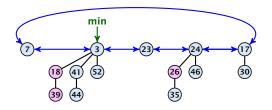
Collection of trees that fulfill the heap property.

Structure is much more relaxed than binomial heaps.

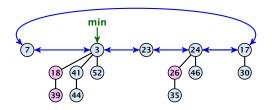


Additional implementation details:

- Every node x stores its degree in a field x. degree. Note that this can be updated in constant time when adding a child to x.
- Every node stores a boolean value x. marked that specifies whether x is marked or not.

The potential function:

- ightharpoonup t(S) denotes the number of trees in the heap.
- \blacktriangleright m(S) denotes the number of marked nodes.
- We use the potential function $\Phi(S) = t(S) + 2m(S)$.



The potential is $\Phi(S) = 5 + 2 \cdot 3 = 11$.

We assume that one unit of potential can pay for a constant amount of work, where the constant is chosen "big enough" (to take care of the constants that occur).

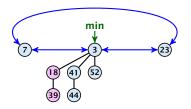
To make this more explicit we use \boldsymbol{c} to denote the amount of work that a unit of potential can pay for.

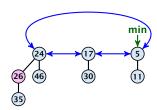
S. minimum()

- Access through the min-pointer.
- Actual cost $\mathcal{O}(1)$.
- No change in potential.
- Amortized cost $\mathcal{O}(1)$.

S. merge(S')

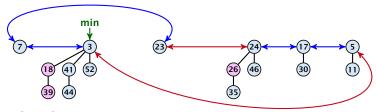
- Merge the root lists.
- Adjust the min-pointer





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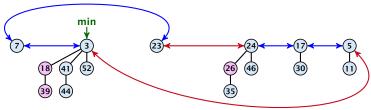


Running time:

Actual cost $\mathcal{O}(1)$.

S. merge(S')

- Merge the root lists.
- Adjust the min-pointer

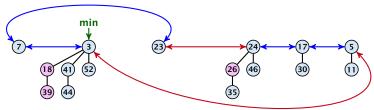


Running time:

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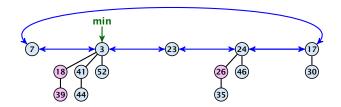
Running time:

- Actual cost $\mathcal{O}(1)$.
- No change in potential.
- \blacktriangleright Hence, amortized cost is $\mathcal{O}(1)$.



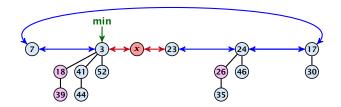
S.insert(x)

- ightharpoonup Create a new tree containing x.
- ▶ Insert *x* into the root-list.
- Update min-pointer, if necessary.



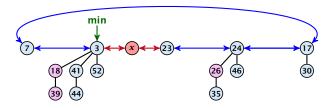
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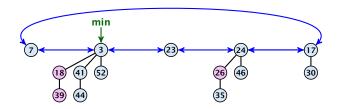


Running time:

- ightharpoonup Actual cost $\mathcal{O}(1)$.
- ightharpoonup Change in potential is +1.
- ▶ Amortized cost is c + O(1) = O(1).

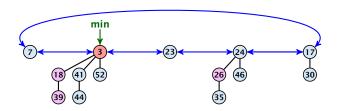


S. delete-min(x)



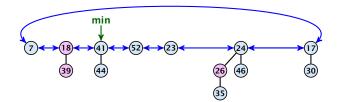
S. delete-min(x)

▶ Delete minimum; add child-trees to heap; time: $D(\min) \cdot O(1)$.



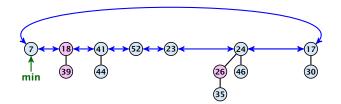
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- Update min-pointer; time: $(t + D(\min)) \cdot \mathcal{O}(1)$.



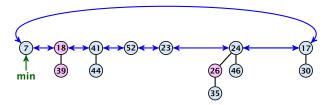
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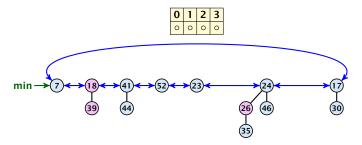


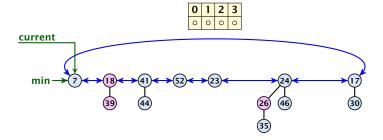
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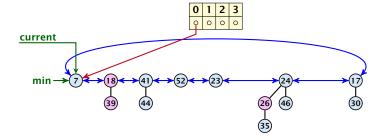
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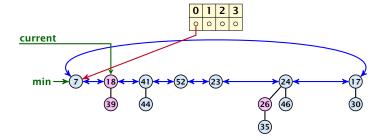


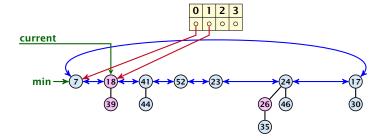
Consolidate root-list so that no roots have the same degree. Time $t\cdot\mathcal{O}(1)$ (see next slide).

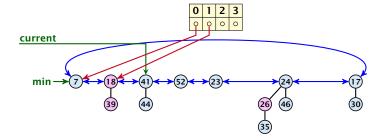


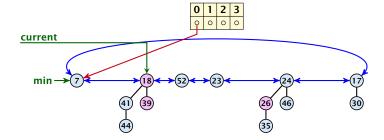


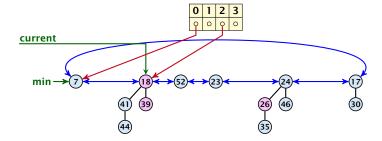


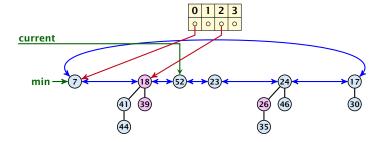


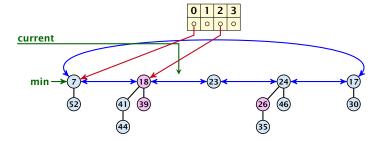


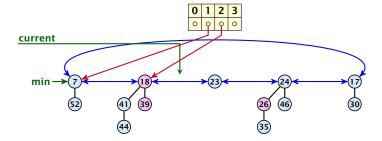


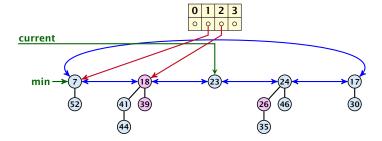


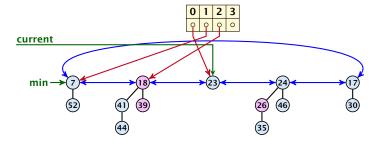


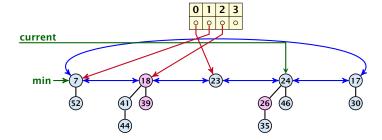


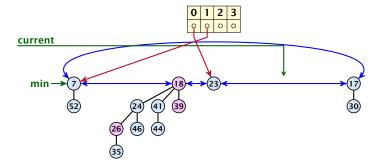


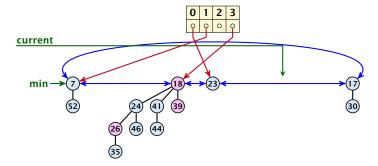


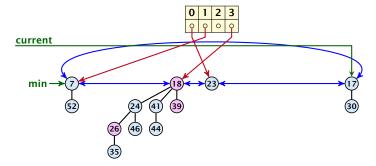


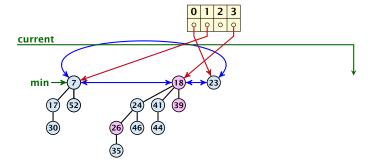


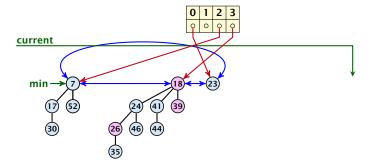




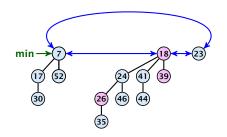








Consolidate:



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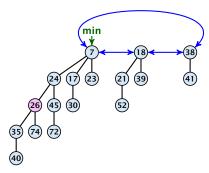
$$\leq (c_1 + c)D_n + (c_1 - c)t + c \leq 2c(D_n + 1) \leq \mathcal{O}(D_n)$$

for $c \ge c_1$.

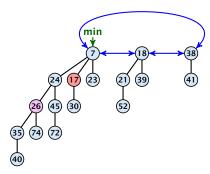
If the input trees of the consolidation procedure are binomial trees (for example only singleton vertices) then the output will be a set of distinct binomial trees, and, hence, the Fibonacci heap will be (more or less) a Binomial heap right after the consolidation.

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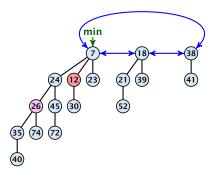
If we do not have delete or decrease-key operations then $D_n \leq \log n$.



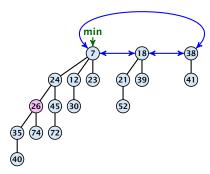
Case 1: decrease-key does not violate heap-property



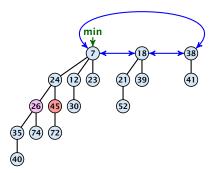
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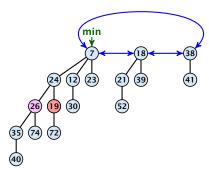


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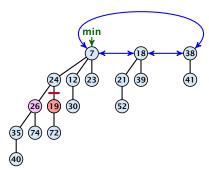
- Decrease key-value of element x reference by h.
- If the heap-property is violated, cut the parent edge of x, and make x into a root.
- Adjust min-pointers, if necessary.
- \blacktriangleright Mark the (previous) parent of x (unless it's a root).





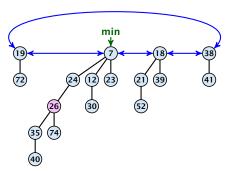
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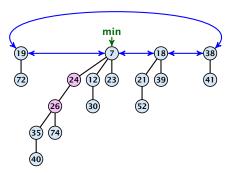
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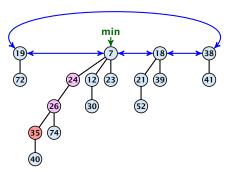
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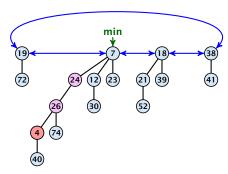
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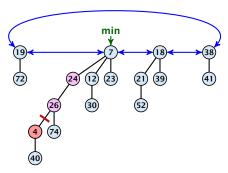
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- Continue cutting the parent until you arrive at an unmarked node.





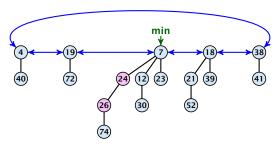
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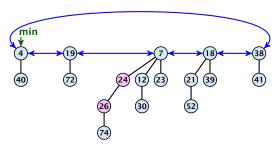
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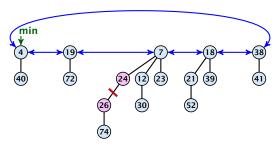
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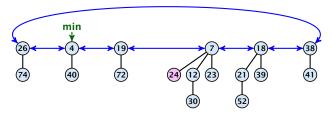
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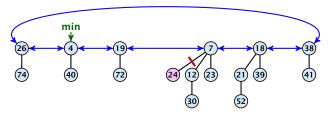
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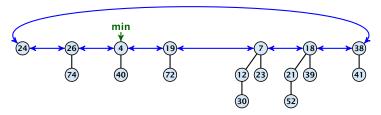
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- Execute the following:

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- ▶ Constant cost for each of ℓ cuts.
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- $ightharpoonup t'=t+\ell$, as every cut creates one new root.
- ▶ $m' \le m (\ell 1) + 1 = m \ell + 2$, since all but the first cut unmarks a node; the last cut may mark a node.

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- Constant cost for decreasing the value.
- ▶ Constant cost for each of ℓ cuts.
- ▶ Hence, cost is at most $c_2 \cdot (\ell + 1)$, for some constant c_2 .

- $t'=t+\ell$, as every cut creates one new root.
- ▶ $m' \le m (\ell 1) + 1 = m \ell + 2$, since all but the first cut unmarks a node; the last cut may mark a node.
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$$c_2(\ell+1) + c(4-\ell) \le (c_2-c)\ell + 4c + c_2 = \mathcal{O}(1)$$
, if $c \ge c_2$.

Delete node

H. delete(x):

- ▶ decrease value of x to $-\infty$.
- delete-min.

Amortized cost: $\mathcal{O}(D_n)$

- \triangleright $\mathcal{O}(1)$ for decrease-key.
- \triangleright $\mathcal{O}(D_n)$ for delete-min.

Lemma 1

Let x be a node with degree k and let y_1, \ldots, y_k denote the children of x in the order that they were linked to x. Then

$$\operatorname{degree}(y_i) \geq \left\{ \begin{array}{ll} 0 & \textit{if } i = 1 \\ i - 2 & \textit{if } i > 1 \end{array} \right.$$

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- Since, then y_i has lost at most one child.
- ▶ Therefore, degree(y_i) ≥ i 2.

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$$= 2 + \sum_{i=2}^{k-2} s_i$$

Definition 2

Consider the following non-standard Fibonacci type sequence:

$$F_k = \begin{cases} 1 & \text{if } k = 0 \\ 2 & \text{if } k = 1 \\ F_{k-1} + F_{k-2} & \text{if } k \ge 2 \end{cases}$$

Facts:

- 1. $F_k \geq \phi^k$.
- **2.** For $k \ge 2$: $F_k = 2 + \sum_{i=0}^{k-2} F_i$.

The above facts can be easily proved by induction. From this it follows that $s_k \ge F_k \ge \phi^k$, which gives that the maximum degree in a Fibonacci heap is logarithmic.

k=0:
$$1 = F_0 \ge \Phi^0 = 1$$

k=1: $2 = F_1 \ge \Phi^1 \approx 1.61$
k-2,k-1 \rightarrow k: $F_k = F_{k-1} + F_{k-2} \ge \Phi^{k-1} + \Phi^{k-2} = \Phi^{k-2}(\Phi^{+1}) = \Phi^k$

k=2:
$$3 = F_2 = 2 + 1 = 2 + F_0$$

k-1 \rightarrow k: $F_k = F_{k-1} + F_{k-2} = 2 + \sum_{i=0}^{k-3} F_i + F_{k-2} = 2 + \sum_{i=0}^{k-2} F_i$