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- $\boldsymbol{S}.\textbf{insert}(\boldsymbol{x})$: Insert an element x.
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Hashing tries to directly compute the memory location from the given key. The goal is to have constant search time.

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Definitions:

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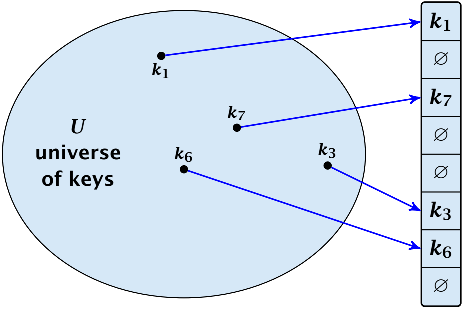
The hash-function h should fulfill:

- Fast to evaluate.
- Small storage requirement.
- Good distribution of elements over the whole table.

Direct Addressing

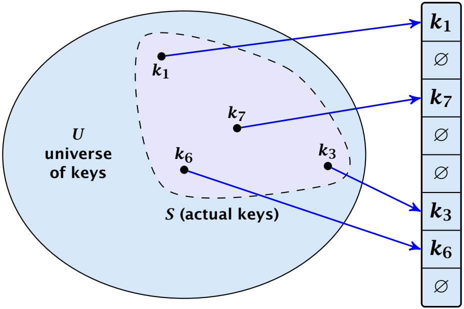
Ideally the hash function maps all keys to different memory locations.



This special case is known as Direct Addressing. It is usually very unrealistic as the universe of keys typically is quite large, and in particular larger than the available memory.

Perfect Hashing

Suppose that we know the set S of actual keys (no insert/no delete). Then we may want to design a simple hash-function that maps all these keys to different memory locations.



Such a hash function h is called a perfect hash function for set S.

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If we do not know the keys in advance, the best we can hope for is that the hash function distributes keys evenly across the table.

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Hence, there may be two elements k_1, k_2 from the set S that map to the same memory location (i.e., $h(k_1) = h(k_2)$). This is called a collision.

Collisions

Typically, collisions do not appear once the size of the set S of actual keys gets close to n, but already when $|S| \geq \omega(\sqrt{n})$.

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Lemma 1
The probability of having a collision when hashing m elements into a table of size n under uniform hashing is at least

$$1 - e^{-\frac{m(m-1)}{2n}} \approx 1 - e^{-\frac{m^2}{2n}} \; .$$

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Uniform hashing:

Choose a hash function uniformly at random from all functions $f : U \to [0, \ldots, n-1]$.

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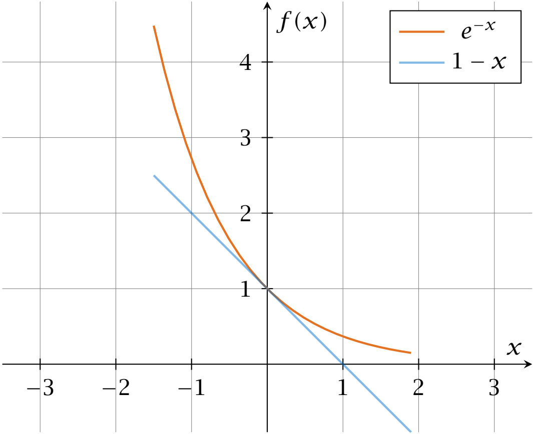
$$\begin{aligned}\Pr[A_{m,n}] &= \prod_{\ell=1}^{m} \frac{n-\ell+1}{n} = \prod_{j=0}^{m-1} \left(1 - \frac{j}{n}\right) \\ &\le \prod_{j=0}^{m-1} e^{-j/n} = e^{-\sum_{j=0}^{m-1} \frac{j}{n}} = e^{-\frac{m(m-1)}{2n}} \enspace .\end{aligned}$$

Here the first equality follows since the ℓ-th element that is hashed has a probability of $\frac{n-\ell+1}{n}$ to not generate a collision under the condition that the previous elements did not induce collisions. □

Collisions



The inequality $1 - x \le e^{-x}$ is derived by stopping the Taylor-expansion of e^{-x} after the second term.

Resolving Collisions

The methods for dealing with collisions can be classified into the two main types

- **open addressing**, aka. closed hashing
- **hashing with chaining**, aka. closed addressing, open hashing.

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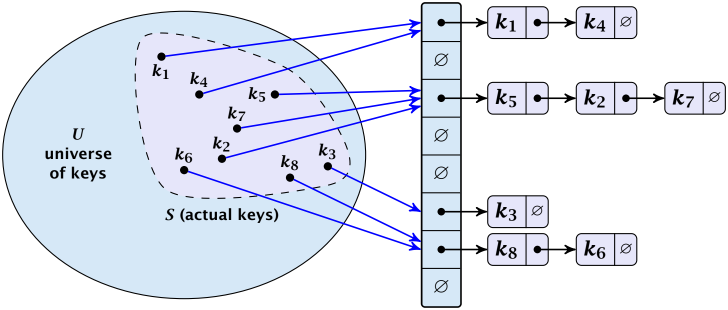
There are applications e.g. computer chess where you do not resolve collisions at all.

Hashing with Chaining

Arrange elements that map to the same position in a linear list.

- Access: compute $h(x)$ and search list for $\text{key}[x]$.
- Insert: insert at the front of the list.



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We assume uniform hashing for the following analysis.

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$$A^{-} = 1 + \alpha \ .$$

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For a successful search observe that we do **not** choose a list at random, but we consider a random key k in the hash-table and ask for the search-time for k.

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Hence, the expected cost for a successful search is $A^+ \leq 1+\frac{\alpha}{2}$.

Hashing with Chaining

Disadvantages:

- pointers increase memory requirements
- pointers may lead to bad cache efficiency

Advantages:

- no à priori limit on the number of elements
- deletion can be implemented efficiently
- by using balanced trees instead of linked list one can also obtain worst-case guarantees.

Open Addressing

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All objects are stored in the table itself.

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Search(k): Try position $h(k, 0)$; if it is empty your search fails; otw. continue with $h(k, 1)$, $h(k, 2)$, ...

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Search(k): Try position $h(k, 0)$; if it is empty your search fails; otw. continue with $h(k, 1)$, $h(k, 2)$, ...

Insert(x): Search until you find an empty slot; insert your element there. If your search reaches $h(k, n-1)$, and this slot is non-empty then your table is full.

Open Addressing

Choices for $h(k,j)$:

- Linear probing:
  $h(k,i) = h(k) + i \mod n$
  (sometimes: $h(k,i) = h(k) + ci \mod n$).

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For quadratic probing and double hashing one has to ensure that the search covers all positions in the table (i.e., for double hashing $h_2(k)$ must be relatively prime to n (teilerfremd); for quadratic probing c_1 and c_2 have to be chosen carefully).

Linear Probing

- Advantage: Cache-efficiency. The new probe position is very likely to be in the cache.

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Lemma 2

Let L be the method of linear probing for resolving collisions:

$$L^+ \approx \frac{1}{2}\left(1 + \frac{1}{1-\alpha}\right)$$

$$L^- \approx \frac{1}{2}\left(1 + \frac{1}{(1-\alpha)^2}\right)$$

Quadratic Probing

- Not as cache-efficient as Linear Probing.
- Secondary clustering: caused by the fact that all keys mapped to the same position have the same probe sequence.

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Lemma 3

Let Q be the method of quadratic probing for resolving collisions:

$$Q^+ \approx 1 + \ln\left(\frac{1}{1-\alpha}\right) - \frac{\alpha}{2}$$

$$Q^- \approx \frac{1}{1-\alpha} + \ln\left(\frac{1}{1-\alpha}\right) - \alpha$$

Double Hashing

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Lemma 4

Let D be the method of double hashing for resolving collisions:

$$D^+ \approx \frac{1}{\alpha} \ln\left(\frac{1}{1-\alpha}\right)$$

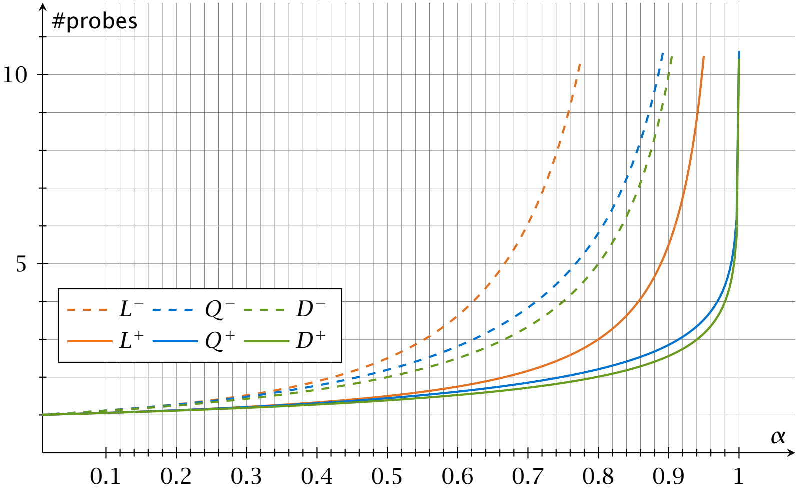
$$D^- \approx \frac{1}{1-\alpha}$$

Open Addressing

Some values:

α	*Linear Probing*		*Quadratic Probing*		*Double Hashing*	
	L^+	L^-	Q^+	Q^-	D^+	D^-
0.5	1.5	2.5	1.44	2.19	1.39	2
0.9	5.5	50.5	2.85	11.40	2.55	10
0.95	10.5	200.5	3.52	22.05	3.15	20

Open Addressing



Analysis of Idealized Open Address Hashing

We analyze the time for a search in a very idealized Open Addressing scheme.

- The probe sequence $h(k,0), h(k,1), h(k,2), \ldots$ is equally likely to be any permutation of $\langle 0, 1, \ldots, n-1 \rangle$.

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Analysis of Idealized Open Address Hashing

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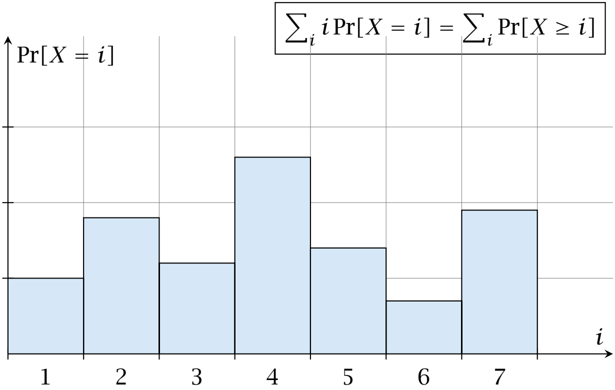
$$\frac{1}{1-\alpha} = 1 + \alpha + \alpha^2 + \alpha^3 + \ldots$$

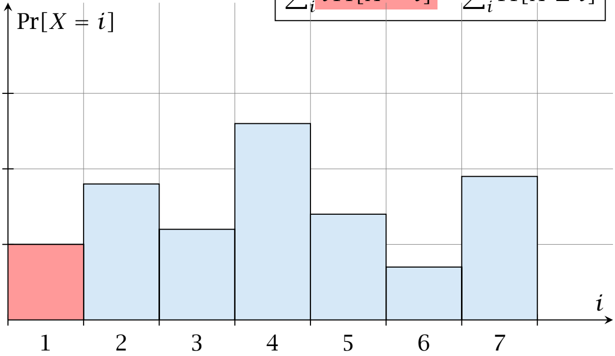
Analysis of Idealized Open Address Hashing



Analysis of Idealized Open Address Hashing

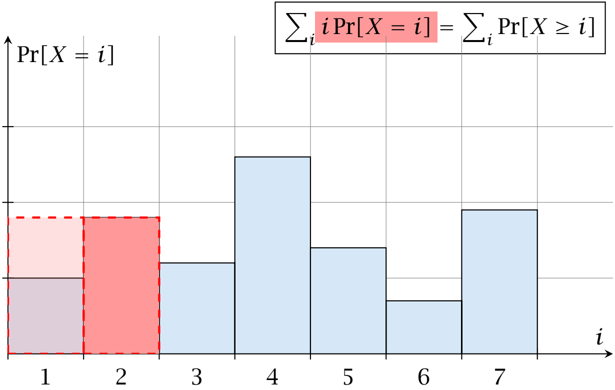
$i = 1$

$\sum_i i \Pr[X = i] = \sum_i \Pr[X \geq i]$



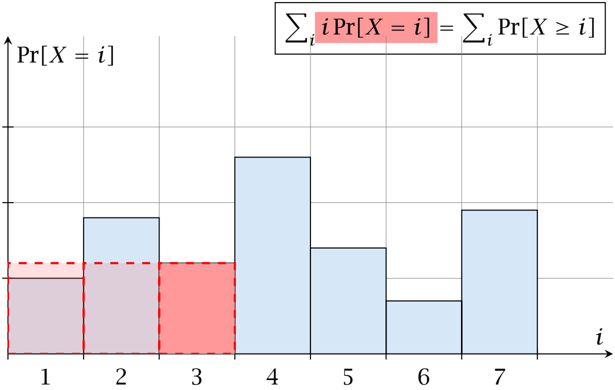
Analysis of Idealized Open Address Hashing

$i = 2$



Analysis of Idealized Open Address Hashing

$i = 3$



Analysis of Idealized Open Address Hashing

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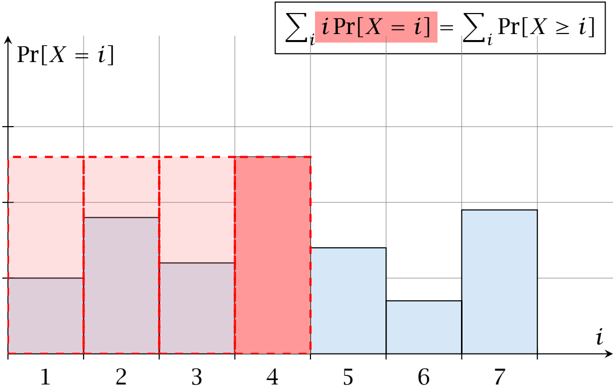

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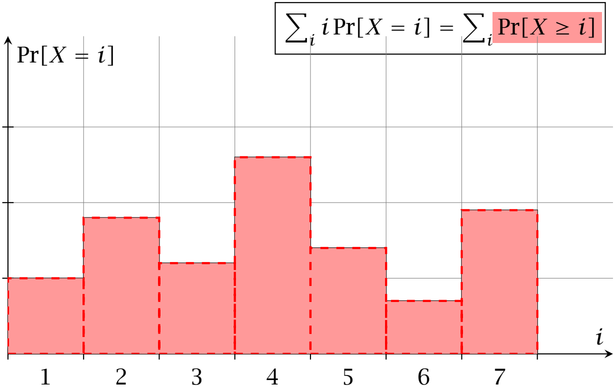

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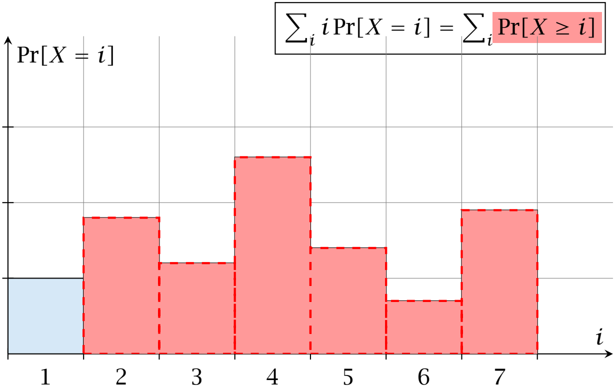

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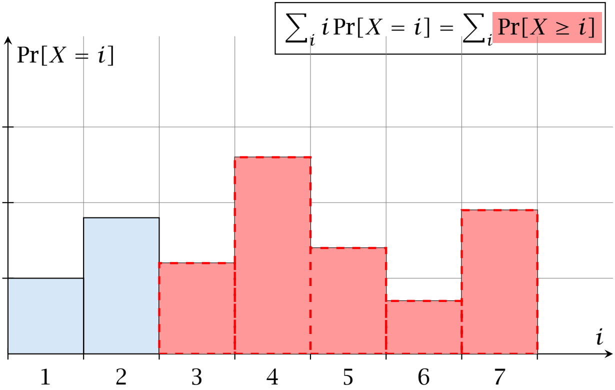

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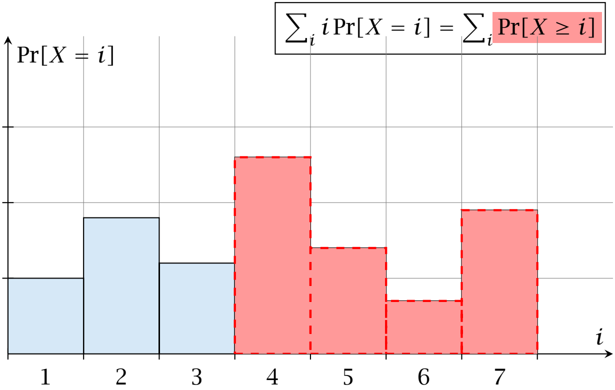

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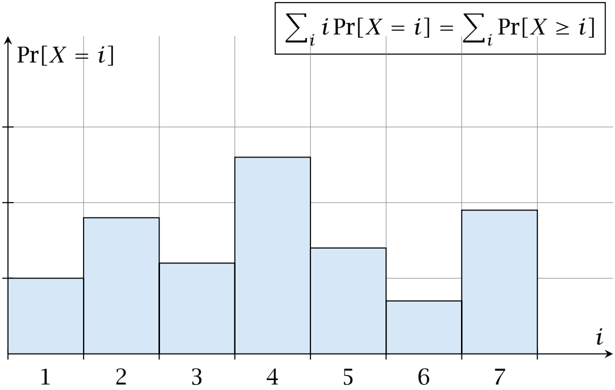

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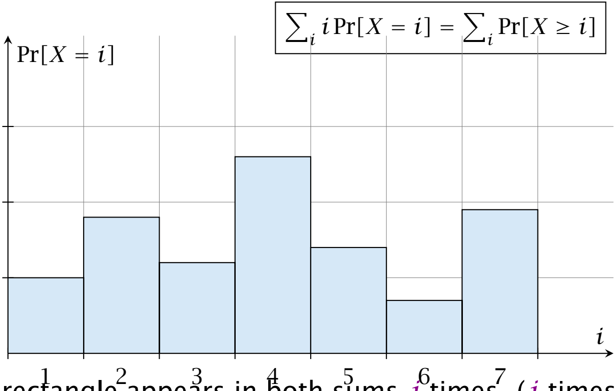

The j-th rectangle appears in both sums j times. (j times in the first due to multiplication with j; and j times in the second for summands $i = 1, 2, \ldots, j$)

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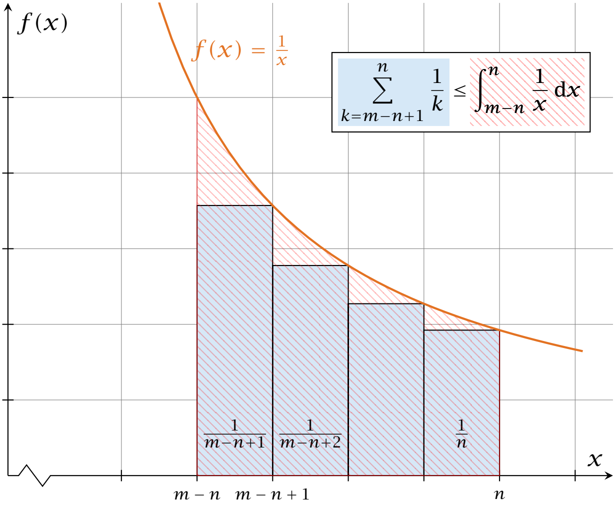

Deletions in Hashtables

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- For open addressing this is difficult.

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- The table could fill up with deleted-markers leading to bad performance.
- If a table contains many deleted-markers (linear fraction of the keys) one can rehash the whole table and amortize the cost for this rehash against the cost for the deletions.

Deletions for Linear Probing

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- Upon a deletion elements that are further down in the probe-sequence may be moved to guarantee that they are still found during a search.

Deletions for Linear Probing

Algorithm 37 delete(p)

```
T[p] ← null
p ← succ(p)
while T[p] ≠ null do
    y ← T[p]
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p is the index into the table-cell that contains the object to be deleted.

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Pointers into the hash-table become invalid.

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However, the assumption of uniform hashing that h is chosen randomly from all functions $f : U \to [0, \ldots, n-1]$ is clearly unrealistic as there are $n^{|U|}$ such functions. Even writing down such a function would take $|U| \log n$ bits.

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Universal hashing tries to define a set $\mathcal{H}$ of functions that is much smaller but still leads to good average case behaviour when selecting a hash-function uniformly at random from $\mathcal{H}$.

Universal Hashing

Definition 5
A class $\mathcal{H}$ of hash-functions from the universe U into the set $\{0, \dots, n-1\}$ is called universal if for all $u_1, u_2 \in U$ with $u_1 \neq u_2$

$$\Pr[h(u_1) = h(u_2)] \leq \frac{1}{n} ,$$

where the probability is w. r. t. the choice of a random hash-function from set $\mathcal{H}$.

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Note that this means that the probability of a collision between two arbitrary elements is at most $\frac{1}{n}$.

Universal Hashing

Definition 6

A class $\mathcal{H}$ of hash-functions from the universe U into the set $\{0, \ldots, n-1\}$ is called 2-independent (pairwise independent) if the following two conditions hold

- For any key $u \in U$, and $t \in \{0, \ldots, n-1\}$ $\Pr[h(u) = t] = \frac{1}{n}$, i.e., a key is distributed uniformly within the hash-table.
- For all $u_1, u_2 \in U$ with $u_1 \neq u_2$, and for any two hash-positions t_1, t_2:

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This requirement clearly implies a universal hash-function.

Universal Hashing

Definition 7
A class $\mathcal{H}$ of hash-functions from the universe U into the set $\{0, \dots, n-1\}$ is called k-independent if for any choice of $\ell \leq k$ distinct keys $u_1, \dots, u_\ell \in U$, and for any set of ℓ not necessarily distinct hash-positions $t_1, \dots, t_\ell$:

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where the probability is w. r. t. the choice of a random hash-function from set $\mathcal{H}$.

Universal Hashing

Definition 8

A class $\mathcal{H}$ of hash-functions from the universe U into the set $\{0, \dots, n-1\}$ is called (μ, k)-independent if for any choice of $\ell \le k$ distinct keys $u_1, \dots, u_\ell \in U$, and for any set of ℓ not necessarily distinct hash-positions $t_1, \dots, t_\ell$:

$$\Pr[h(u_1) = t_1 \wedge \cdots \wedge h(u_\ell) = t_\ell] \le \frac{\mu}{n^\ell} \ ,$$

where the probability is w. r. t. the choice of a random hash-function from set $\mathcal{H}$.

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Let $U := \{0, \ldots, p-1\}$ for a prime p. Let $\mathbb{Z}_p := \{0, \ldots, p-1\}$, and let $\mathbb{Z}_p^* := \{1, \ldots, p-1\}$ denote the set of invertible elements in $\mathbb{Z}_p$.

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$$h_{a,b}(x) := (ax + b \bmod p) \bmod n$$

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Lemma 9

The class

$$\mathcal{H} = \{h_{a,b} \mid a \in \mathbb{Z}_p^*, b \in \mathbb{Z}_p\}$$

is a universal class of hash-functions from U to $\{0, \dots, n-1\}$.

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  where we use that $\mathbb{Z}_p$ is a field (Körper) and, hence, has no zero divisors (nullteilerfrei).

Universal Hashing

- The hash-function does not generate collisions before the $\pmod{n}$-operation. Furthermore, every choice (a, b) is mapped to a different pair (t_x, t_y) with $t_x := ax + b$ and $t_y := ay + b$.

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t_x &\equiv ax + b && \pmod{p} \\
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$$

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$$t_x - t_y \equiv a(x - y) \pmod{p}$$
$$t_y \equiv ay + b \pmod{p}$$

$$a \equiv (t_x - t_y)(x - y)^{-1} \pmod{p}$$
$$b \equiv t_y - ay \pmod{p}$$

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Fix a value t_x. There are $p - 1$ possible values for choosing t_y.

From the range $0, \dots, p-1$ the values $t_x, t_x + n, t_x + 2n, \dots$ map to t_x after the modulo-operation. These are at most $\lceil p/n \rceil$ values.

Universal Hashing

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This happens with probability at most $\frac{1}{n}$.

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It is also possible to show that $\mathcal{H}$ is an (almost) pairwise independent class of hash-functions.

$$\Pr_{t_x \neq t_y \in \mathbb{Z}_p^2}\left[\begin{array}{c} t_x \bmod n = h_1 \\ \wedge \\ t_y \bmod n = h_2 \end{array}\right]$$

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$$\frac{\left\lfloor \frac{p}{n} \right\rfloor^2}{p(p-1)} \le \Pr_{t_x \ne t_y \in \mathbb{Z}_p^2}\left[\begin{array}{c} t_x \bmod n = h_1 \\ \wedge \\ t_y \bmod n = h_2 \end{array}\right] \le \frac{\left\lceil \frac{p}{n} \right\rceil^2}{p(p-1)}$$

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Note that the middle is the probability that $h(x) = h_1$ and $h(y) = h_2$. The total number of choices for (t_x, t_y) is $p(p-1)$. The number of choices for t_x (t_y) such that $t_x \bmod n = h_1$ ($t_y \bmod n = h_2$) lies between $\lfloor \frac{p}{n} \rfloor$ and $\lceil \frac{p}{n} \rceil$.

Universal Hashing

Definition 10
Let $d \in \mathbb{N}$; $q \geq (d+1)n$ be a prime; and let $\bar{a} \in \{0, \ldots, q-1\}^{d+1}$. Define for $x \in \{0, \ldots, q-1\}$

$$h_{\bar{a}}(x) := \Big(\sum_{i=0}^{d} a_i x^i \bmod q \Big) \bmod n \ .$$

Let $\mathcal{H}_n^d := \{h_{\bar{a}} \mid \bar{a} \in \{0, \ldots, q-1\}^{d+1}\}$. The class $\mathcal{H}_n^d$ is $(e, d+1)$-independent.

Note that in the previous case we had $d = 1$ and chose $a_d \neq 0$.

Universal Hashing

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For the coefficients $\bar{a} \in \{0, \ldots, q-1\}^{d+1}$ let $f_{\bar{a}}$ denote the polynomial

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The polynomial is defined by $d+1$ distinct points.

Universal Hashing

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Fix $\ell \leq d+1$; let $x_1, \ldots, x_\ell \in \{0, \ldots, q-1\}$ be keys, and let $t_1, \ldots, t_\ell$ denote the corresponding hash-function values.

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$$h_{\bar{a}} \in A^\ell \Leftrightarrow h_{\bar{a}} = f_{\bar{a}} \bmod n \text{ and}$$

$$f_{\bar{a}}(x_i) \in \underbrace{\{t_i + \alpha \cdot n \mid \alpha \in \{0, \dots, \lceil \tfrac{q}{n} \rceil - 1\}\}}_{=: B_i}$$

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We have

$$|B_1| \cdot \ldots \cdot |B_\ell|$$

possibilities to do this (so that $h_{\bar{a}}(x_i) = t_i$).

Universal Hashing

Now, we choose $d - \ell + 1$ other inputs and choose their value arbitrarily. We have $q^{d-\ell+1}$ possibilities to do this.

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Therefore we have

$$|B_1| \cdot \ldots \cdot |B_\ell| \cdot q^{d-\ell+1} \leq \lceil \frac{q}{n} \rceil^\ell \cdot q^{d-\ell+1}$$

possibilities to choose $\bar{a}$ such that $h_{\bar{a}} \in A_\ell$.

Universal Hashing

Therefore the probability of choosing $h_{\bar{a}}$ from A_ℓ is only

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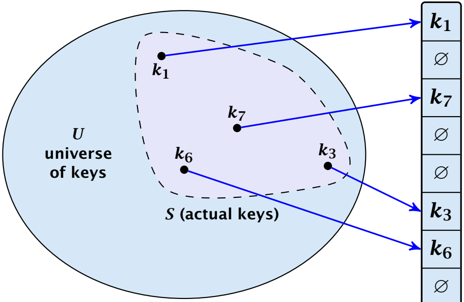
$$\le \Big(1+\frac{1}{\ell}\Big)^\ell \cdot \frac{1}{n^\ell} \le \frac{e}{n^\ell} \enspace .$$

This shows that the $\mathcal{H}$ is $(e, d+1)$-universal.

The last step followed from $q \ge (d+1)n$, and $\ell \le d+1$.

Perfect Hashing

Suppose that we **know** the set S of actual keys (no insert/no delete). Then we may want to design a **simple** hash-function that maps all these keys to different memory locations.



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Can we get an upper bound on the probability of having collisions?

The probability of having 1 or more collisions can be at most $\frac{1}{2}$ as otherwise the expectation would be larger than $\frac{1}{2}$.

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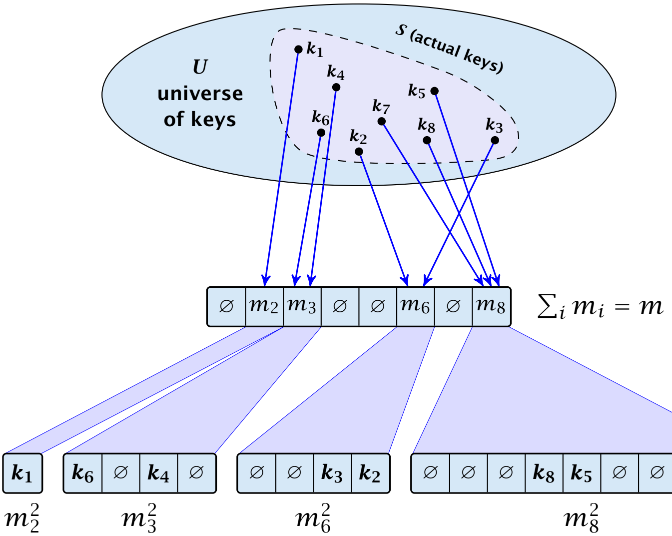
Let m_j denote the number of items that are hashed to the j-th bucket. For each bucket we choose a second hash-function that maps the elements of the bucket into a table of size m_j^2. The second function can be chosen such that all elements are mapped to different locations.

Perfect Hashing



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The first expectation is simply the expected number of collisions, for the first level. Since we use universal hashing we have

$$= 2\binom{m}{2}\frac{1}{m} + m = 2m - 1\ .$$

Perfect Hashing

We need only $\mathcal{O}(m)$ time to construct a hash-function h with $\sum_j m_j^2 = \mathcal{O}(4m)$, because with probability at least $1/2$ a random function from a universal family will have this property.

Then we construct a hash-table h_j for every bucket. This takes expected time $\mathcal{O}(m_j)$ for every bucket. A random function h_j is collision-free with probability at least $1/2$. We need $\mathcal{O}(m_j)$ to test this.

We only need that the hash-functions are chosen from a universal family!!!

Cuckoo Hashing

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Goal:
Try to generate a hash-table with constant worst-case search time in a dynamic scenario.

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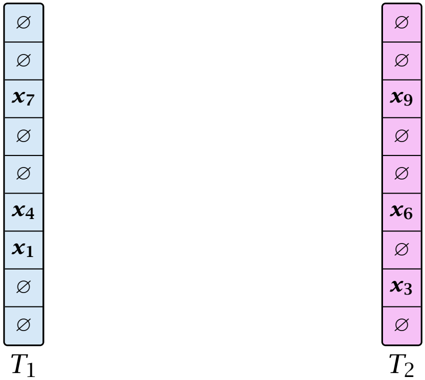
- An object x is either stored at location $T_1[h_1(x)]$ or $T_2[h_2(x)]$.
- A search clearly takes constant time if the above constraint is met.

Cuckoo Hashing

Insert:



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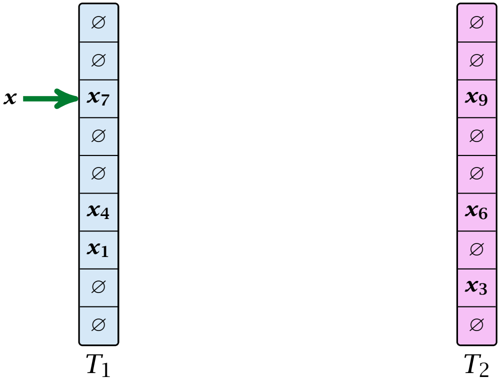

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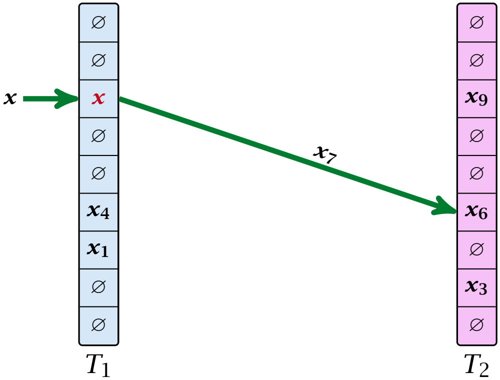

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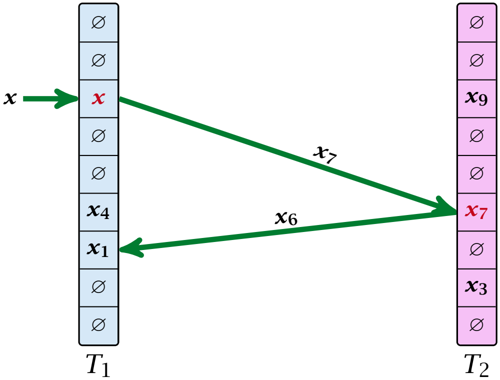

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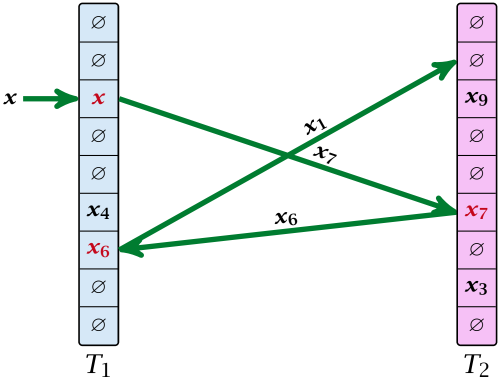

Cuckoo Hashing

Algorithm 38 Cuckoo-Insert(x)

if $T_1[h_1(x)] = x \vee T_2[h_2(x)] = x$ **then return**
steps $\leftarrow 1$
while steps $\leq$ maxsteps **do**
    exchange x and $T_1[h_1(x)]$
    if $x = \text{null}$ **then return**
    exchange x and $T_2[h_2(x)]$
    if $x = \text{null}$ **then return**
    steps $\leftarrow$ steps $+1$
rehash() // change hash-functions; rehash everything
Cuckoo-Insert(x)

Cuckoo Hashing

- We call one iteration through the while-loop a step of the algorithm.

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- We call a sequence of iterations through the while-loop without the termination condition becoming true a phase of the algorithm.
- We say a phase is successful if it is not terminated by the maxstep-condition, but the while loop is left because $x = \text{null}$.

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We first analyze the probability that we end-up in an infinite loop (that is then terminated after maxsteps steps).

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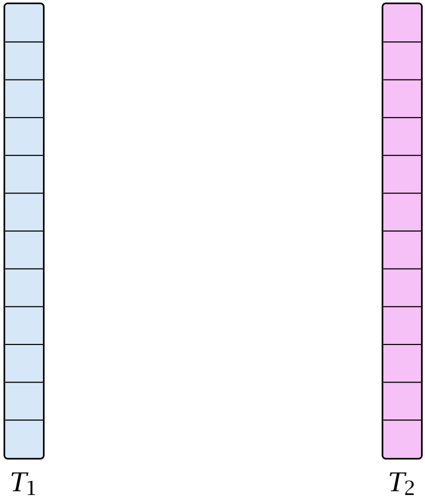
We first analyze the probability that we end-up in an infinite loop (that is then terminated after $\mathrm{maxsteps}$ steps).

Formally what is the probability to enter an infinite loop that touches s different keys?

Cuckoo Hashing: Insert



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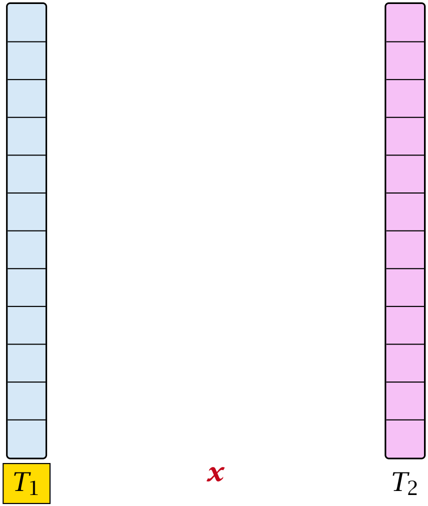

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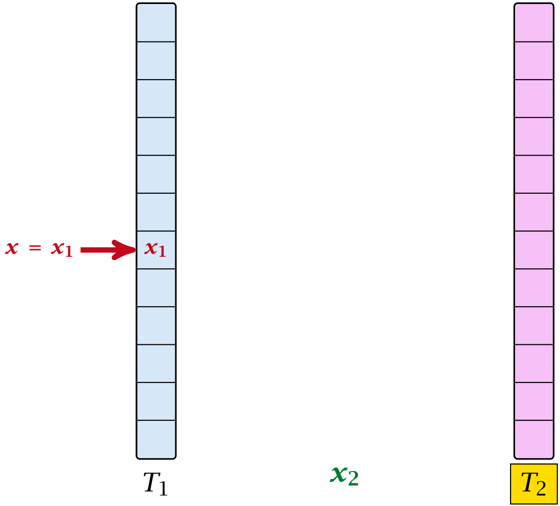

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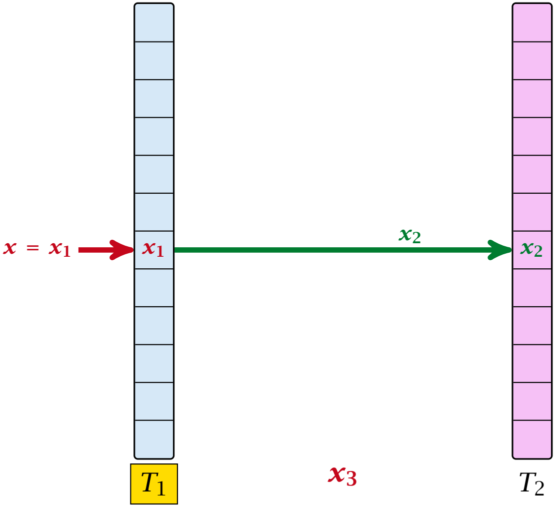

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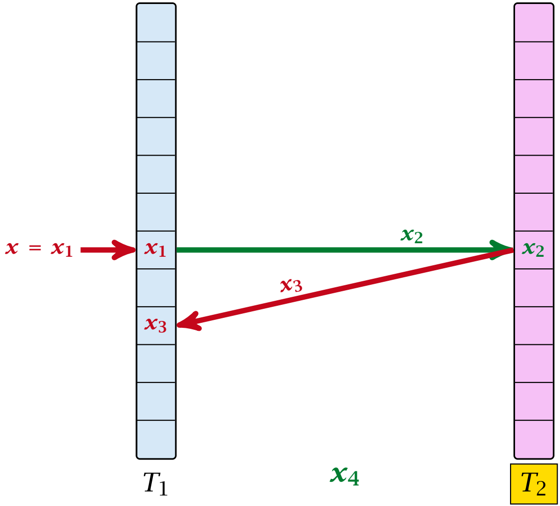

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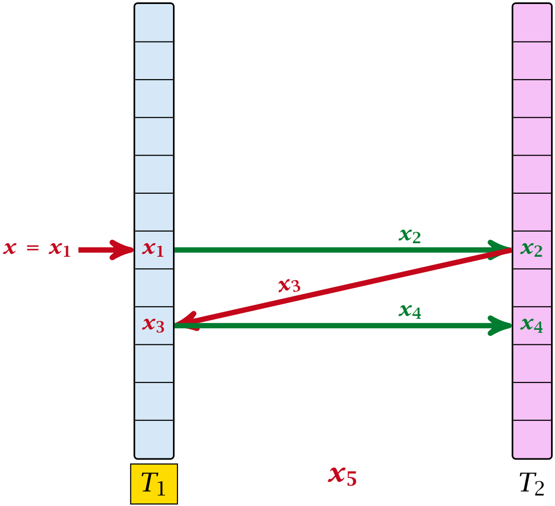

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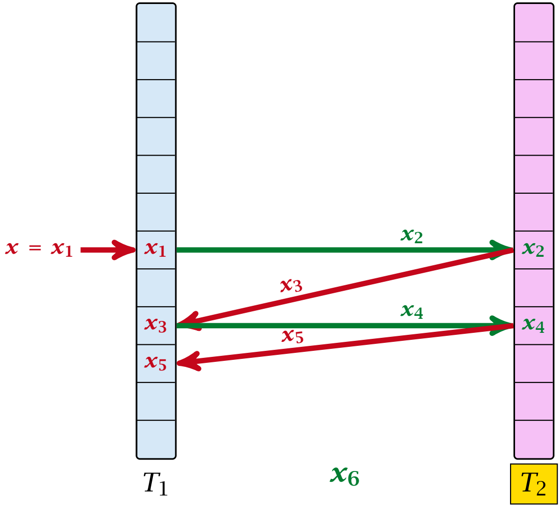
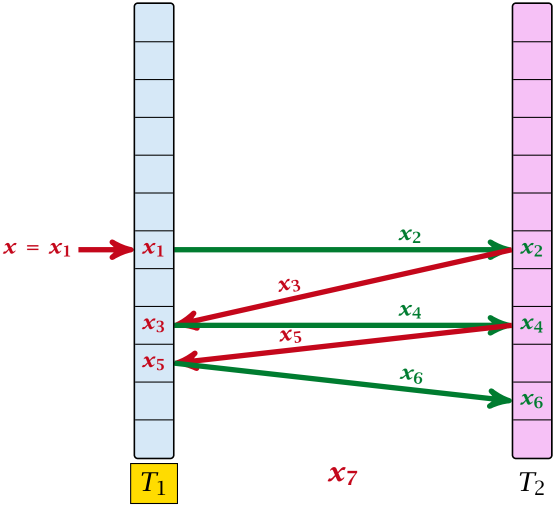


Cuckoo Hashing: Insert



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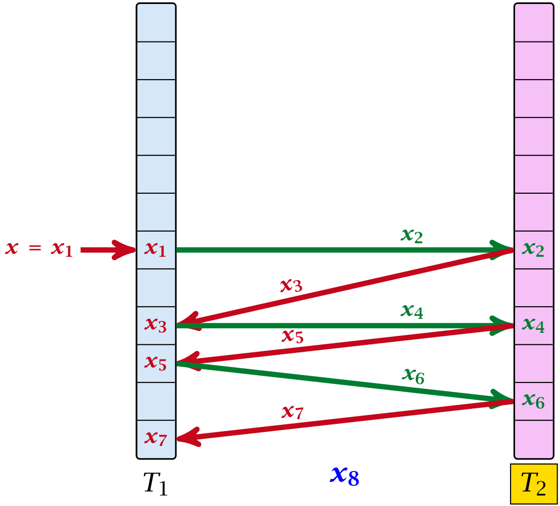

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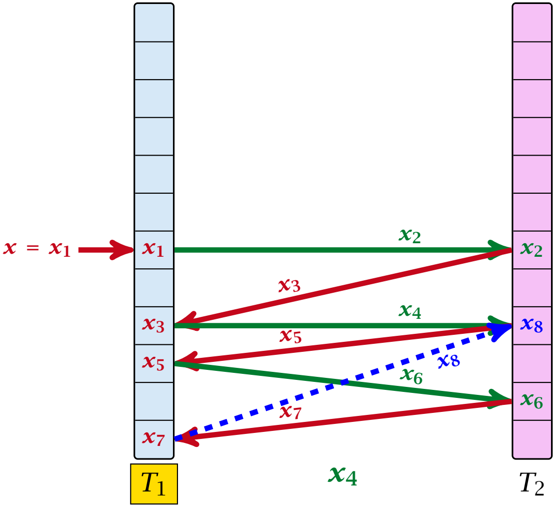

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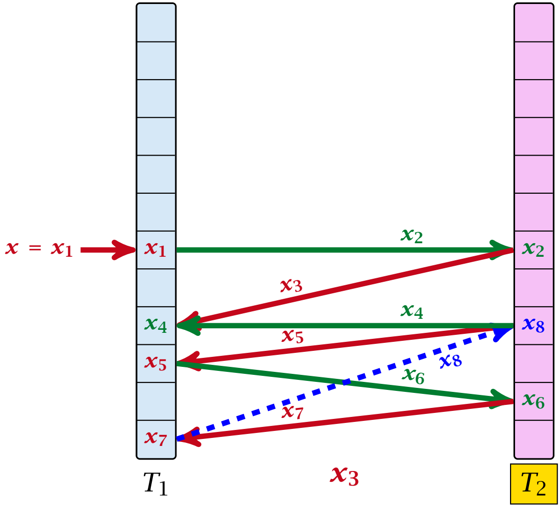

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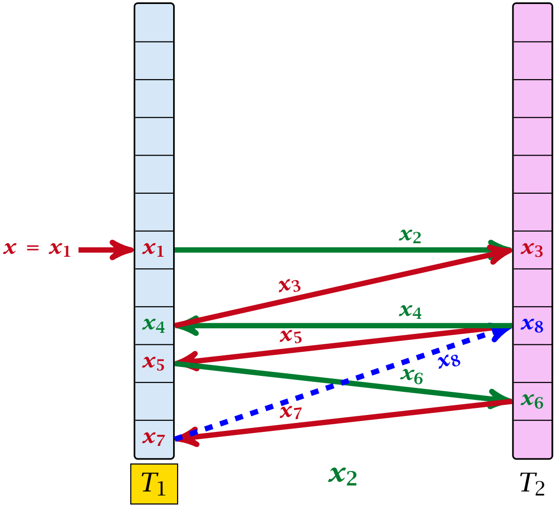

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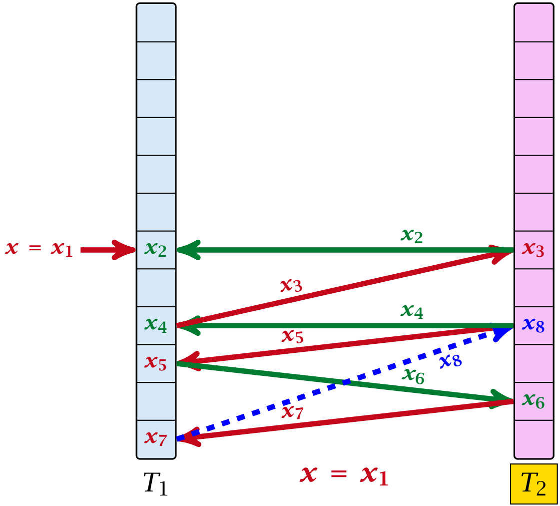

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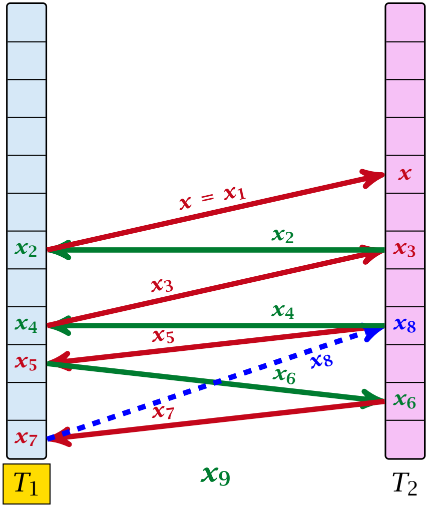

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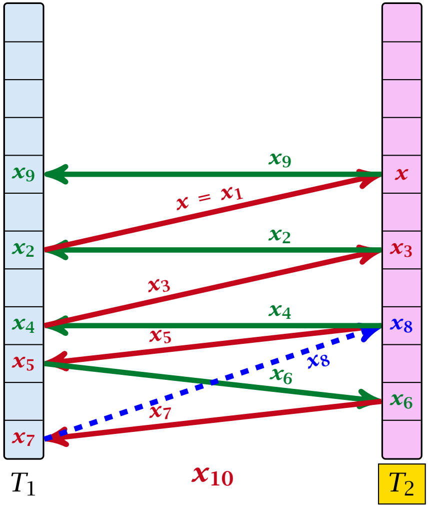

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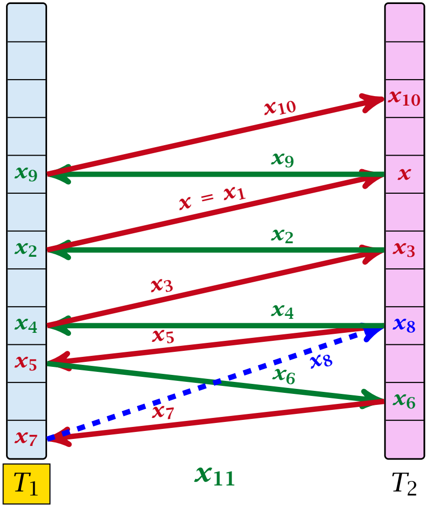
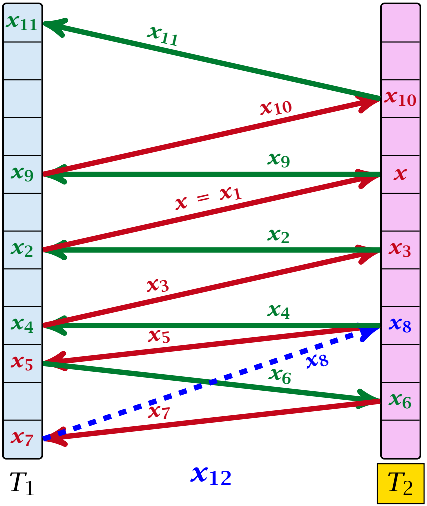


Cuckoo Hashing: Insert



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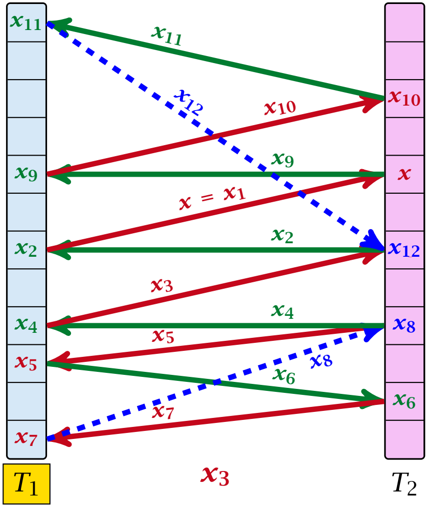

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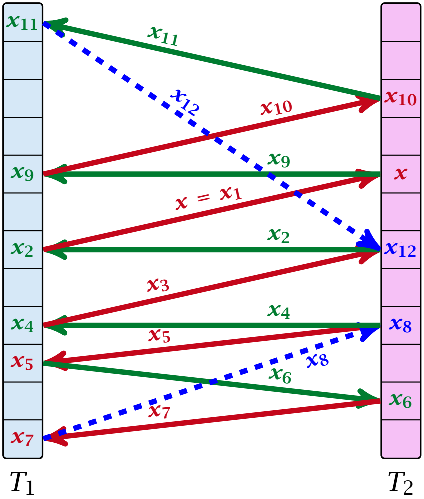

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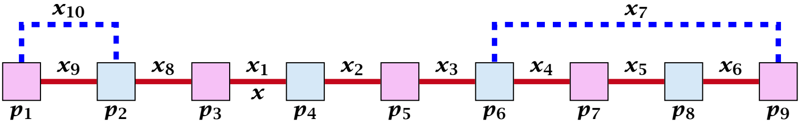

A cycle-structure of size s is defined by

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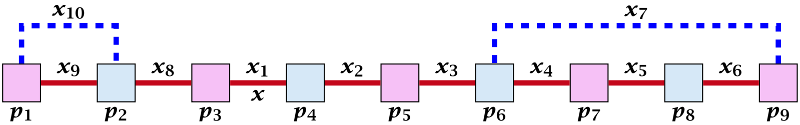

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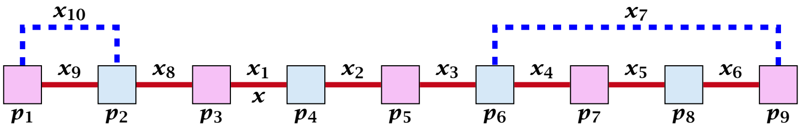
Cuckoo Hashing



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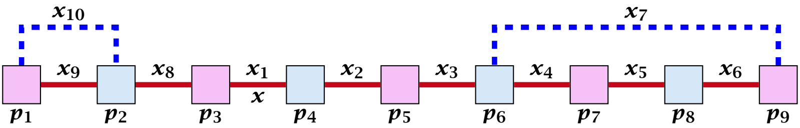

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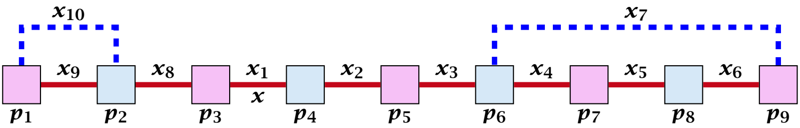

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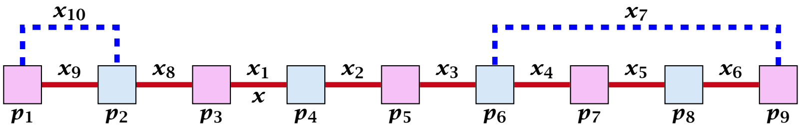

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- The leftmost cell is "linked forward" to some cell on the right.
- The rightmost cell is "linked backward" to a cell on the left.
- One link represents key x; this is where the counting starts.

Cuckoo Hashing

A cycle-structure is active if for every key x_ℓ (linking a cell p_i from T_1 and a cell p_j from T_2) we have

$$h_1(x_\ell) = p_i \qquad \text{and} \qquad h_2(x_\ell) = p_j$$

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Observation:
If during a phase the insert-procedure runs into a cycle there must exist an active cycle structure of size $s \geq 3$.

Cuckoo Hashing

What is the probability that all keys in a cycle-structure of size s correctly map into their T_1-cell?

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These events are independent.

Cuckoo Hashing

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- There are at most s possibilities to choose where to place key x.
- There are m^{s-1} possibilities to choose the keys apart from x.
- There are n^{s-1} possibilities to choose the cells.

Cuckoo Hashing

The probability that there exists an active cycle-structure is therefore at most

$$\sum_{s=3}^{\infty} s^3 \cdot n^{s-1} \cdot m^{s-1} \cdot \frac{\mu^2}{n^{2s}}$$

Cuckoo Hashing

The probability that there exists an active cycle-structure is therefore at most

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Here we used the fact that $(1+\epsilon)m \leq n$.

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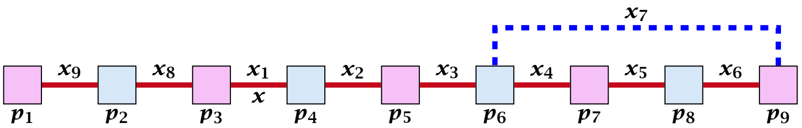
Hence,

$$\Pr[\text{cycle}] = \mathcal{O}\left(\frac{1}{m^2}\right) \quad .$$

Cuckoo Hashing

Now, we analyze the probability that a phase is not successful without running into a closed cycle.

Cuckoo Hashing



Sequence of visited keys:

$x = x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_3, x_2, x_1 = x, x_8, x_9, \ldots$

Cuckoo Hashing

Consider the sequence of not necessarily distinct keys starting with x in the order that they are visited during the phase.

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Lemma 11

If the sequence is of length p then there exists a sub-sequence of at least $\frac{p+2}{3}$ keys starting with x of distinct keys.

Cuckoo Hashing

Proof.

Let i be the number of keys (including x) that we see before the first repeated key. Let j denote the total number of distinct keys.

The sequence is of the form:

$x = x_1 \to x_2 \to \cdots \to x_i \to x_r \to x_{r-1} \to \cdots \to x_1 \to x_{i+1} \to \cdots \to x_j$

As $r \le i - 1$ the length p of the sequence is

$$p = i + r + (j - i) \le i + j - 1 \ .$$

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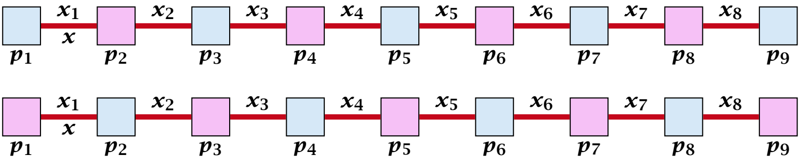
Either sub-sequence $x_1 \to x_2 \to \cdots \to x_i$ or sub-sequence $x_1 \to x_{i+1} \to \cdots \to x_j$ has at least $\frac{p+2}{3}$ elements. □

Cuckoo Hashing



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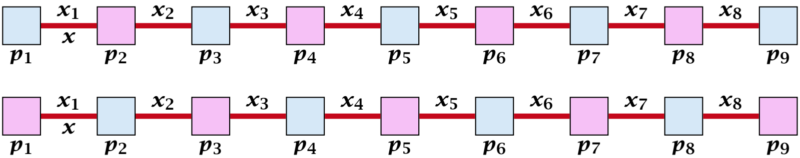
Cuckoo Hashing



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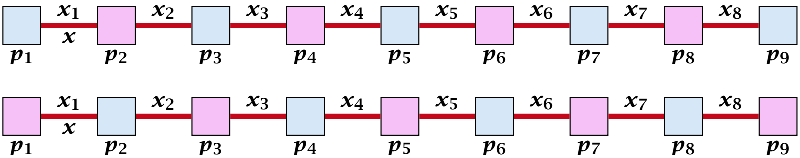

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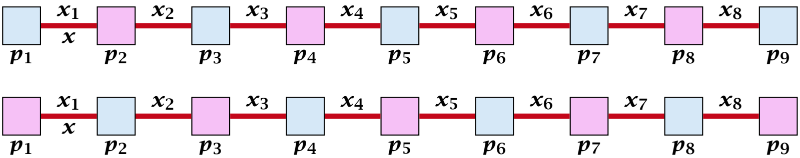

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Observation:
If a phase takes at least t steps without running into a cycle there must exist an active path-structure of size $(2t+2)/3$.

Cuckoo Hashing

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by choosing $\ell \geq \log\left(\frac{1}{2\mu^2 m^2}\right) / \log\left(\frac{1}{1+\epsilon}\right) = \log\left(2\mu^2 m^2\right) / \log\left(1+\epsilon\right)$

Cuckoo Hashing

We choose $\text{maxsteps} \geq 3\ell/2 + 1/2$. Then the probability that a phase terminates unsuccessfully without running into a cycle is at most

$$\begin{aligned}
&\Pr[\text{unsuccessful} \mid \text{no cycle}] \\
&\quad \leq \Pr[\exists \text{ active path-structure of size at least } \tfrac{2\text{maxsteps}+2}{3}] \\
&\quad \leq \Pr[\exists \text{ active path-structure of size at least } \ell + 1] \\
&\quad \leq \Pr[\exists \text{ active path-structure of size exactly } \ell + 1] \\
&\quad \leq 2\mu^2\Big(\frac{1}{1+\epsilon}\Big)^{\ell} \leq \frac{1}{m^2}
\end{aligned}$$

by choosing $\ell \geq \log\left(\frac{1}{2\mu^2 m^2}\right)/\log\left(\frac{1}{1+\epsilon}\right) = \log\left(2\mu^2 m^2\right)/\log\left(1+\epsilon\right)$

This gives $\text{maxsteps} = \Theta(\log m)$.

Cuckoo Hashing

So far we estimated

$$\Pr[\text{cycle}] \le \mathcal{O}\Big(\frac{1}{m^2}\Big)$$

and

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Cuckoo Hashing

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This means the expected cost for a successful phase is constant (even after accounting for the cost of the incomplete step that finishes the phase).

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Therefore the expected cost for re-hashes is $\mathcal{O}(m) \cdot \mathcal{O}(p) = \mathcal{O}(1)$.

Formal Proof

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Therefore, it is sufficient to have $(\mu, \Theta(\log m))$-independent hash-functions.

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- Note that right after a change in table-size we have $m = \alpha n/2$. In order for a table-expand to occur at least $\alpha n/2$ insertions are required. Similar, for a table-shrink at least $\alpha n/4$ deletions must occur.
- Therefore we can amortize the rehash cost after a change in table-size against the cost for insertions and deletions.

Cuckoo Hashing

Lemma 12

Cuckoo Hashing has an expected constant insert-time and a worst-case constant search-time.

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Cuckoo Hashing has an expected constant insert-time and a worst-case constant search-time.

Note that the above lemma only holds if the fill-factor (number of keys/total number of hash-table slots) is at most $\frac{1}{2(1+\epsilon)}$.