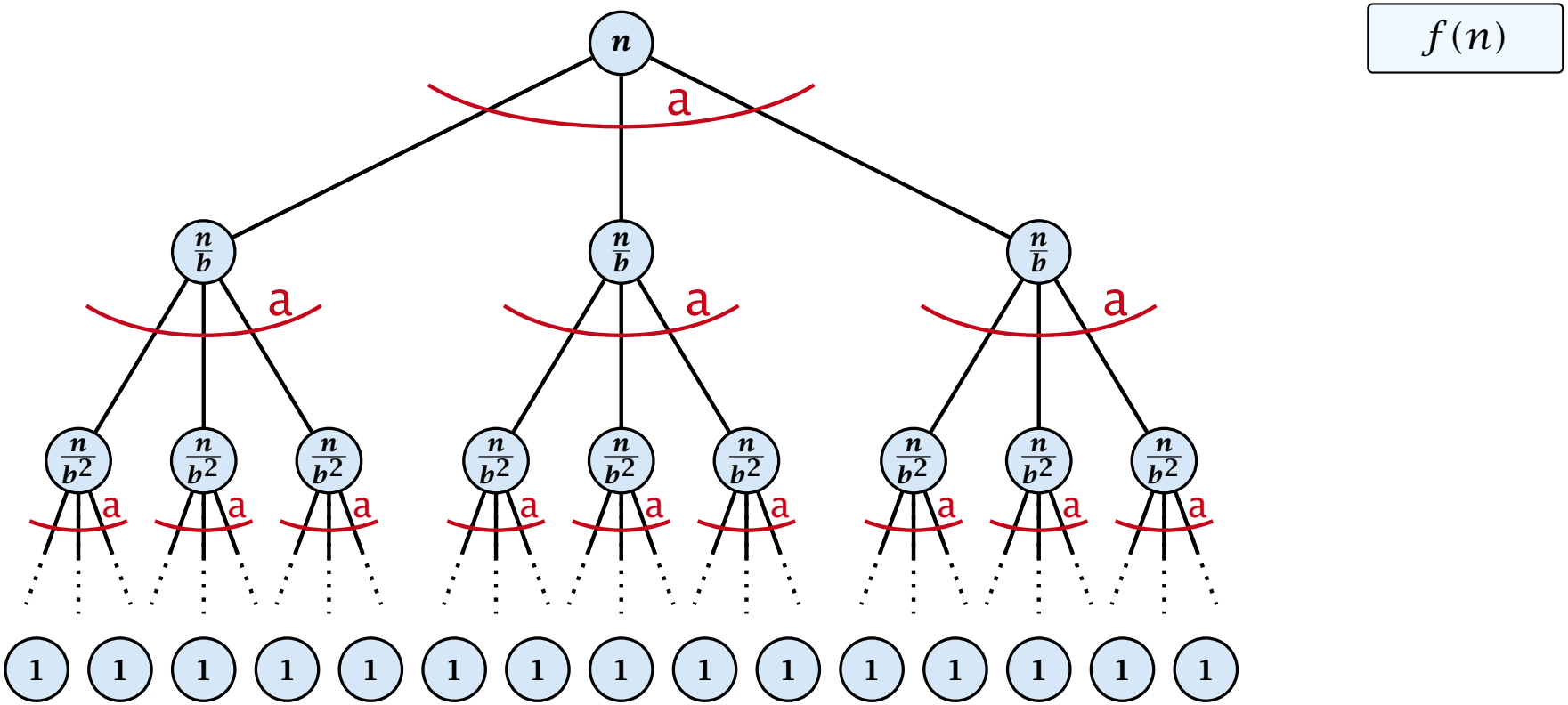

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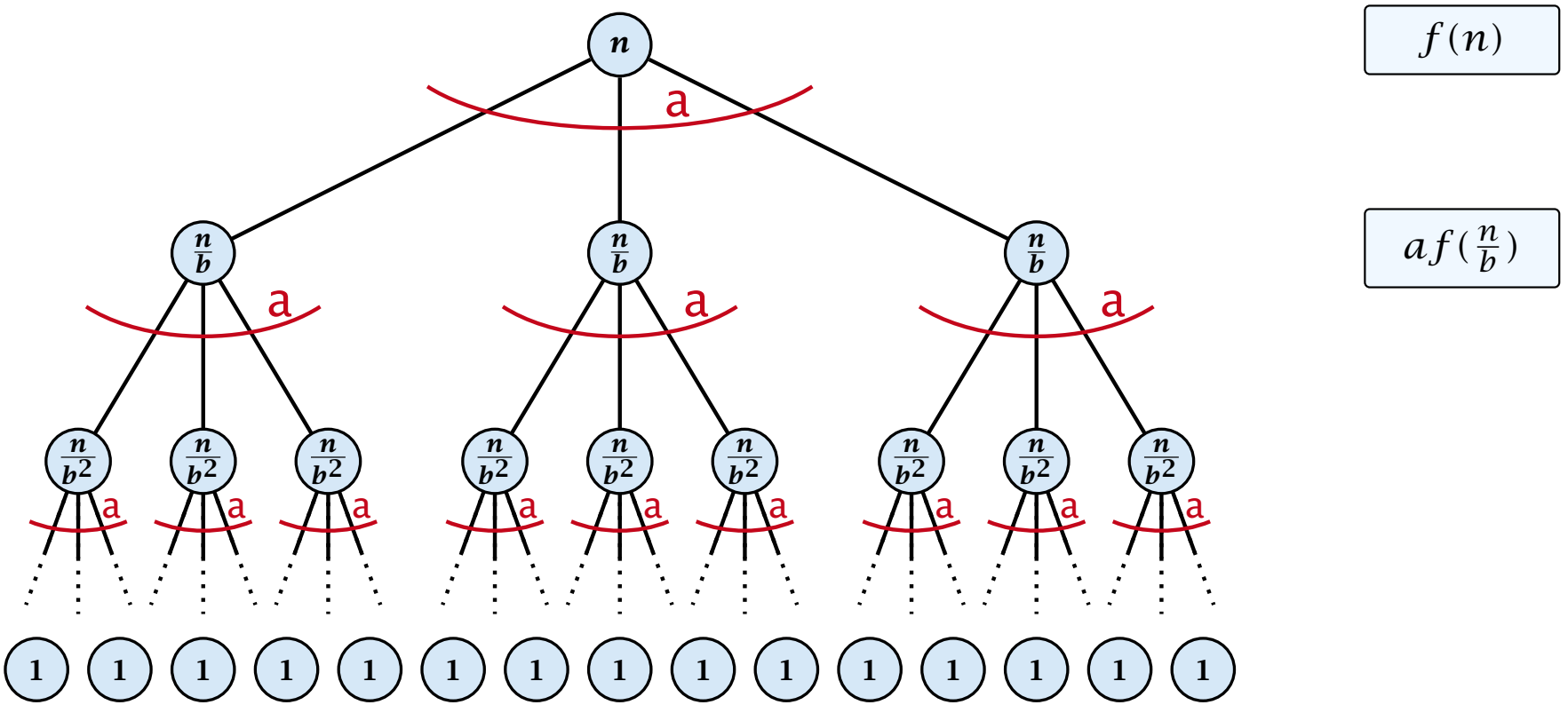

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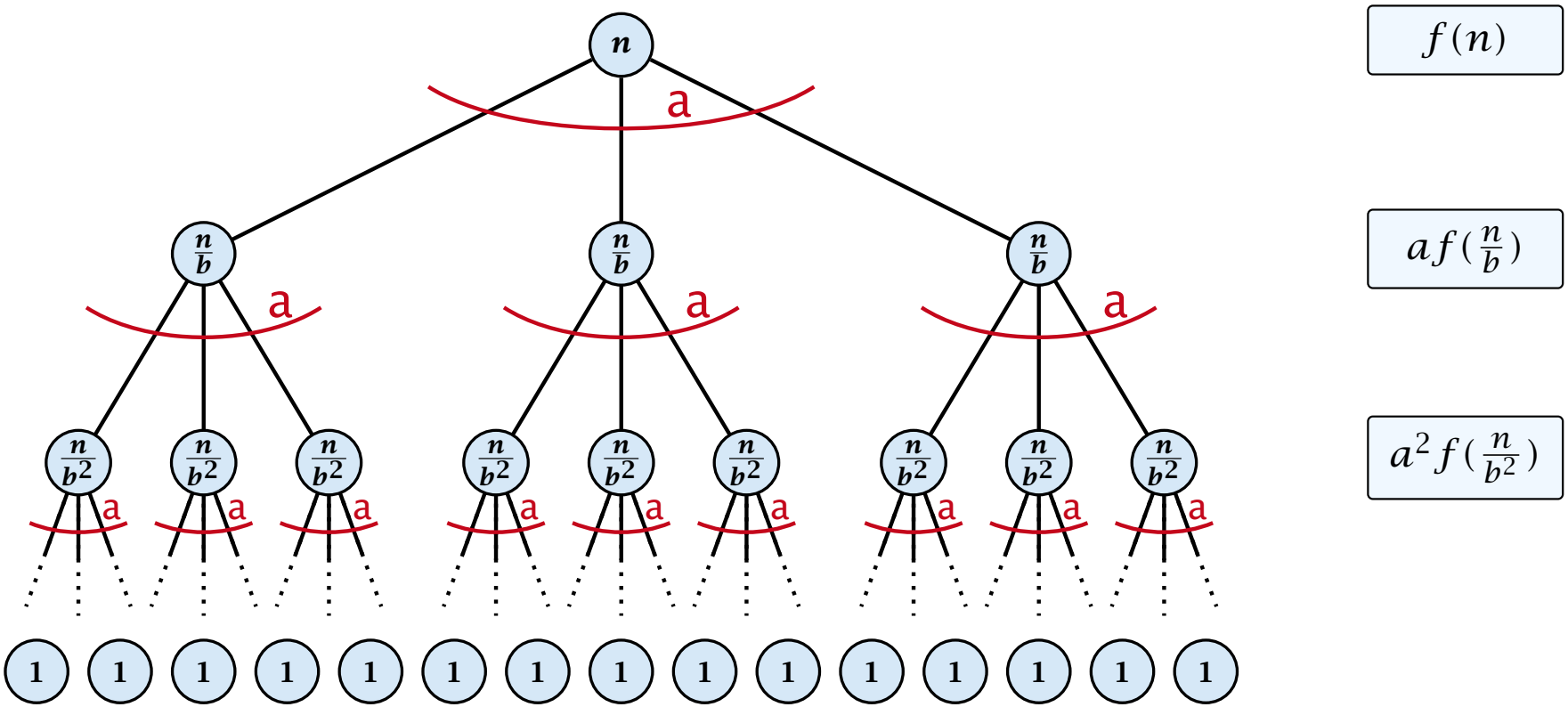

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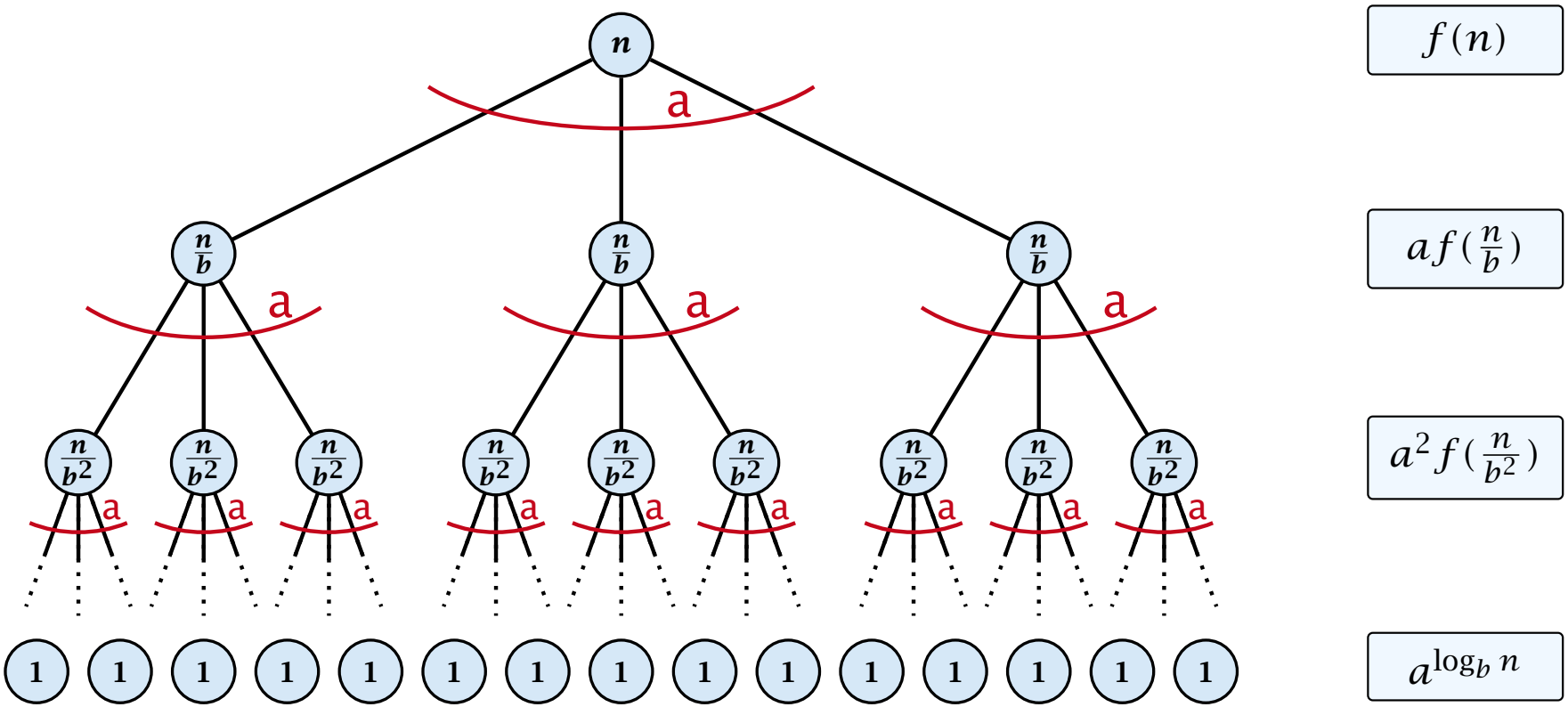

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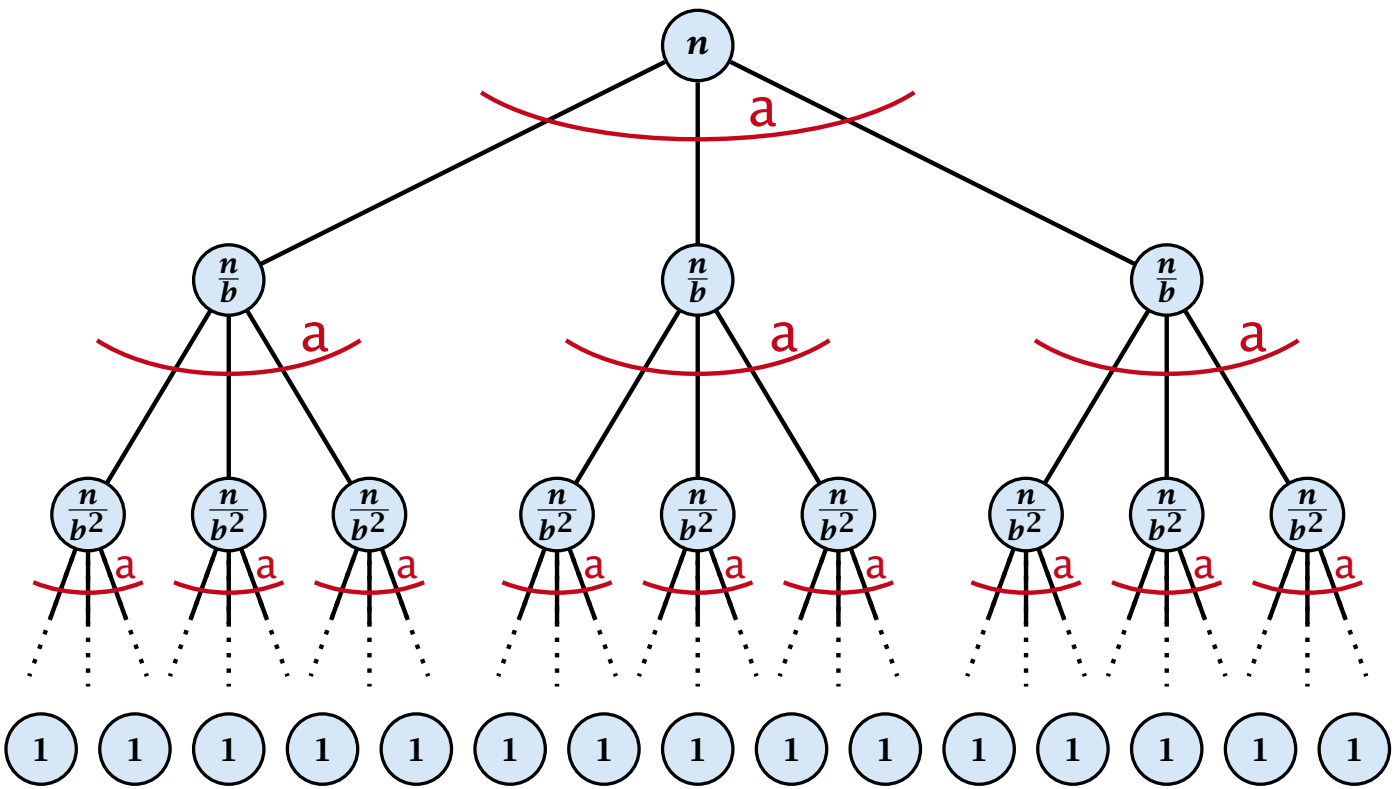

$f(n)$

$af(\frac{n}{b})$

$a^2 f(\frac{n}{b^2})$

$a^{\log_b n}$

$=$

$n^{\log_b a}$

6.2 Master Theorem

This gives

$$T(n) = n^{\log_b a} + \sum_{i=0}^{\log_b n - 1} a^i f\left(\frac{n}{b^i}\right) .$$

Case 1. Now suppose that $f(n) \leq cn^{\log_b a - \epsilon}$.

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$$T(n) - n^{\log_b a}$$

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$$T(n) - n^{\log_b a} = \sum_{i=0}^{\log_b n-1} a^i f\left(\frac{n}{b^i}\right)$$

$$\leq c \sum_{i=0}^{\log_b n-1} a^i \left(\frac{n}{b^i}\right)^{\log_b a-\epsilon}$$

Case 1. Now suppose that $f(n) \leq c n^{\log_b a - \epsilon}$.

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$$b^{-i(\log_b a - \epsilon)} = b^{\epsilon i} (b^{\log_b a})^{-i} = b^{\epsilon i} a^{-i}$$

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$$= cn^{\log_b a-\epsilon}(b^{\epsilon \log_b n} - 1)/(b^\epsilon - 1)$$

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Hence,

$$T(n) \leq \left(\frac{c}{b^\epsilon - 1} + 1\right) n^{\log_b(a)}$$

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Hence,

$$T(n) \leq \left(\frac{c}{b^\epsilon - 1} + 1\right) n^{\log_b(a)}$$

$$\Rightarrow T(n) = \mathcal{O}(n^{\log_b a}).$$

Case 2. Now suppose that $f(n) \leq cn^{\log_b a}$.

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$$
\begin{aligned}
T(n) - n^{\log_b a} &= \sum_{i=0}^{\log_b n - 1} a^i f\left(\frac{n}{b^i}\right) \\
&\leq c \sum_{i=0}^{\log_b n - 1} a^i \left(\frac{n}{b^i}\right)^{\log_b a} \\
&= cn^{\log_b a} \sum_{i=0}^{\log_b n - 1} 1
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Hence,

$$T(n) = \mathcal{O}(n^{\log_b a} \log_b n) \qquad \boxed{\Rightarrow T(n) = \mathcal{O}(n^{\log_b a} \log n).}$$

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\end{aligned}
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Hence,

$$T(n) = \Omega(n^{\log_b a} \log_b n) \qquad \boxed{\Rightarrow T(n) = \Omega(n^{\log_b a} \log n).}$$

Case 2. Now suppose that $f(n) \leq cn^{\log_b a}(\log_b(n))^k$.

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$n = b^\ell \Rightarrow \ell = \log_b n$

$$= cn^{\log_b a} \sum_{i=0}^{\ell-1} (\ell - i)^k$$

$$= cn^{\log_b a} \boxed{\sum_{i=1}^{\ell} i^k} \approx \tfrac{1}{k}\ell^{k+1}$$

Case 2. Now suppose that $f(n) \leq cn^{\log_b a}(\log_b(n))^k$.

$$T(n) - n^{\log_b a} = \sum_{i=0}^{\log_b n - 1} a^i f\left(\frac{n}{b^i}\right)$$

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$\Rightarrow T(n) = \mathcal{O}(n^{\log_b a} \log^{k+1} n)$.

Case 3. Now suppose that $f(n) \geq dn^{\log_b a+\epsilon}$, and that for sufficiently large n: $af(n/b) \leq cf(n)$, for $c < 1$.

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$$T(n) \leq \mathcal{O}(f(n)) \qquad \Rightarrow T(n) = \Theta(f(n)).$$

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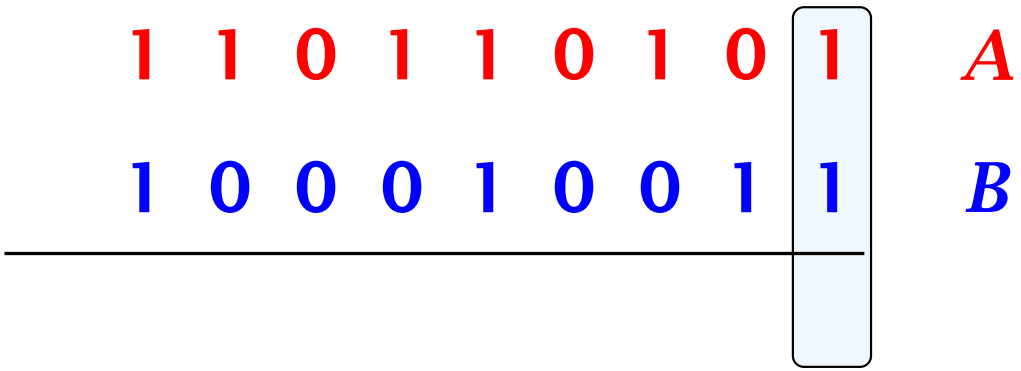

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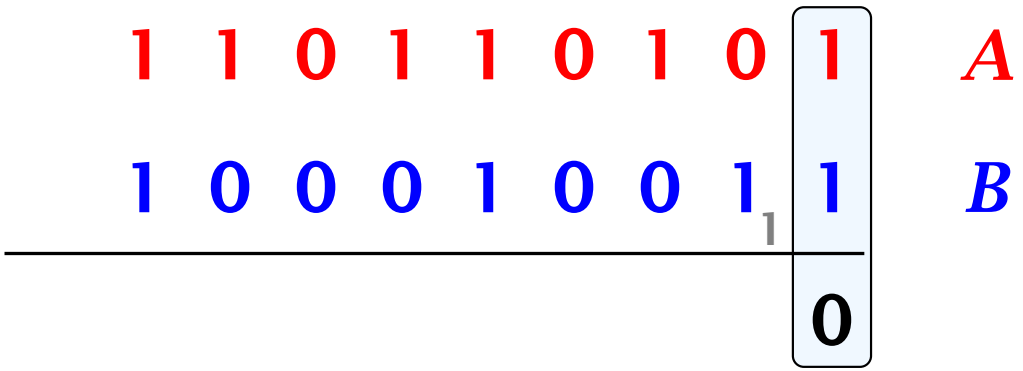

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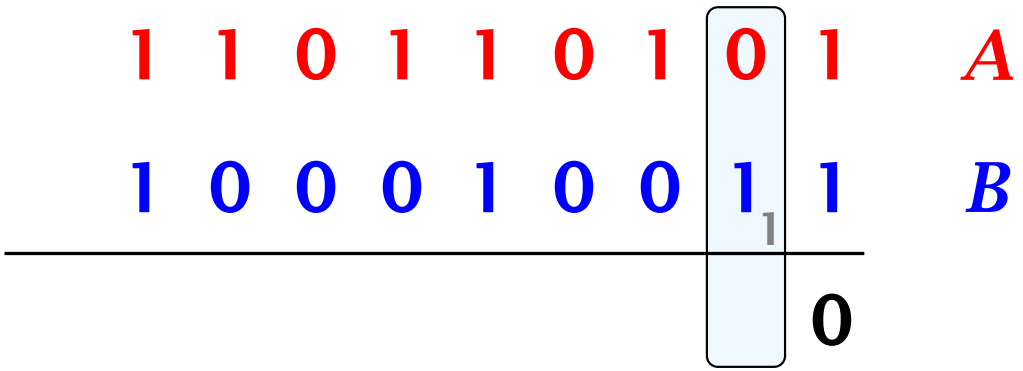

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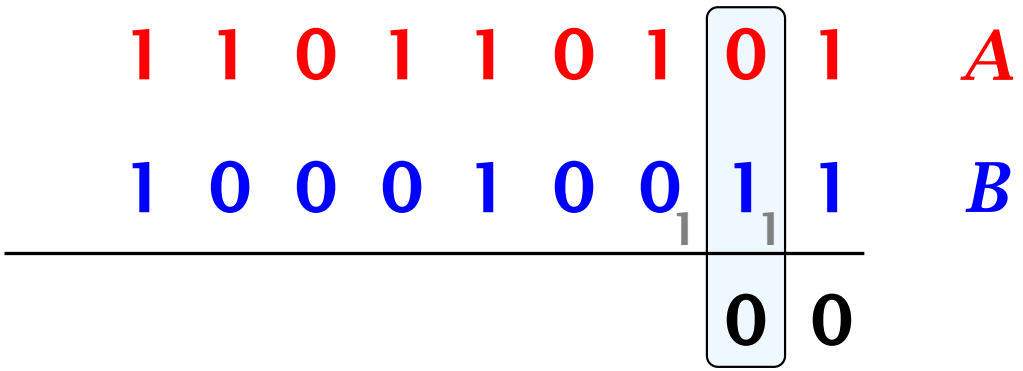

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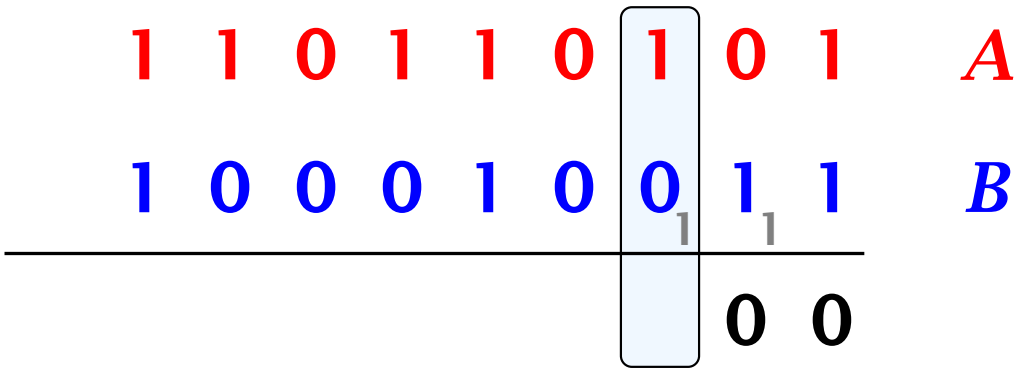

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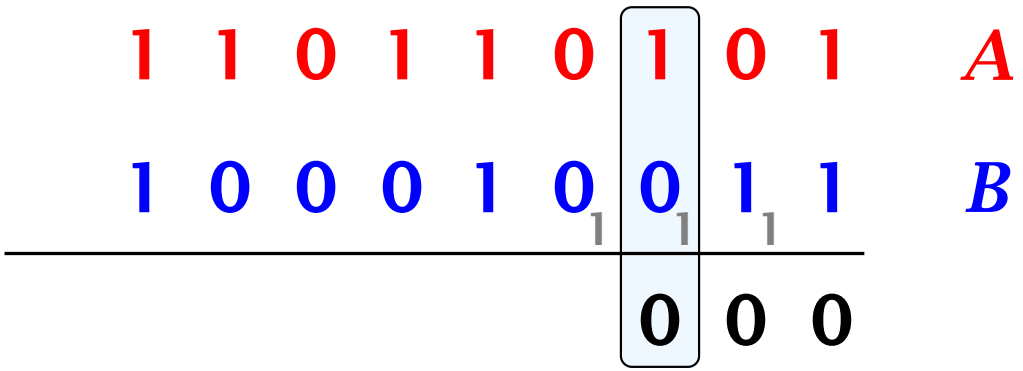

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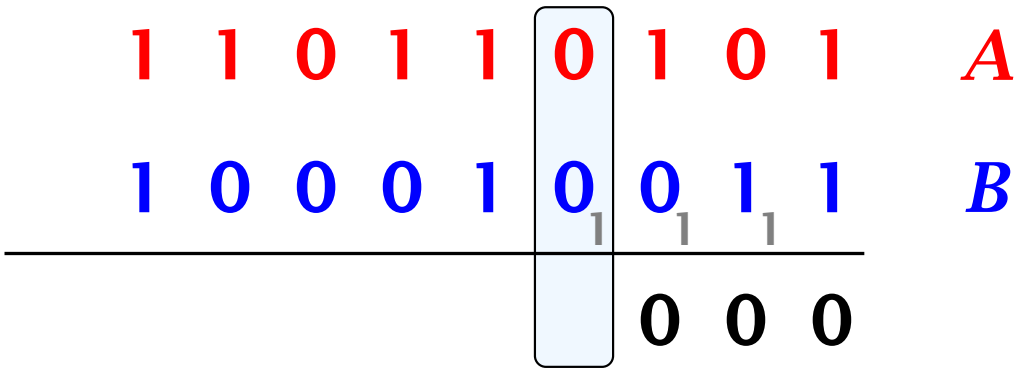

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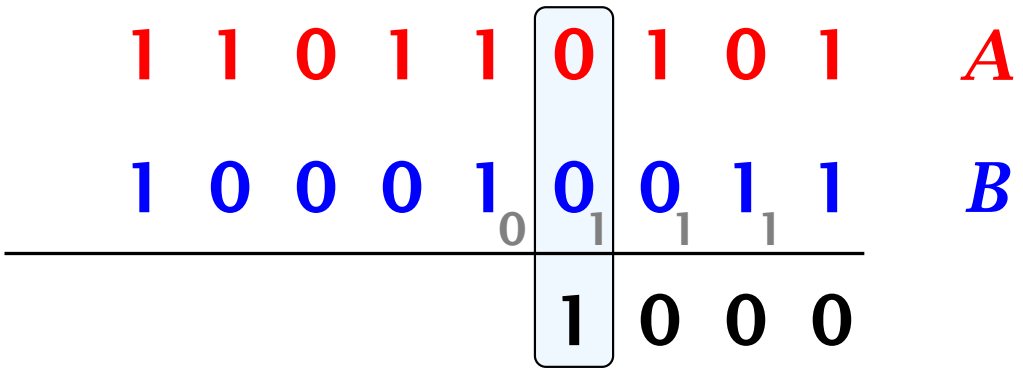

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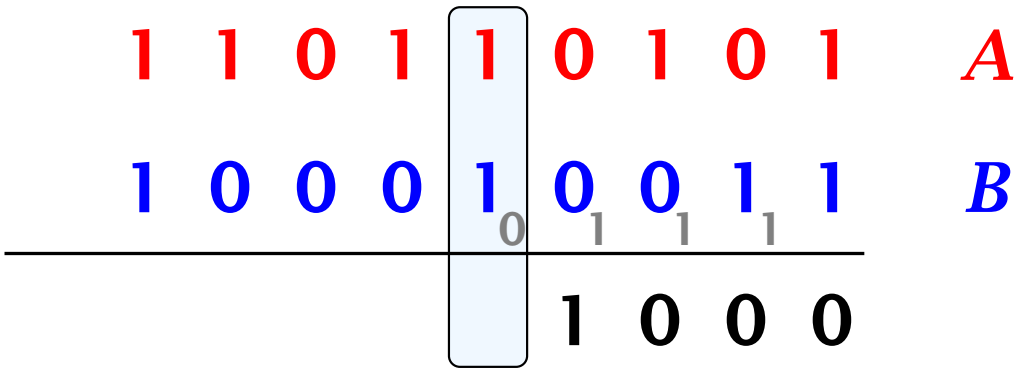

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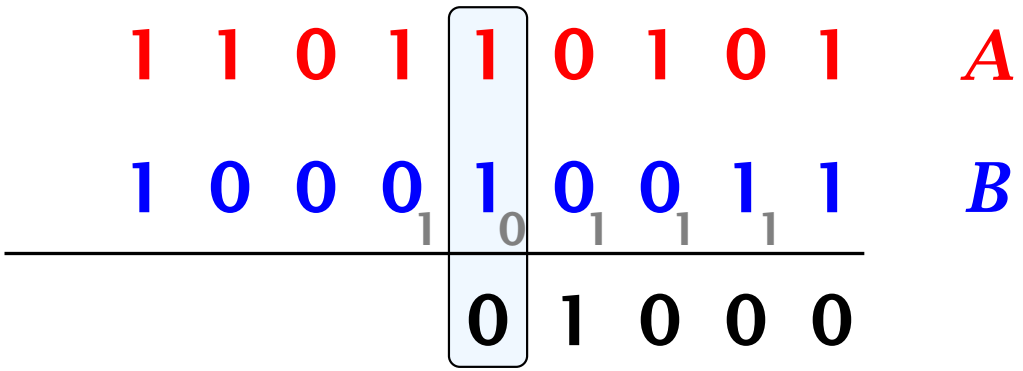

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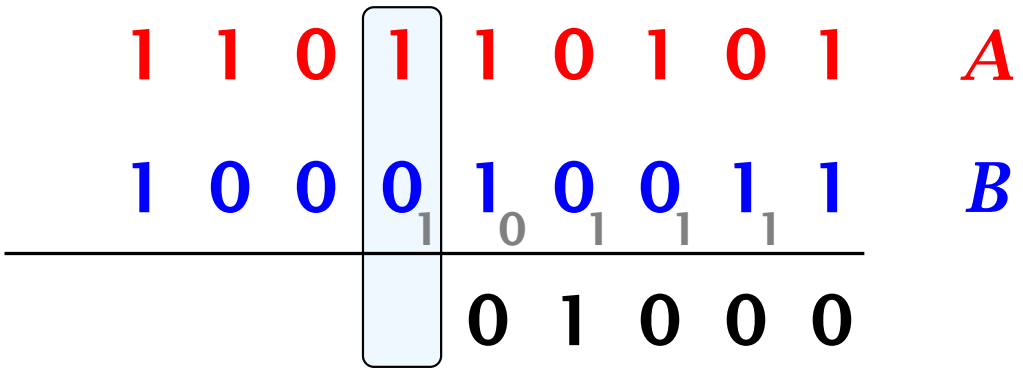

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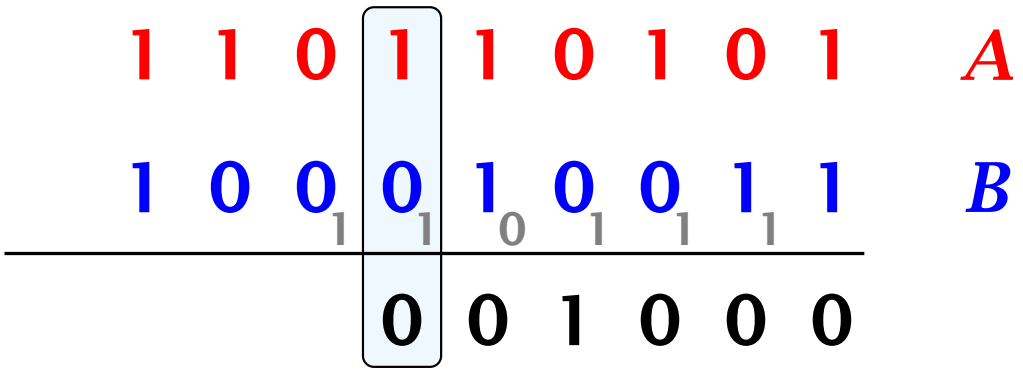

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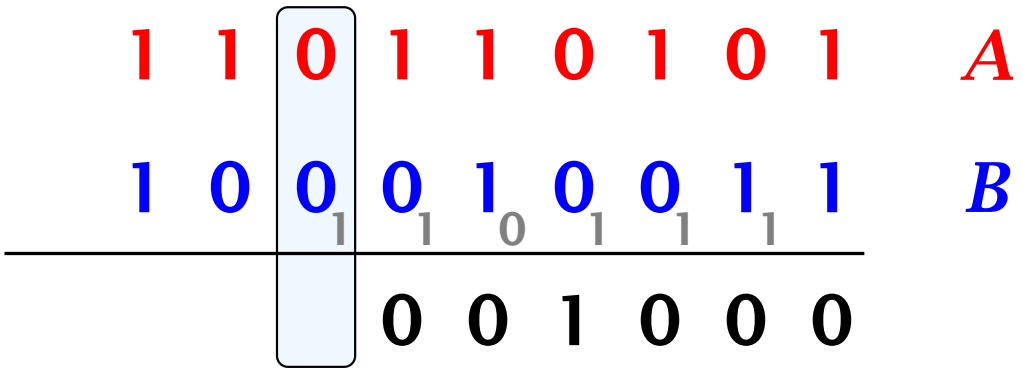

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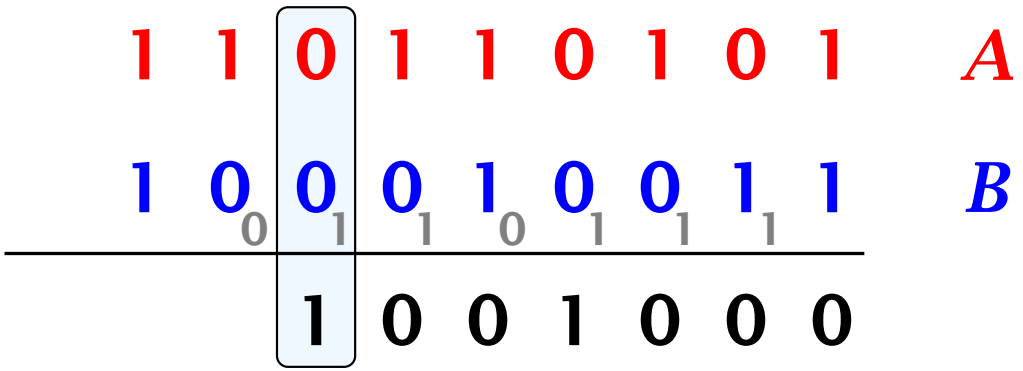

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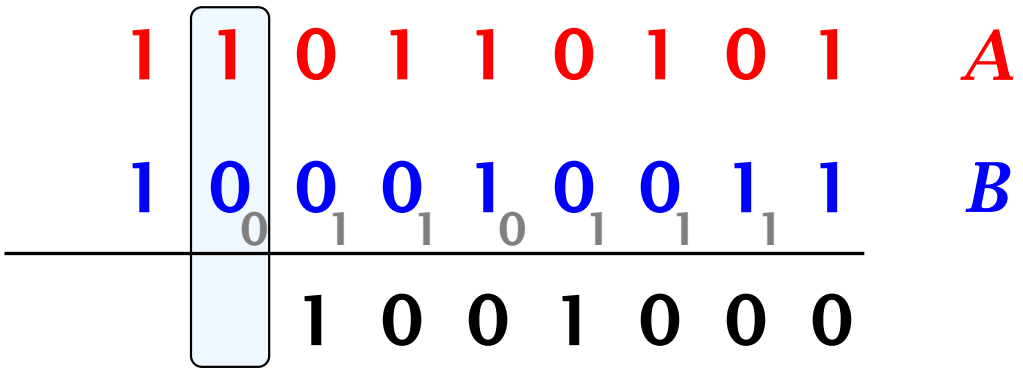

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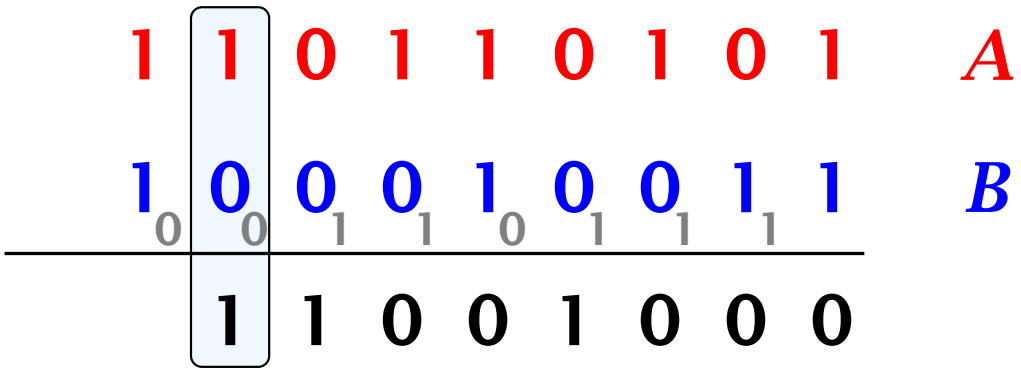

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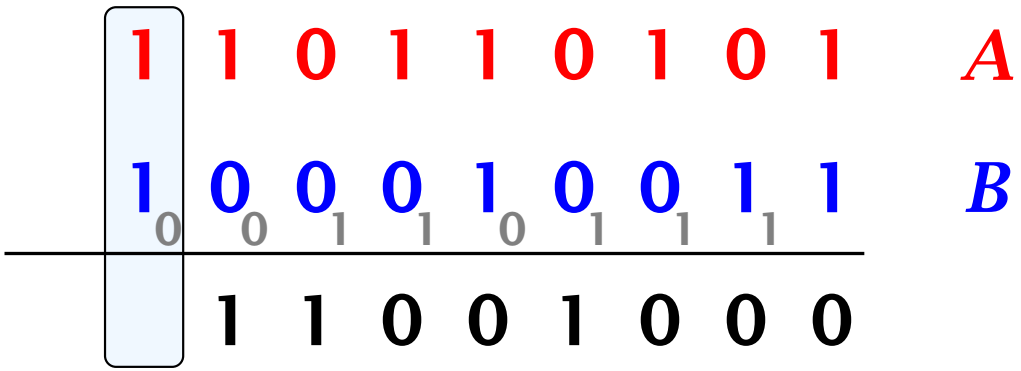

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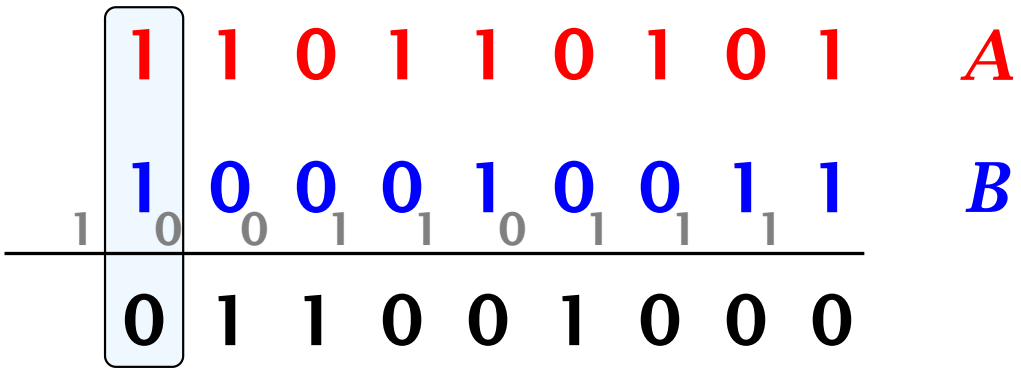

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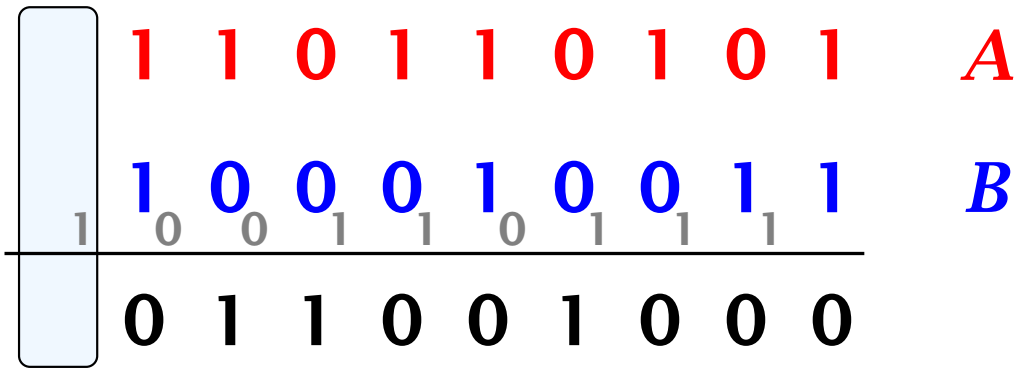

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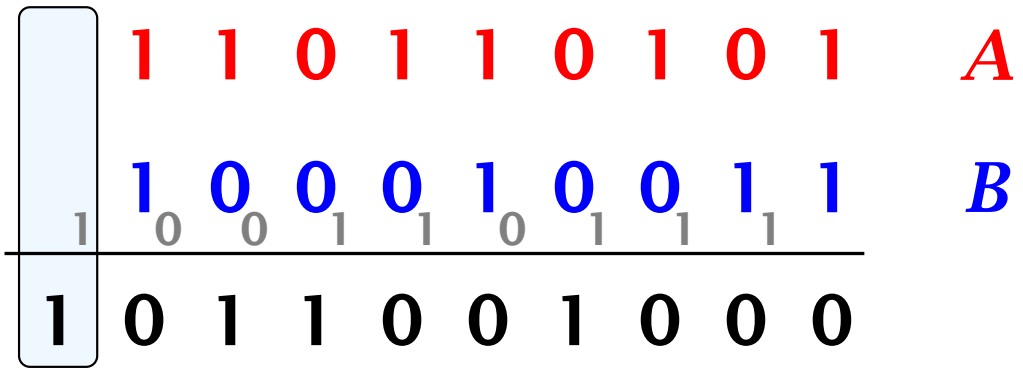

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This gives that two n-bit integers can be added in time $\mathcal{O}(n)$.

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Suppose that we want to multiply an n-bit integer A and an m-bit integer B ($m \leq n$).

- This is also nown as the "school method" for multiplying integers.
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\hline
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\hline
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 & & & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & & \\
\hline
1 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & &
\end{array}
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- This is also nown as the "school method" for multiplying integers.
- Note that the intermediate numbers that are generated can have at most $m + n \leq 2n$ bits.

Example: Multiplying Two Integers

Suppose that we want to multiply an n-bit integer A and an m-bit integer B ($m \leq n$).

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- Computing intermediate results: $\mathcal{O}(nm)$.
- Adding m numbers of length $\leq 2n$: $\mathcal{O}((m+n)m) = \mathcal{O}(nm)$.

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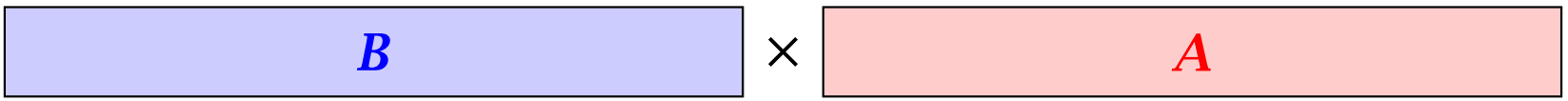

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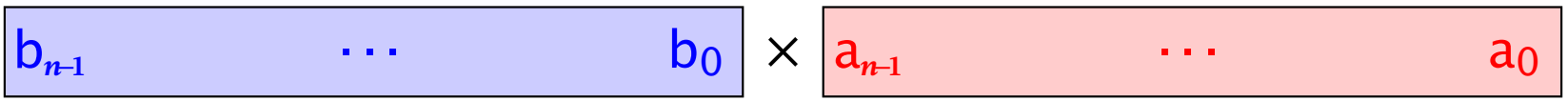

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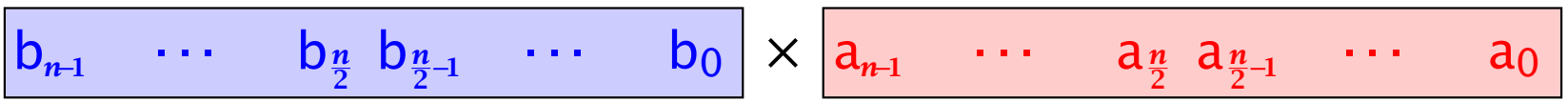

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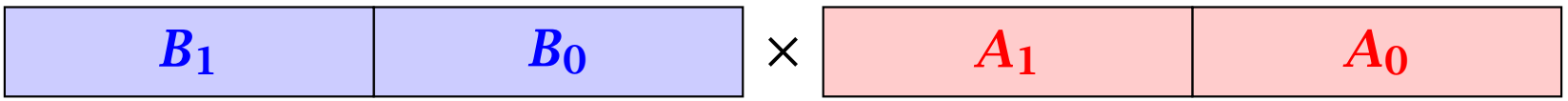

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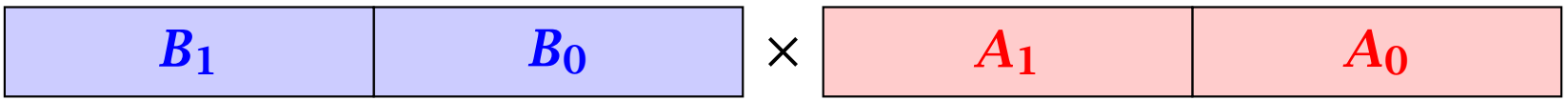

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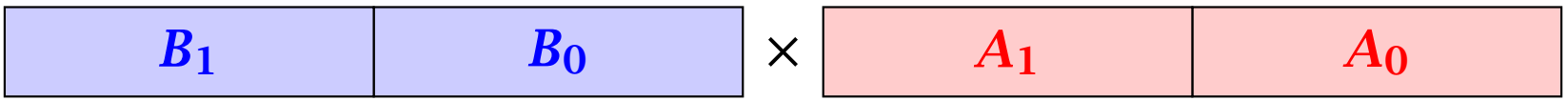

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We get the following recurrence:

$$T(n) = 4T\left(\frac{n}{2}\right) + \mathcal{O}(n) \ .$$

Example: Multiplying Two Integers

Master Theorem: Recurrence: $T[n] = aT(\frac{n}{b}) + f(n)$.

- Case 1: $f(n) = \mathcal{O}(n^{\log_b a - \epsilon})$ $\quad T(n) = \Theta(n^{\log_b a})$
- Case 2: $f(n) = \Theta(n^{\log_b a} \log^k n)$ $\quad T(n) = \Theta(n^{\log_b a} \log^{k+1} n)$
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In our case $a = 4$, $b = 2$, and $f(n) = \Theta(n)$. Hence, we are in Case 1, since $n = \mathcal{O}(n^{2-\epsilon}) = \mathcal{O}(n^{\log_b a - \epsilon})$.

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$\Longrightarrow$ Not better then the "school method".

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We can use the following identity to compute Z_1:

A more precise (correct) analysis would say that computing Z_1 needs time $T(\frac{n}{2}+1)+\mathcal{O}(n)$.

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2:	**return** $a_0 \cdot b_0$	$\mathcal{O}(1)$
3:	split A into A_0 and A_1	$\mathcal{O}(n)$
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5:	$Z_2 \leftarrow \text{mult}(A_1, B_1)$	$T(\frac{n}{2})$
6:	$Z_0 \leftarrow \text{mult}(A_0, B_0)$	
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A more precise (correct) analysis would say that computing Z_1 needs time $T(\frac{n}{2} + 1) + \mathcal{O}(n)$.

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- Case 2: $f(n) = \Theta(n^{\log_b a} \log^k n)$ $\quad T(n) = \Theta(n^{\log_b a} \log^{k+1} n)$
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A huge improvement over the "school method".