

WS 2024/25

Efficient Algorithms

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TU München

<https://www.moodle.tum.de/course/view.php?id=100478>

Winter Term 2024/25

Part I

Organizational Matters

Part I

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- ▶ Modul: IN2003
- ▶ Name: “Efficient Algorithms and Data Structures”
“Effiziente Algorithmen und Datenstrukturen”
- ▶ ECTS: 8 Credit points
- ▶ Lectures:
 - ▶ 4 SWS
Mon 10:00–12:00 (Room Interim2)
Fri 10:00–12:00 (Room Interim2)
- ▶ Webpage:
<https://www.moodle.tum.de/course/view.php?id=100478>

- ▶ Required knowledge:
 - ▶ IN0001, IN0003
“**Introduction to Informatics 1/2**”
“Einführung in die Informatik 1/2”
 - ▶ IN0007
“**Fundamentals of Algorithms and Data Structures**”
“Grundlagen: Algorithmen und Datenstrukturen” (GAD)
 - ▶ IN0011
“**Basic Theoretic Informatics**”
“Einführung in die Theoretische Informatik” (THEO)
 - ▶ IN0015
“**Discrete Structures**”
“Diskrete Strukturen” (DS)
 - ▶ IN0018
“**Discrete Probability Theory**”
“Diskrete Wahrscheinlichkeitstheorie” (DWT)

The Lecturer

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- ▶ Room: 03.09.042
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1 Contents

- ▶ Foundations
 - ▶ Machine models
 - ▶ Efficiency measures
 - ▶ Asymptotic notation
 - ▶ Recursion

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 - ▶ Union/Find data structures




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- ▶ Cuts/Flows
- ▶ Matchings

2 Literatur

-  Alfred V. Aho, John E. Hopcroft, Jeffrey D. Ullman:
The design and analysis of computer algorithms,
Addison-Wesley Publishing Company: Reading (MA), 1974
-  Thomas H. Cormen, Charles E. Leiserson, Ron L. Rivest,
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The Algorithm Design Manual,

Springer, 1998

Part II

Foundations

3 Goals

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- ▶ Learn how to analyze and judge the efficiency of algorithms.
- ▶ Learn how to design efficient algorithms.

4 Modelling Issues

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- ▶ Memory requirement

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- ▶ Implementing and testing on representative inputs
 - ▶ How do you choose your inputs?
 - ▶ May be very time-consuming.
 - ▶ Very reliable results if done correctly.
 - ▶ Results only hold for a specific machine and for a specific set of inputs.

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 - ▶ May be very time-consuming.
 - ▶ Very reliable results if done correctly.
 - ▶ Results only hold for a specific machine and for a specific set of inputs.

- ▶ Theoretical analysis in a specific **model of computation**.
 - ▶ Gives **asymptotic bounds** like “this algorithm always runs in time $\mathcal{O}(n^2)$ ”.
 - ▶ Typically focuses on the **worst case**.
 - ▶ Can give lower bounds like “any comparison-based sorting algorithm needs at least $\Omega(n \log n)$ comparisons in the worst case”.

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The theoretical bounds are usually given by a function $f : \mathbb{N} \rightarrow \mathbb{N}$ that maps the **input length** to the running time (or storage space, comparisons, multiplications, program size etc.).

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Example 1

Suppose n numbers from the interval $\{1, \dots, N\}$ have to be sorted. In this case we usually say that the input length is n instead of e.g. $n \log N$, which would be the number of bits required to encode the input.

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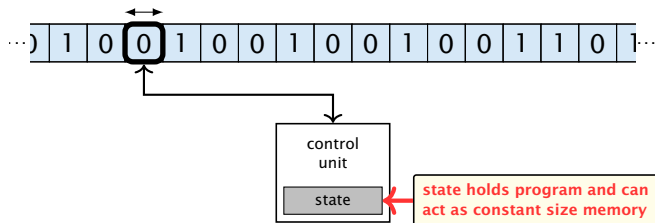
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Version 2. is often easier, but focusing on one type of operation makes it more difficult to obtain meaningful results.

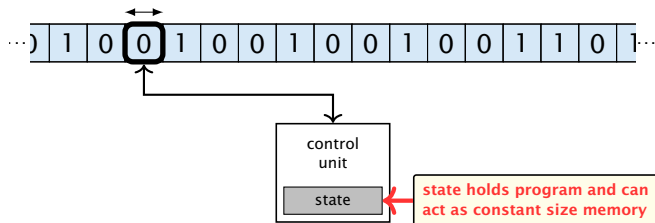
Turing Machine

- ▶ Very simple model of computation.



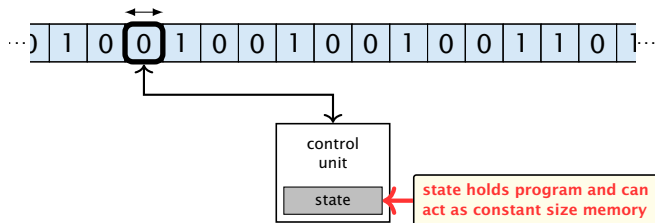
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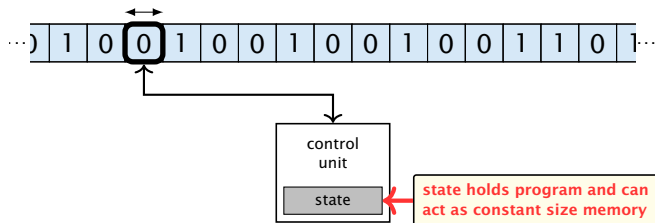
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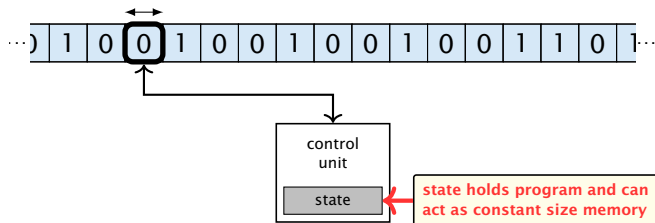
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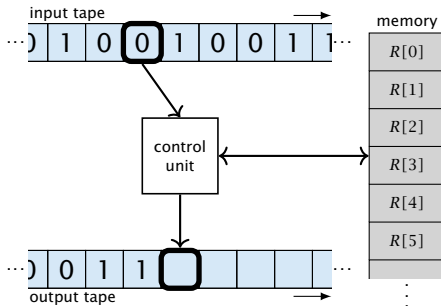
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⇒ **Not a good model for developing efficient algorithms.**



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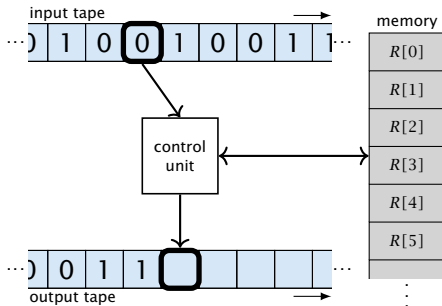
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Note that in the picture on the right the tapes are one-directional, and that a READ- or WRITE-operation always advances its tape.

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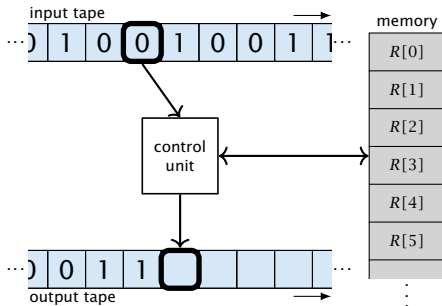
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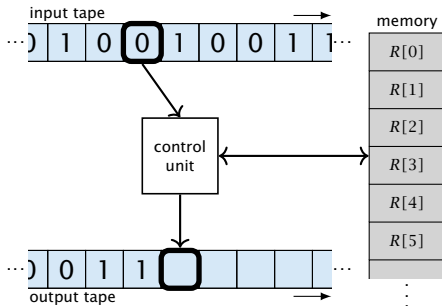
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 - ▶ $R[i] := R[j] + R[k]$;
 - ▶ $R[i] := -R[k]$;

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Model of Computation

- ▶ **uniform** cost model
Every operation takes time 1.

The latter model is quite realistic as the word-size of a standard computer that handles a problem of size n must be at least $\log_2 n$ as otherwise the computer could either not store the problem instance or not address all its memory.

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Bounded word RAM model: cost is uniform but the largest value stored in a register may not exceed 2^w , where usually $w = \log_2 n$.

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Example 2

Algorithm 1 RepeatedSquaring(n)

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Usually easy to analyze, but not very meaningful.

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more general: probability measure μ

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▶ **amortized** complexity:

The average cost of data structure operations over a worst case sequence of operations.

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The algorithm may use random bits. Expected running time (over all possible choices of random bits) for a fixed input x . Then take the worst-case over all x with $|x| = n$.

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- ▶ Running time should be expressed by simple functions.

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Formal Definition

Let f, g denote functions from \mathbb{N} to \mathbb{R}^+ .

- ▶ $\mathcal{O}(f) = \{g \mid \exists c > 0 \exists n_0 \in \mathbb{N}_0 \forall n \geq n_0 : [g(n) \leq c \cdot f(n)]\}$
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There is an equivalent definition using limes notation (**assuming that the respective limes exists**). f and g are functions from \mathbb{N}_0 to \mathbb{R}_0^+ .

$$\blacktriangleright g \in \mathcal{O}(f): 0 \leq \lim_{n \rightarrow \infty} \frac{g(n)}{f(n)} < \infty$$

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Abuse of notation

1. People write $f = \mathcal{O}(g)$, when they mean $f \in \mathcal{O}(g)$. This is **not** an equality (how could a function be equal to a set of functions).

2. In this context $f(n)$ does **not** mean the function f evaluated at n , but instead it is a shorthand for the function itself (leaving out domain and codomain and only giving the rule of correspondence of the function).

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4. People write $\mathcal{O}(f(n)) = \mathcal{O}(g(n))$, when they mean $\mathcal{O}(f(n)) \subseteq \mathcal{O}(g(n))$. Again this is not an equality.

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Asymptotic Notation in Equations

How do we interpret an expression like:

$$2n^2 + 3n + 1 = 2n^2 + \Theta(n)$$

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Note that $\Theta(n)$ is on the right hand side, otw. this interpretation is wrong.

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How do we interpret an expression like:

$$2n^2 + \mathcal{O}(n) = \Theta(n^2)$$

Regardless of how we choose the anonymous function $f(n) \in \mathcal{O}(n)$ there is an anonymous function $g(n) \in \Theta(n^2)$ that makes the expression true.

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How do we interpret an expression like:

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Careful!

Asymptotic Notation in Equations

The $\Theta(i)$ -symbol on the left represents **one** anonymous function $f : \mathbb{N} \rightarrow \mathbb{R}^+$, and then $\sum_i f(i)$ is computed.

How do we interpret an expression like:

$$\sum_{i=1}^n \Theta(i) = \Theta(n^2)$$

Careful!

“It is understood” that every occurrence of an Θ -symbol (or Ω, o, ω) on the left represents **one anonymous function**.

Hence, the left side is **not** equal to

$$\Theta(1) + \Theta(2) + \dots + \Theta(n-1) + \Theta(n)$$

$\Theta(1) + \Theta(2) + \dots + \Theta(n-1) + \Theta(n)$ does not really have a reasonable interpretation.

Asymptotic Notation in Equations

We can view an expression containing asymptotic notation as generating a set:

$$n^2 \cdot \mathcal{O}(n) + \mathcal{O}(\log n)$$

represents

$$\{f : \mathbb{N} \rightarrow \mathbb{R}^+ \mid f(n) = n^2 \cdot g(n) + h(n)\}$$

with $g(n) \in \mathcal{O}(n)$ and $h(n) \in \mathcal{O}(\log n)$

Recall that according to the previous slide e.g. the expressions $\sum_{i=1}^n \mathcal{O}(i)$ and $\sum_{i=1}^{n/2} \mathcal{O}(i) + \sum_{i=n/2+1}^n \mathcal{O}(i)$ generate different sets.

Asymptotic Notation in Equations

Then an asymptotic equation can be interpreted as containment btw. two sets:

$$n^2 \cdot \mathcal{O}(n) + \mathcal{O}(\log n) = \Theta(n^2)$$

represents

$$n^2 \cdot \mathcal{O}(n) + \mathcal{O}(\log n) \subseteq \Theta(n^2)$$

Note that the equation does not hold.

Asymptotic Notation

Lemma 3

Let f, g be functions with the property

$\exists n_0 > 0 \forall n \geq n_0 : f(n) > 0$ (the same for g). Then

- ▶ $c \cdot f(n) \in \Theta(f(n))$ for any constant c

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The expressions also hold for Ω . Note that this means that $f(n) + g(n) \in \Theta(\max\{f(n), g(n)\})$.

Asymptotic Notation

Comments

- ▶ Do not use asymptotic notation within induction proofs.

Asymptotic Notation

Comments

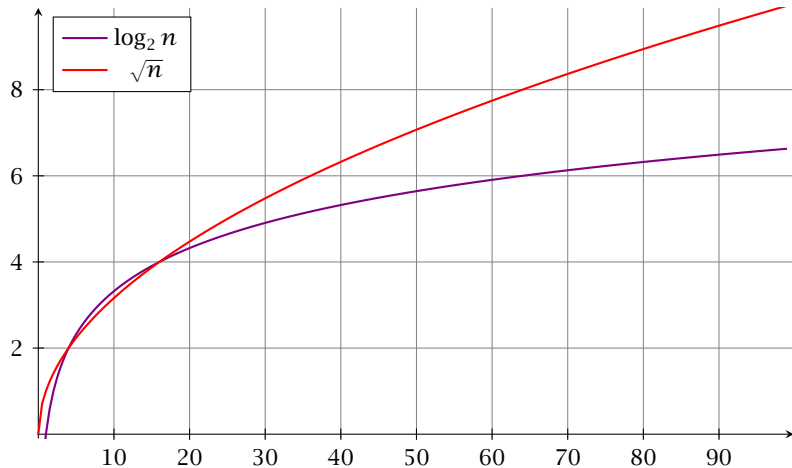
- ▶ Do not use asymptotic notation within induction proofs.
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Therefore, we will usually ignore the base of a logarithm within asymptotic notation.

Asymptotic Notation

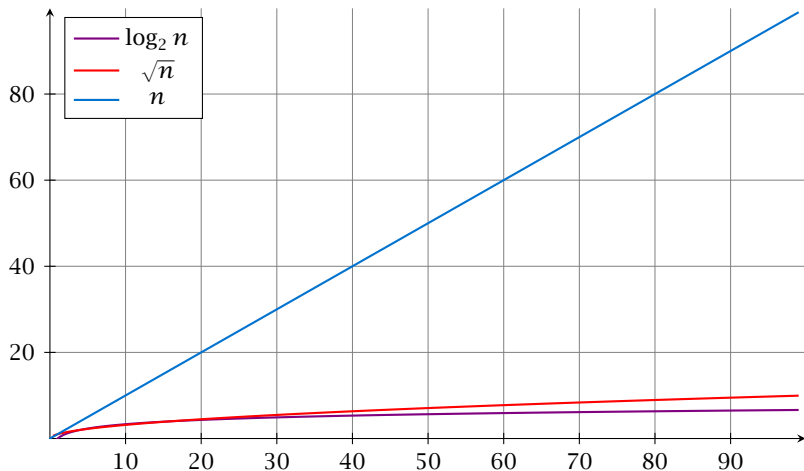
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- ▶ Do not use asymptotic notation within induction proofs.
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Therefore, we will usually ignore the base of a logarithm within asymptotic notation.
- ▶ In general $\log n = \log_2 n$, i.e., we use 2 as the default base for the logarithm.

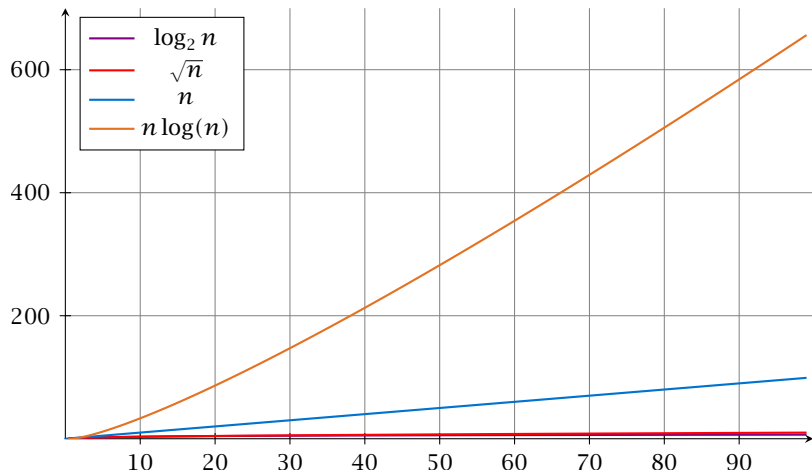
Funktionen



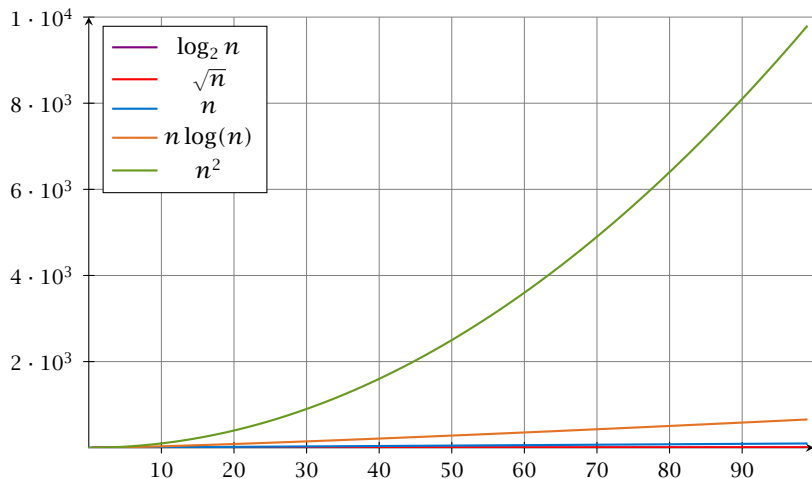
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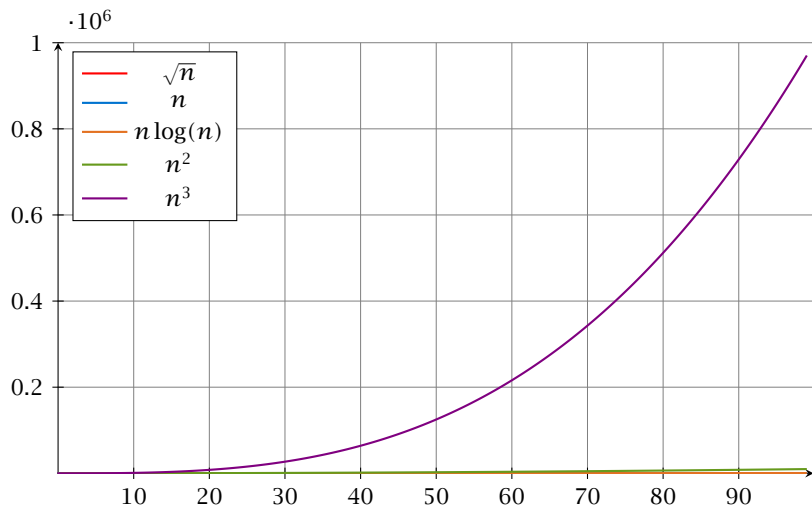
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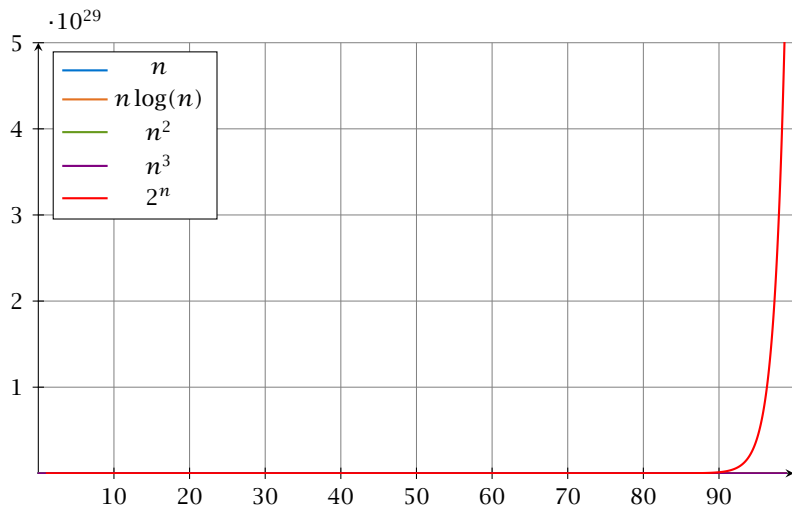
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Funktionen



Laufzeiten

Funktion	Eingabelänge n							
	10	10^2	10^3	10^4	10^5	10^6	10^7	10^8
$\log n$	33ns	66ns	0.1 μ s	0.1 μ s	0.2 μ s	0.2 μ s	0.2 μ s	0.3 μ s
\sqrt{n}	32ns	0.1 μ s	0.3 μ s	1 μ s	3.1 μ s	10 μ s	31 μ s	0.1ms
n	100ns	1 μ s	10 μ s	0.1ms	1ms	10ms	0.1s	1s
$n \log n$	0.3 μ s	6.6 μ s	0.1ms	1.3ms	16ms	0.2s	2.3s	27s
$n^{3/2}$	0.3 μ s	10 μ s	0.3ms	10ms	0.3s	10s	5.2min	2.7h
n^2	1 μ s	0.1ms	10ms	1s	1.7min	2.8h	11d	3.2y
n^3	10 μ s	10ms	10s	2.8h	115d	317y	$3.2 \cdot 10^5$ y	
1.1^n	26ns	0.1ms	$7.8 \cdot 10^{25}$ y					
2^n	10 μ s	$4 \cdot 10^{14}$ y						
$n!$	36ms	$3 \cdot 10^{142}$ y						

1 Operation = 10ns; 100MHz

Alter des Universums: ca. $13.8 \cdot 10^9$ y

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In general asymptotic classification of running times is a good measure for comparing algorithms:

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Clearly $f = o(g)$. However, as long as $\log n \leq 1000$ Algorithm B will be more efficient.

Multiple Variables in Asymptotic Notation

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(set of functions that asymptotically grow **not faster** than f)

Multiple Variables in Asymptotic Notation

Example 4

- ▶ $f : \mathbb{N} \rightarrow \mathbb{R}_0^+$, $f(n, m) = 1$ und $g : \mathbb{N} \rightarrow \mathbb{R}_0^+$, $g(n, m) = n - 1$

Multiple Variables in Asymptotic Notation

Example 4

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6 Recurrences

Algorithm 2 mergesort(list L)

```
1:  $n \leftarrow \text{size}(L)$ 
2: if  $n \leq 1$  return  $L$ 
3:  $L_1 \leftarrow L[1 \cdots \lfloor \frac{n}{2} \rfloor]$ 
4:  $L_2 \leftarrow L[\lfloor \frac{n}{2} \rfloor + 1 \cdots n]$ 
5: mergesort( $L_1$ )
6: mergesort( $L_2$ )
7:  $L \leftarrow \text{merge}(L_1, L_2)$ 
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This algorithm requires

$$T(n) = T\left(\left\lfloor \frac{n}{2} \right\rfloor\right) + T\left(\left\lfloor \frac{n}{2} \right\rfloor\right) + \mathcal{O}(n) \leq 2T\left(\left\lfloor \frac{n}{2} \right\rfloor\right) + \mathcal{O}(n)$$

comparisons when $n > 1$ and 0 comparisons when $n \leq 1$.

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How do we bring the expression for the number of comparisons (\approx running time) into a **closed form**?

For this we need to **solve** the recurrence.

Methods for Solving Recurrences

1. Guessing+Induction

Guess the right solution and prove that it is correct via induction. It needs experience to make the right guess.

2. Master Theorem

For a lot of recurrences that appear in the analysis of algorithms this theorem can be used to obtain tight asymptotic bounds. It does not provide exact solutions.

3. Characteristic Polynomial

Linear homogenous recurrences can be solved via this method.

4. Generating Functions

A more general technique that allows to solve certain types of linear inhomogenous relations and also sometimes non-linear recurrence relations.

5. Transformation of the Recurrence

Sometimes one can transform the given recurrence relations so that it e.g. becomes linear and can therefore be solved with one of the other techniques.

6.1 Guessing+Induction

First we need to get rid of the \mathcal{O} -notation in our recurrence:

$$T(n) \leq \begin{cases} 2T(\lceil \frac{n}{2} \rceil) + cn & n \geq 2 \\ 0 & \text{otherwise} \end{cases}$$

Informal way:

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One way of solving such a recurrence is to **guess** a solution, and check that it is correct by plugging it in.

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Formally, this is not correct if n is not a power of 2. Also even in this case one would need to do an induction proof.

6.1 Guessing+Induction

$$T(n) \leq \begin{cases} 2T(\frac{n}{2}) + cn & n \geq 16 \\ b & \text{otw.} \end{cases}$$

- Note that this proves the statement for $n = 2^k$, $k \in \mathbb{N}_{\geq 1}$, as the statement is wrong for $n = 1$.
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Hence, statement is **true** if we choose $d \geq c$.

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Note that we can do this as for constant-sized inputs the running time is always some constant (b in the above case).

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$$\boxed{\log n \leq \frac{n}{4}} \leq dn \log n + (\log 9 - 3.5)dn + cn$$

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$$T(n) \leq 2T\left(\left\lceil \frac{n}{2} \right\rceil\right) + cn$$

$$\leq 2\left(d\left\lceil \frac{n}{2} \right\rceil \log \left\lceil \frac{n}{2} \right\rceil\right) + cn$$

$$\left\lceil \frac{n}{2} \right\rceil \leq \frac{n}{2} + 1$$

$$\leq 2\left(d\left(\frac{n}{2} + 1\right) \log\left(\frac{n}{2} + 1\right)\right) + cn$$

$$\frac{n}{2} + 1 \leq \frac{9}{16}n$$

$$\leq dn \log\left(\frac{9}{16}n\right) + 2d \log n + cn$$

$$\log \frac{9}{16}n = \log n + (\log 9 - 4)$$

$$= dn \log n + (\log 9 - 4)dn + 2d \log n + cn$$

$$\log n \leq \frac{n}{4}$$

$$\leq dn \log n + (\log 9 - 3.5)dn + cn$$

$$\leq dn \log n - 0.33dn + cn$$

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$$\left\lceil \frac{n}{2} \right\rceil \leq \frac{n}{2} + 1$$

$$\leq 2\left(d\left(\frac{n}{2} + 1\right) \log\left(\frac{n}{2} + 1\right)\right) + cn$$

$$\frac{n}{2} + 1 \leq \frac{9}{16}n$$

$$\leq dn \log\left(\frac{9}{16}n\right) + 2d \log n + cn$$

$$\log \frac{9}{16}n = \log n + (\log 9 - 4)$$

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$$\leq dn \log n - 0.33dn + cn$$

$$\leq dn \log n$$

for a suitable choice of d .

6.2 Master Theorem

Note that the cases do not cover all possibilities.

Lemma 5

Let $a \geq 1$, $b > 1$ and $\epsilon > 0$ denote constants. Consider the recurrence

$$T(n) = aT\left(\frac{n}{b}\right) + f(n) .$$

Case 1.

If $f(n) = \mathcal{O}(n^{\log_b(a)-\epsilon})$ then $T(n) = \Theta(n^{\log_b a})$.

Case 2.

If $f(n) = \Theta(n^{\log_b(a)} \log^k n)$ then $T(n) = \Theta(n^{\log_b a} \log^{k+1} n)$,
 $k \geq 0$.

Case 3.

If $f(n) = \Omega(n^{\log_b(a)+\epsilon})$ and for sufficiently large n
 $af\left(\frac{n}{b}\right) \leq cf(n)$ for some constant $c < 1$ then $T(n) = \Theta(f(n))$.

6.2 Master Theorem

We prove the Master Theorem for the case that n is of the form b^{ℓ} , and we assume that the non-recursive case occurs for problem size 1 and incurs cost 1.

The Recursion Tree

The running time of a recursive algorithm can be visualized by a recursion tree:

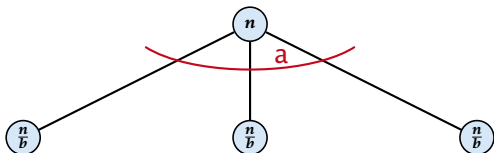
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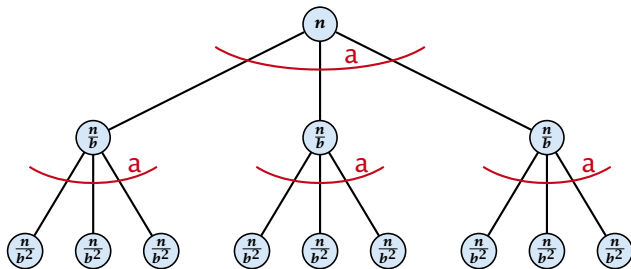
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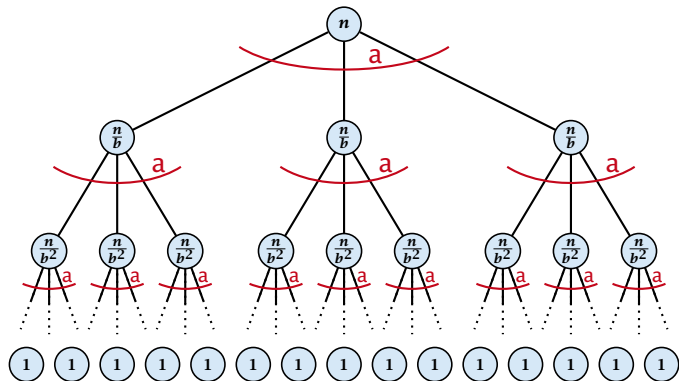
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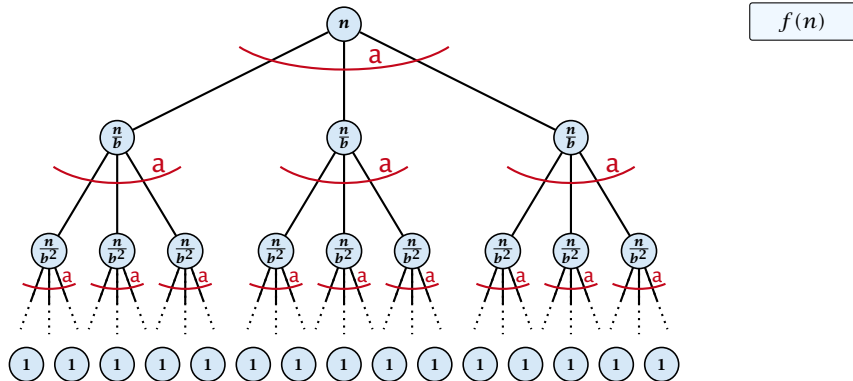
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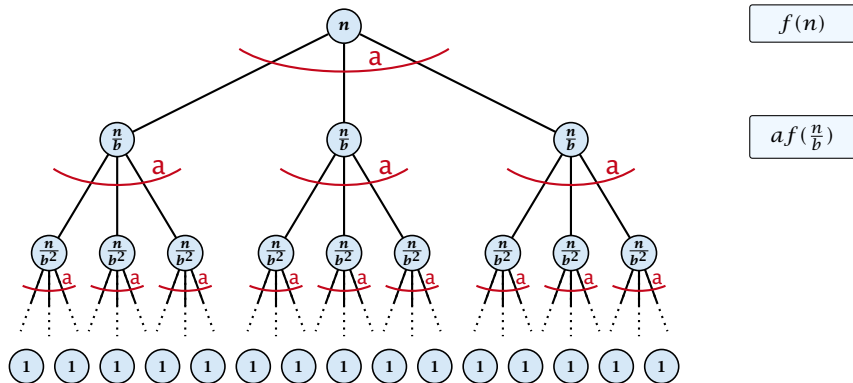
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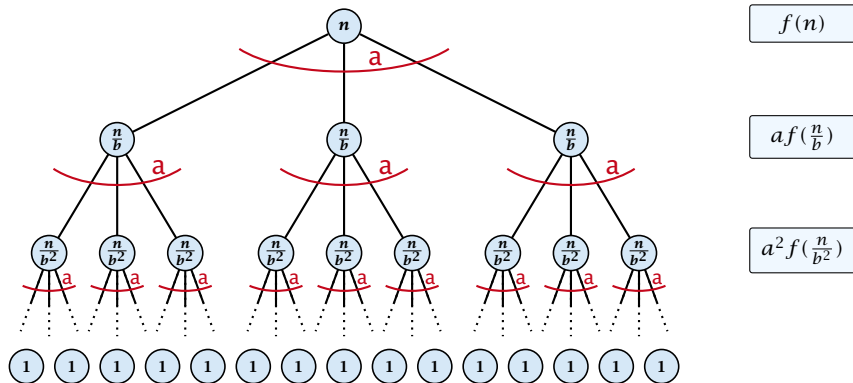
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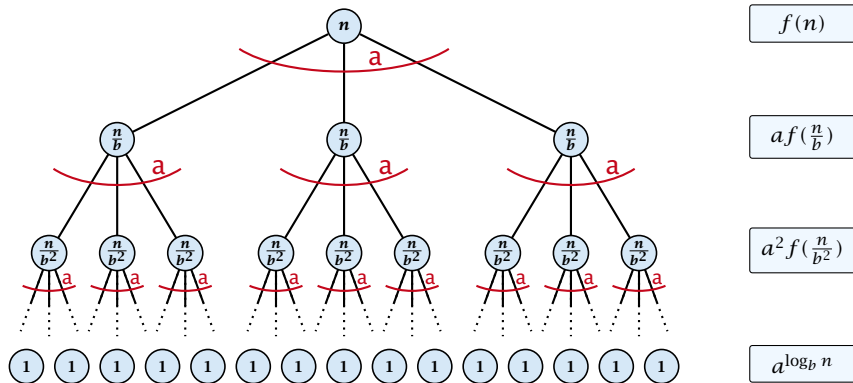
The Recursion Tree

The running time of a recursive algorithm can be visualized by a recursion tree:



The Recursion Tree

The running time of a recursive algorithm can be visualized by a recursion tree:



6.2 Master Theorem

This gives

$$T(n) = n^{\log_b a} + \sum_{i=0}^{\log_b n - 1} a^i f\left(\frac{n}{b^i}\right).$$

Case 1. Now suppose that $f(n) \leq cn^{\log_b a - \epsilon}$.

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$$T(n) - n^{\log_b a} = \sum_{i=0}^{\log_b n - 1} a^i f\left(\frac{n}{b^i}\right)$$

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$$\begin{aligned} T(n) - n^{\log_b a} &= \sum_{i=0}^{\log_b n - 1} a^i f\left(\frac{n}{b^i}\right) \\ &\leq c \sum_{i=0}^{\log_b n - 1} a^i \left(\frac{n}{b^i}\right)^{\log_b a - \epsilon} \end{aligned}$$

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$$\boxed{\sum_{i=0}^k q^i = \frac{q^{k+1} - 1}{q - 1}}$$

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$$\boxed{\sum_{i=0}^k q^i = \frac{q^{k+1} - 1}{q - 1}} = cn^{\log_b a - \epsilon} (b^{\epsilon \log_b n} - 1) / (b^{\epsilon} - 1)$$

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Hence,

$$T(n) \leq \left(\frac{c}{b^{\epsilon} - 1} + 1 \right) n^{\log_b(a)}$$

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Hence,

$$T(n) \leq \left(\frac{c}{b^{\epsilon} - 1} + 1 \right) n^{\log_b(a)} \quad \Rightarrow T(n) = \mathcal{O}(n^{\log_b a}).$$

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Hence,

$$T(n) = \mathcal{O}(n^{\log_b a} \log_b n)$$

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Hence,

$$T(n) = \mathcal{O}(n^{\log_b a} \log_b n)$$

$$\Rightarrow T(n) = \mathcal{O}(n^{\log_b a} \log n).$$

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Hence,

$$T(n) = \Omega(n^{\log_b a} \log_b n)$$

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Hence,

$$T(n) = \Omega(n^{\log_b a} \log_b n)$$

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$$n = b^\ell \Rightarrow \ell = \log_b n$$

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$n = b^\ell \Rightarrow \ell = \log_b n$	$= cn^{\log_b a} \sum_{i=0}^{\ell-1} \left(\log_b\left(\frac{b^\ell}{b^i}\right)\right)^k$
--	--

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$$n = b^\ell \Rightarrow \ell = \log_b n$$

$$= cn^{\log_b a} \sum_{i=0}^{\ell-1} \left(\log_b\left(\frac{b^\ell}{b^i}\right)\right)^k$$

$$= cn^{\log_b a} \sum_{i=0}^{\ell-1} (\ell - i)^k$$

$$= cn^{\log_b a} \sum_{i=1}^{\ell} i^k \approx \frac{1}{k} \ell^{k+1}$$

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$$n = b^\ell \Rightarrow \ell = \log_b n$$

$$\begin{aligned} &= cn^{\log_b a} \sum_{i=0}^{\ell-1} \left(\log_b\left(\frac{b^\ell}{b^i}\right)\right)^k \\ &= cn^{\log_b a} \sum_{i=0}^{\ell-1} (\ell - i)^k \\ &= cn^{\log_b a} \sum_{i=1}^{\ell} i^k \\ &\approx \frac{c}{k} n^{\log_b a} \ell^{k+1} \end{aligned}$$

Case 2. Now suppose that $f(n) \leq cn^{\log_b a} (\log_b(n))^k$.

$$\begin{aligned} T(n) - n^{\log_b a} &= \sum_{i=0}^{\log_b n - 1} a^i f\left(\frac{n}{b^i}\right) \\ &\leq c \sum_{i=0}^{\log_b n - 1} a^i \left(\frac{n}{b^i}\right)^{\log_b a} \cdot \left(\log_b\left(\frac{n}{b^i}\right)\right)^k \end{aligned}$$

$$n = b^\ell \Rightarrow \ell = \log_b n$$

$$= cn^{\log_b a} \sum_{i=0}^{\ell-1} \left(\log_b\left(\frac{b^\ell}{b^i}\right)\right)^k$$

$$= cn^{\log_b a} \sum_{i=0}^{\ell-1} (\ell - i)^k$$

$$= cn^{\log_b a} \sum_{i=1}^{\ell} i^k$$

$$\approx \frac{c}{k} n^{\log_b a} \ell^{k+1}$$

$$\Rightarrow T(n) = \mathcal{O}(n^{\log_b a} \log^{k+1} n).$$

Case 3. Now suppose that $f(n) \geq dn^{\log_b a + \epsilon}$, and that for sufficiently large n : $af(n/b) \leq cf(n)$, for $c < 1$.

Where did we use $f(n) \geq \Omega(n^{\log_b a + \epsilon})$?

Case 3. Now suppose that $f(n) \geq dn^{\log_b a + \epsilon}$, and that for sufficiently large n : $af(n/b) \leq cf(n)$, for $c < 1$.

From this we get $a^i f(n/b^i) \leq c^i f(n)$, where we assume that $n/b^{i-1} \geq n_0$ is still sufficiently large.

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From this we get $a^i f(n/b^i) \leq c^i f(n)$, where we assume that $n/b^{i-1} \geq n_0$ is still sufficiently large.

$$T(n) - n^{\log_b a} = \sum_{i=0}^{\log_b n - 1} a^i f\left(\frac{n}{b^i}\right)$$

Where did we use $f(n) \geq \Omega(n^{\log_b a + \epsilon})$?

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From this we get $a^i f(n/b^i) \leq c^i f(n)$, where we assume that $n/b^{i-1} \geq n_0$ is still sufficiently large.

$$\begin{aligned} T(n) - n^{\log_b a} &= \sum_{i=0}^{\log_b n - 1} a^i f\left(\frac{n}{b^i}\right) \\ &\leq \sum_{i=0}^{\log_b n - 1} c^i f(n) + \mathcal{O}(n^{\log_b a}) \end{aligned}$$

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$$\begin{aligned} T(n) - n^{\log_b a} &= \sum_{i=0}^{\log_b n - 1} a^i f\left(\frac{n}{b^i}\right) \\ &\leq \sum_{i=0}^{\log_b n - 1} c^i f(n) + \mathcal{O}(n^{\log_b a}) \end{aligned}$$

$$q < 1 : \sum_{i=0}^n q^i = \frac{1 - q^{n+1}}{1 - q} \leq \frac{1}{1 - q}$$

Where did we use $f(n) \geq \Omega(n^{\log_b a + \epsilon})$?

Case 3. Now suppose that $f(n) \geq dn^{\log_b a + \epsilon}$, and that for sufficiently large n : $af(n/b) \leq cf(n)$, for $c < 1$.

From this we get $a^i f(n/b^i) \leq c^i f(n)$, where we assume that $n/b^{i-1} \geq n_0$ is still sufficiently large.

$$\begin{aligned} T(n) - n^{\log_b a} &= \sum_{i=0}^{\log_b n - 1} a^i f\left(\frac{n}{b^i}\right) \\ &\leq \sum_{i=0}^{\log_b n - 1} c^i f(n) + \mathcal{O}(n^{\log_b a}) \\ &\leq \frac{1}{1-c} f(n) + \mathcal{O}(n^{\log_b a}) \end{aligned}$$

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From this we get $a^i f(n/b^i) \leq c^i f(n)$, where we assume that $n/b^{i-1} \geq n_0$ is still sufficiently large.

$$\begin{aligned} T(n) - n^{\log_b a} &= \sum_{i=0}^{\log_b n - 1} a^i f\left(\frac{n}{b^i}\right) \\ &\leq \sum_{i=0}^{\log_b n - 1} c^i f(n) + \mathcal{O}(n^{\log_b a}) \\ &\leq \frac{1}{1-c} f(n) + \mathcal{O}(n^{\log_b a}) \end{aligned}$$

$$q < 1 : \sum_{i=0}^n q^i = \frac{1-q^{n+1}}{1-q} \leq \frac{1}{1-q}$$

Hence,

$$T(n) \leq \mathcal{O}(f(n))$$

Where did we use $f(n) \geq \Omega(n^{\log_b a + \epsilon})$?

Case 3. Now suppose that $f(n) \geq dn^{\log_b a + \epsilon}$, and that for sufficiently large n : $af(n/b) \leq cf(n)$, for $c < 1$.

From this we get $a^i f(n/b^i) \leq c^i f(n)$, where we assume that $n/b^{i-1} \geq n_0$ is still sufficiently large.

$$\begin{aligned} T(n) - n^{\log_b a} &= \sum_{i=0}^{\log_b n - 1} a^i f\left(\frac{n}{b^i}\right) \\ &\leq \sum_{i=0}^{\log_b n - 1} c^i f(n) + \mathcal{O}(n^{\log_b a}) \\ &\leq \frac{1}{1-c} f(n) + \mathcal{O}(n^{\log_b a}) \end{aligned}$$

$$q < 1 : \sum_{i=0}^n q^i = \frac{1-q^{n+1}}{1-q} \leq \frac{1}{1-q}$$

Hence,

$$T(n) \leq \mathcal{O}(f(n))$$

$$\Rightarrow T(n) = \Theta(f(n)).$$

Where did we use $f(n) \geq \Omega(n^{\log_b a + \epsilon})$?

Example: Multiplying Two Integers

Suppose we want to multiply two n -bit Integers, but our registers can only perform operations on integers of constant size.

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Example: Multiplying Two Integers

Suppose we want to multiply two n -bit Integers, but our registers can only perform operations on integers of constant size.

For this we first need to be able to add two integers A and B :

$$\begin{array}{r} 1\ 1\ 0\ 1\ 1\ 0\ 1\ 0\ 1\ A \\ 1\ 0\ 0\ 0\ 1\ 0\ 0\ 1\ 1\ B \\ \hline \end{array}$$

Example: Multiplying Two Integers

Suppose we want to multiply two n -bit Integers, but our registers can only perform operations on integers of constant size.

For this we first need to be able to add two integers A and B :

$$\begin{array}{r} 110110101 \\ 100010011 \\ \hline \end{array}$$

The diagram shows two 9-bit integers, A and B , aligned for addition. Integer A is represented by the red bits 110110101 and integer B by the blue bits 100010011. A horizontal line is drawn under the bits, and a vertical light blue box highlights the rightmost bit (the least significant bit) of both numbers, which is a 1 in each case.

Example: Multiplying Two Integers

Suppose we want to multiply two n -bit Integers, but our registers can only perform operations on integers of constant size.

For this we first need to be able to add two integers A and B :

1	1	0	1	1	0	1	0	1	A
1	0	0	0	1	0	0	1	1	B
<hr/>									
								0	

Example: Multiplying Two Integers

Suppose we want to multiply two n -bit Integers, but our registers can only perform operations on integers of constant size.

For this we first need to be able to add two integers A and B :

1	1	0	1	1	0	1	0	1	A
1	0	0	0	1	0	0	1	1	B
<hr/>								1	
									0

Example: Multiplying Two Integers

Suppose we want to multiply two n -bit Integers, but our registers can only perform operations on integers of constant size.

For this we first need to be able to add two integers A and B :

$$\begin{array}{r} 1\ 1\ 0\ 1\ 1\ 0\ 1\ 0\ 1\ A \\ 1\ 0\ 0\ 0\ 1\ 0\ 0\ 1\ 1\ B \\ \hline 0\ 0 \end{array}$$

The diagram shows the addition of two 10-bit integers, A and B. The bits of A are 1, 1, 0, 1, 1, 0, 1, 0, 1 and the bits of B are 1, 0, 0, 0, 1, 0, 0, 1, 1. A horizontal line is drawn under the bits of B. The result of the addition is shown below the line, with the bits 0 and 0 highlighted in a light blue box. Small '1' characters are placed below the 7th and 8th bits of the result, indicating carry bits.

Example: Multiplying Two Integers

Suppose we want to multiply two n -bit Integers, but our registers can only perform operations on integers of constant size.

For this we first need to be able to add two integers A and B :

$$\begin{array}{rcccccccc} 1 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & A \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & B \\ \hline & & & & & & 1 & 1 & & \\ & & & & & & & & 0 & 0 \end{array}$$

The diagram illustrates the addition of two 9-bit integers, A and B. A vertical line is drawn between the 7th and 8th bits from the right. A light blue shaded box highlights the 7th bit of A (1) and the 7th bit of B (0). Below the line, the 7th bit of the sum is 1, and the 8th bit of the sum is 1. Below the 8th bit of the sum, the 8th and 9th bits are 0 and 0.

Example: Multiplying Two Integers

Suppose we want to multiply two n -bit Integers, but our registers can only perform operations on integers of constant size.

For this we first need to be able to add two integers A and B :

$$\begin{array}{rcccccccc} 1 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & A \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & B \\ \hline & & & & & & 0 & 0 & 0 & \end{array}$$

The diagram illustrates the addition of two 9-bit integers, A and B. The numbers are written in binary, with A in red and B in blue. A horizontal line is drawn under the numbers. The result of the addition is shown below the line, with the carry bits (0, 0, 0) highlighted in a light blue box. The carry bits are shown as 0, 0, 0, indicating that the sum of the two numbers is 100110101.

Example: Multiplying Two Integers

Suppose we want to multiply two n -bit Integers, but our registers can only perform operations on integers of constant size.

For this we first need to be able to add two integers A and B :

1	1	0	1	1	0	1	0	1	A
1	0	0	0	1	0	0	1	1	B
<hr/>									
					1	1	1		
						0	0	0	

Example: Multiplying Two Integers

Suppose we want to multiply two n -bit Integers, but our registers can only perform operations on integers of constant size.

For this we first need to be able to add two integers A and B :

1	1	0	1	1	0	1	0	1	A
1	0	0	0	1	0	0	1	1	B
					0	1	1	1	
					1	0	0	0	

Example: Multiplying Two Integers

Suppose we want to multiply two n -bit Integers, but our registers can only perform operations on integers of constant size.

For this we first need to be able to add two integers A and B :

$$\begin{array}{rcccccccc} 1 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & A \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & B \\ \hline & & & & 0 & 1 & 1 & 1 & & \\ & & & & & 1 & 0 & 0 & 0 & \end{array}$$

Example: Multiplying Two Integers

Suppose we want to multiply two n -bit Integers, but our registers can only perform operations on integers of constant size.

For this we first need to be able to add two integers A and B :

$$\begin{array}{rcccccccc} 1 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & A \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & B \\ \hline & & & & 0 & 1 & 0 & 0 & 0 & \end{array}$$

The diagram illustrates the addition of two 9-bit integers, A and B. The bits of A are 1 1 0 1 1 0 1 0 1 and the bits of B are 1 0 0 0 1 0 0 1 1. A horizontal line is drawn under the bits of B. The result of the addition is shown below the line, with the bits 0 1 0 0 0. A vertical box highlights the 5th bit from the right (the 5th bit from the left in the image), which is 0. This bit is the result of adding the 5th bits of A and B (1 + 1) and the carry-in from the 4th bit (1). The carry-out from the 5th bit is 1, which is the carry-in for the 6th bit.

Example: Multiplying Two Integers

Suppose we want to multiply two n -bit Integers, but our registers can only perform operations on integers of constant size.

For this we first need to be able to add two integers A and B :

1	1	0	1	1	0	1	0	1	A
1	0	0	0	1	0	0	1	1	B
<hr/>									
			1	0	1	1	1		
			0	1	0	0	0		

Example: Multiplying Two Integers

Suppose we want to multiply two n -bit Integers, but our registers can only perform operations on integers of constant size.

For this we first need to be able to add two integers A and B :

$$\begin{array}{rcccccccc} 1 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & A \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & B \\ \hline & & & 0 & 0 & 1 & 0 & 0 & 0 & \end{array}$$

The diagram illustrates the addition of two 9-bit integers, A and B, to produce a 9-bit result. The numbers are aligned to the right. A vertical light blue box highlights the 4th bit position (from the right) in both A and B, and the resulting 0 in the 4th position of the sum. Small subscripts are placed below the bits in the second row: 1 under the 3rd bit, 1 under the 4th bit, 0 under the 5th bit, 1 under the 6th bit, 1 under the 7th bit, and 1 under the 8th bit.

Example: Multiplying Two Integers

Suppose we want to multiply two n -bit Integers, but our registers can only perform operations on integers of constant size.

For this we first need to be able to add two integers A and B :

$$\begin{array}{rcccccccc} 1 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & A \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & B \\ \hline & & & 1 & 1 & 0 & 1 & 1 & 1 & \\ & & & & 0 & 0 & 1 & 0 & 0 & 0 \end{array}$$

The diagram illustrates the addition of two 9-bit integers, A and B. The numbers are written in binary. A vertical blue box highlights the third bit position (from the right) of both numbers, which are both 0. Below the horizontal line, the carry bits are shown as subscripts under the corresponding bit positions: 1 under the 3rd bit, 1 under the 4th, 0 under the 5th, 1 under the 6th, 1 under the 7th, and 1 under the 8th. The final result of the addition is shown in black below the line.

Example: Multiplying Two Integers

Suppose we want to multiply two n -bit Integers, but our registers can only perform operations on integers of constant size.

For this we first need to be able to add two integers A and B :

$$\begin{array}{rcccccccc} 1 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & A \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & B \\ \hline & & 1 & 0 & 0 & 1 & 0 & 0 & 0 & \end{array}$$

The diagram illustrates the addition of two 9-bit integers, A and B, using a ripple carry method. The numbers are aligned by their least significant bits. A horizontal line is drawn under the second row. The result of the addition is shown in the third row. The carry bits are indicated by small subscripts below the digits in the second row: 0 under the first 0, 1 under the first 0, 1 under the first 1, 0 under the second 0, 1 under the second 0, 1 under the second 1, and 1 under the second 1. The digit '0' in the second row, second column, and the digit '1' in the third row, second column, are highlighted with a light blue background.

Example: Multiplying Two Integers

Suppose we want to multiply two n -bit Integers, but our registers can only perform operations on integers of constant size.

For this we first need to be able to add two integers A and B :

1	1	0	1	1	0	1	0	1	A
1	0	0	0	1	0	0	1	1	B
<hr/>									
		1	0	0	1	0	0	0	

The diagram illustrates the addition of two 9-bit integers, A and B. The bits of A are 1 1 0 1 1 0 1 0 1 and the bits of B are 1 0 0 0 1 0 0 1 1. A vertical blue box highlights the first bit of A (1) and the first bit of B (1). Below the horizontal line, the result of the addition is shown as 1 0 0 1 0 0 0. Small subscripts are placed below the bits of B: 0, 1, 1, 0, 1, 1, 1.

Example: Multiplying Two Integers

Suppose we want to multiply two n -bit Integers, but our registers can only perform operations on integers of constant size.

For this we first need to be able to add two integers A and B :

	1	1	0	1	1	0	1	0	1	A
	1	0	0	0	1	0	0	1	1	B
	<hr/>									
	1	1	0	0	1	0	0	0		

The diagram illustrates the addition of two integers, A and B, using a carry propagation method. The numbers are represented as binary strings. A vertical box highlights the first two bits of A (1 and 1) and the first bit of B (1), which are being added together. The result of this addition is 1, which is shown in the first column of the result row below the horizontal line. The rest of the result row shows the bits of the sum: 1, 1, 0, 0, 1, 0, 0, 0.

Example: Multiplying Two Integers

Suppose we want to multiply two n -bit Integers, but our registers can only perform operations on integers of constant size.

For this we first need to be able to add two integers A and B :

	1	1	0	1	1	0	1	0	1	A
	1	0	0	0	1	0	0	1	1	B
	0	0	1	1	0	1	1	1		
		1	1	0	0	1	0	0	0	

Example: Multiplying Two Integers

Suppose we want to multiply two n -bit Integers, but our registers can only perform operations on integers of constant size.

For this we first need to be able to add two integers A and B :

	1	1	0	1	1	0	1	0	1	A
	1	0	0	0	1	0	0	1	1	B
1	0	0	1	1	0	1	1	1		
	0	1	1	0	0	1	0	0	0	

Example: Multiplying Two Integers

Suppose we want to multiply two n -bit Integers, but our registers can only perform operations on integers of constant size.

For this we first need to be able to add two integers A and B :

		1	1	0	1	1	0	1	0	1	A
		1	0	0	0	1	0	0	1	1	B
		<hr/>									
		0	1	1	0	0	1	0	0	0	

Carry bits (shown in light blue boxes):

	1	0	0	1	1	0	1	1	1	
--	---	---	---	---	---	---	---	---	---	--

Example: Multiplying Two Integers

Suppose we want to multiply two n -bit Integers, but our registers can only perform operations on integers of constant size.

For this we first need to be able to add two integers A and B :

$$\begin{array}{r}
 \boxed{1} \quad 1 \quad 1 \quad 0 \quad 1 \quad 1 \quad 0 \quad 1 \quad 0 \quad 1 \quad A \\
 \quad 1 \quad 0 \quad 0 \quad 1 \quad 1 \quad 0 \quad 1 \quad 0 \quad 1 \quad 1 \quad B \\
 \hline
 \boxed{1} \quad 0 \quad 1 \quad 1 \quad 0 \quad 0 \quad 1 \quad 0 \quad 0 \quad 0
 \end{array}$$

Example: Multiplying Two Integers

Suppose we want to multiply two n -bit Integers, but our registers can only perform operations on integers of constant size.

For this we first need to be able to add two integers A and B :

$$\begin{array}{r} 1\ 1\ 0\ 1\ 1\ 0\ 1\ 0\ 1\ A \\ 1\ 0\ 0\ 0\ 1\ 0\ 0\ 1\ 1\ B \\ \hline 1\ 0\ 1\ 1\ 0\ 0\ 1\ 0\ 0\ 0 \end{array}$$

This gives that two n -bit integers can be added in time $\mathcal{O}(n)$.

Example: Multiplying Two Integers

Suppose that we want to multiply an n -bit integer A and an m -bit integer B ($m \leq n$).

- This is also known as the “school method” for multiplying integers.
- Note that the intermediate numbers that are generated can have at most $m + n \leq 2n$ bits.

Example: Multiplying Two Integers

Suppose that we want to multiply an n -bit integer A and an m -bit integer B ($m \leq n$).

$$\begin{array}{r} 1\ 0\ 0\ 0\ 1 \\ \times 1\ 0\ 1\ 1 \\ \hline \end{array}$$

- This is also known as the “school method” for multiplying integers.
- Note that the intermediate numbers that are generated can have at most $m + n \leq 2n$ bits.

Example: Multiplying Two Integers

Suppose that we want to multiply an n -bit integer A and an m -bit integer B ($m \leq n$).

$$\begin{array}{r} 10001 \times 1011 \\ \hline \end{array}$$

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Example: Multiplying Two Integers

Suppose that we want to multiply an n -bit integer A and an m -bit integer B ($m \leq n$).

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Example: Multiplying Two Integers

Suppose that we want to multiply an n -bit integer A and an m -bit integer B ($m \leq n$).

$$\begin{array}{r} 10001 \times 1011 \\ \hline 10001 \\ 00000 \\ 00000 \\ 10001 \\ \hline 0 \end{array}$$

- This is also known as the “school method” for multiplying integers.
- Note that the intermediate numbers that are generated can have at most $m + n \leq 2n$ bits.

Example: Multiplying Two Integers

Suppose that we want to multiply an n -bit integer A and an m -bit integer B ($m \leq n$).

$$\begin{array}{r} 1\ 0\ 0\ 0\ 1 \times 1\ 0\ 1\ 1 \\ \hline 1\ 0\ 0\ 0\ 1 \\ 1\ 0\ 0\ 0\ 1\ 0 \end{array}$$

- This is also known as the “school method” for multiplying integers.
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$$\begin{array}{r} 1\ 0\ 0\ 0\ 1 \times 1\ 0\ 1\ 1 \\ \hline 1\ 0\ 0\ 0\ 1 \\ 1\ 0\ 0\ 0\ 1\ 0 \end{array}$$

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$$\begin{array}{r} 1\ 0\ 0\ 0\ 1 \times 1\ 0\ 1\ 1 \\ \hline 1\ 0\ 0\ 0\ 1 \\ 1\ 0\ 0\ 0\ 1\ 0 \\ 0\ 0 \end{array}$$

- This is also known as the “school method” for multiplying integers.
- Note that the intermediate numbers that are generated can have at most $m + n \leq 2n$ bits.

Example: Multiplying Two Integers

Suppose that we want to multiply an n -bit integer A and an m -bit integer B ($m \leq n$).

$$\begin{array}{r} 1\ 0\ 0\ 0\ 1 \times 1\ 0\ 1\ 1 \\ \hline 1\ 0\ 0\ 0\ 1 \\ 1\ 0\ 0\ 0\ 1\ 0 \\ 0\ 0\ 0\ 0\ 0\ 0\ 0 \\ \hline \end{array}$$

- This is also known as the “school method” for multiplying integers.
- Note that the intermediate numbers that are generated can have at most $m + n \leq 2n$ bits.

Example: Multiplying Two Integers

Suppose that we want to multiply an n -bit integer A and an m -bit integer B ($m \leq n$).

$$\begin{array}{r} 1\ 0\ 0\ 0\ 1 \times 1\ 0\ 1\ 1 \\ \hline 1\ 0\ 0\ 0\ 1 \\ 1\ 0\ 0\ 0\ 1\ 0 \\ 0\ 0\ 0\ 0\ 0\ 0\ 0 \end{array}$$

- This is also known as the “school method” for multiplying integers.
- Note that the intermediate numbers that are generated can have at most $m + n \leq 2n$ bits.

Example: Multiplying Two Integers

Suppose that we want to multiply an n -bit integer A and an m -bit integer B ($m \leq n$).

$$\begin{array}{r} 10001 \times 1011 \\ \hline 10001 \\ 100010 \\ 0000000 \\ 00000000 \end{array}$$

- This is also known as the “school method” for multiplying integers.
- Note that the intermediate numbers that are generated can have at most $m + n \leq 2n$ bits.

Example: Multiplying Two Integers

Suppose that we want to multiply an n -bit integer A and an m -bit integer B ($m \leq n$).

$$\begin{array}{r} 1\ 0\ 0\ 0\ 1 \times 1\ 0\ 1\ 1 \\ \hline 1\ 0\ 0\ 0\ 1 \\ 1\ 0\ 0\ 0\ 1\ 0 \\ 0\ 0\ 0\ 0\ 0\ 0\ 0 \\ 1\ 0\ 0\ 0\ 1\ 0\ 0\ 0 \end{array}$$

- This is also known as the “school method” for multiplying integers.
- Note that the intermediate numbers that are generated can have at most $m + n \leq 2n$ bits.

Example: Multiplying Two Integers

Suppose that we want to multiply an n -bit integer A and an m -bit integer B ($m \leq n$).

$$\begin{array}{r} 1\ 0\ 0\ 0\ 1 \times 1\ 0\ 1\ 1 \\ \hline 1\ 0\ 0\ 0\ 1 \\ 1\ 0\ 0\ 0\ 1\ 0 \\ 0\ 0\ 0\ 0\ 0\ 0\ 0 \\ 1\ 0\ 0\ 0\ 1\ 0\ 0\ 0 \\ \hline \end{array}$$

- This is also known as the “school method” for multiplying integers.
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- ▶ Computing intermediate results: $\mathcal{O}(nm)$.

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Time requirement:

- ▶ Computing intermediate results: $\mathcal{O}(nm)$.
- ▶ Adding m numbers of length $\leq 2n$: $\mathcal{O}((m+n)m) = \mathcal{O}(nm)$.

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Suppose that integers A and B are of length $n = 2^k$, for some k .

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Hence,

$$A \cdot B = A_1 B_1 \cdot 2^n + (A_1 B_0 + A_0 B_1) \cdot 2^{\frac{n}{2}} + A_0 B_0$$

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Algorithm 3 $\text{mult}(A, B)$

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1: if  $|A| = |B| = 1$  then  
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3: split  $A$  into  $A_0$  and  $A_1$   
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We get the following recurrence:

$$T(n) = 4T\left(\frac{n}{2}\right) + \mathcal{O}(n) .$$

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Master Theorem: Recurrence: $T[n] = aT(\frac{n}{b}) + f(n)$.

- ▶ Case 1: $f(n) = \mathcal{O}(n^{\log_b a - \epsilon})$ $T(n) = \Theta(n^{\log_b a})$
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In our case $a = 4$, $b = 2$, and $f(n) = \Theta(n)$. Hence, we are in Case 1, since $n = \mathcal{O}(n^{2-\epsilon}) = \mathcal{O}(n^{\log_b a - \epsilon})$.

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⇒ Not better than the “school method”.

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We can use the following identity to compute Z_1 :

A more precise
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A more precise (correct) analysis would say that computing Z_1 needs time $T(\frac{n}{2} + 1) + \mathcal{O}(n)$.

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Hence,

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1: **if** $|A| = |B| = 1$ **then**

2: **return** $a_0 \cdot b_0$

3: split A into A_0 and A_1

4: split B into B_0 and B_1

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A huge improvement over the “school method”.

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Consider the recurrence relation:

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Note that we ignore **boundary conditions** for the moment.

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- ▶ First determine all solutions that satisfy recurrence relation.
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- ▶ First consider the homogenous case.

The Homogenous Case

The solution space

$$S = \{ \mathcal{T} = T[1], T[2], T[3], \dots \mid \mathcal{T} \text{ fulfills recurrence relation} \}$$

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How do we find a non-trivial solution?

We guess that the solution is of the form λ^n , $\lambda \neq 0$, and see what happens. In order for this guess to fulfill the recurrence we need

$$c_0\lambda^n + c_1\lambda^{n-1} + c_2 \cdot \lambda^{n-2} + \dots + c_k \cdot \lambda^{n-k} = 0$$

for all $n \geq k$.

The Homogenous Case

Dividing by λ^{n-k} gives that all these constraints are identical to

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Let $\lambda_1, \dots, \lambda_k$ be the k (complex) roots of $P[\lambda]$. Then, because of the vector space property

$$\alpha_1\lambda_1^n + \alpha_2\lambda_2^n + \dots + \alpha_k\lambda_k^n$$

is a solution for arbitrary values α_i .

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Lemma 6

Assume that the characteristic polynomial has k *distinct* roots $\lambda_1, \dots, \lambda_k$. Then *all* solutions to the recurrence relation are of the form

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Proof.

There is one solution for every possible choice of boundary conditions for $T[1], \dots, T[k]$.

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We show that the above set of solutions contains one solution for every choice of boundary conditions.

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Proof (cont.).

Suppose I am given boundary conditions $T[i]$ and I want to see whether I can choose the α'_i 's such that these conditions are met:

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We show that the column vectors are linearly independent. Then the above equation has a solution.

Computing the Determinant

$$\begin{vmatrix} \lambda_1 & \lambda_2 & \cdots & \lambda_{k-1} & \lambda_k \\ \lambda_1^2 & \lambda_2^2 & \cdots & \lambda_{k-1}^2 & \lambda_k^2 \\ \vdots & \vdots & & \vdots & \vdots \\ \lambda_1^k & \lambda_2^k & \cdots & \lambda_{k-1}^k & \lambda_k^k \end{vmatrix} =$$

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Computing the Determinant

$$\begin{vmatrix} 1 & 0 & \cdots & 0 & 0 \\ 1 & (\lambda_2 - \lambda_1) \cdot 1 & \cdots & (\lambda_2 - \lambda_1) \cdot \lambda_2^{k-3} & (\lambda_2 - \lambda_1) \cdot \lambda_2^{k-2} \\ \vdots & \vdots & & \vdots & \vdots \\ 1 & (\lambda_k - \lambda_1) \cdot 1 & \cdots & (\lambda_k - \lambda_1) \cdot \lambda_k^{k-3} & (\lambda_k - \lambda_1) \cdot \lambda_k^{k-2} \end{vmatrix} =$$

$$\prod_{i=2}^k (\lambda_i - \lambda_1) \cdot \begin{vmatrix} 1 & \lambda_2 & \cdots & \lambda_2^{k-3} & \lambda_2^{k-2} \\ \vdots & \vdots & & \vdots & \vdots \\ 1 & \lambda_k & \cdots & \lambda_k^{k-3} & \lambda_k^{k-2} \end{vmatrix}$$

Computing the Determinant

Repeating the above steps gives:

$$\begin{vmatrix} \lambda_1 & \lambda_2 & \cdots & \lambda_{k-1} & \lambda_k \\ \lambda_1^2 & \lambda_2^2 & \cdots & \lambda_{k-1}^2 & \lambda_k^2 \\ \vdots & \vdots & & \vdots & \vdots \\ \lambda_1^k & \lambda_2^k & \cdots & \lambda_{k-1}^k & \lambda_k^k \end{vmatrix} = \prod_{i=1}^k \lambda_i \cdot \prod_{i>\ell} (\lambda_i - \lambda_\ell)$$

Hence, if all λ_i 's are different, then the determinant is non-zero.

The Homogeneous Case

What happens if the roots are not all distinct?

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Suppose we have a root λ_i with multiplicity (**Vielfachheit**) at least 2. Then not only is λ_i^n a solution to the recurrence but also $n\lambda_i^{n-1}$.

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To see this consider the polynomial

$$P[\lambda] \cdot \lambda^{n-k} = c_0\lambda^n + c_1\lambda^{n-1} + c_2\lambda^{n-2} + \dots + c_k\lambda^{n-k}$$

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Since λ_i is a root we can write this as $Q[\lambda] \cdot (\lambda - \lambda_i)^2$. Calculating the derivative gives a polynomial that still has root λ_i .

This means

$$c_0 n \lambda_i^{n-1} + c_1 (n-1) \lambda_i^{n-2} + \dots + c_k (n-k) \lambda_i^{n-k-1} = 0$$

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Hence,

$$\underbrace{c_0 n \lambda_i^n}_{T[n]} + \underbrace{c_1 (n-1) \lambda_i^{n-1}}_{T[n-1]} + \dots + \underbrace{c_k (n-k) \lambda_i^{n-k}}_{T[n-k]} = 0$$

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We can continue $j-1$ times.

Hence, $n^\ell \lambda_i^n$ is a solution for $\ell \in 0, \dots, j-1$.

The Homogeneous Case

Lemma 7

Let $P[\lambda]$ denote the characteristic polynomial to the recurrence

$$c_0T[n] + c_1T[n-1] + \dots + c_kT[n-k] = 0$$

Let $\lambda_i, i = 1, \dots, m$ be the (complex) roots of $P[\lambda]$ with multiplicities ℓ_i . Then the general solution to the recurrence is given by

$$T[n] = \sum_{i=1}^m \sum_{j=0}^{\ell_i-1} \alpha_{ij} \cdot (n^j \lambda_i^n) .$$

The full proof is omitted. We have only shown that any choice of α_{ij} 's is a solution to the recurrence.

Example: Fibonacci Sequence

$$T[0] = 0$$

$$T[1] = 1$$

$$T[n] = T[n - 1] + T[n - 2] \text{ for } n \geq 2$$

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Finding the roots, gives

$$\lambda_{1/2} = \frac{1}{2} \pm \sqrt{\frac{1}{4} + 1} = \frac{1}{2} (1 \pm \sqrt{5})$$

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Hence, the solution is of the form

$$\alpha \left(\frac{1 + \sqrt{5}}{2} \right)^n + \beta \left(\frac{1 - \sqrt{5}}{2} \right)^n$$

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$$\alpha \left(\frac{1 + \sqrt{5}}{2} \right) + \beta \left(\frac{1 - \sqrt{5}}{2} \right) = 1 \implies \alpha - \beta = \frac{2}{\sqrt{5}}$$

Example: Fibonacci Sequence

Hence, the solution is

$$\frac{1}{\sqrt{5}} \left[\left(\frac{1 + \sqrt{5}}{2} \right)^n - \left(\frac{1 - \sqrt{5}}{2} \right)^n \right]$$

The Inhomogeneous Case

Consider the recurrence relation:

$$c_0T(n) + c_1T(n-1) + c_2T(n-2) + \cdots + c_kT(n-k) = f(n)$$

with $f(n) \neq 0$.

While we have a fairly general technique for solving **homogeneous**, linear recurrence relations the inhomogeneous case is different.

The Inhomogeneous Case

The general solution of the recurrence relation is

$$T(n) = T_h(n) + T_p(n) ,$$

where T_h is **any** solution to the homogeneous equation, and T_p is **one** particular solution to the inhomogeneous equation.

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$$T(n) = T_h(n) + T_p(n) ,$$

where T_h is **any** solution to the homogeneous equation, and T_p is **one** particular solution to the inhomogeneous equation.

There is no general method to find a particular solution.

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$$T[n] = T[n - 1] + 1 \quad T[0] = 1$$

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I get a completely determined recurrence if I add $T[0] = 1$ and $T[1] = 2$.

The Inhomogeneous Case

Example: Characteristic polynomial:

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$T[0] = 1$ gives $\alpha = 1$.

$T[1] = 2$ gives $1 + \beta = 2 \Rightarrow \beta = 1$.

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Shift:

$$T[n - 1] = 2T[n - 2] - T[n - 3] + 2(n - 1) - 1$$

$$T[n] = 2T[n - 1] - T[n - 2] + 2n - 1$$

Shift:

$$\begin{aligned} T[n - 1] &= 2T[n - 2] - T[n - 3] + 2(n - 1) - 1 \\ &= 2T[n - 2] - T[n - 3] + 2n - 3 \end{aligned}$$

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$$T[n] = 3T[n - 1] - 3T[n - 2] + T[n - 3] + 2$$

and so on...

6.4 Generating Functions

Definition 8 (Generating Function)

Let $(a_n)_{n \geq 0}$ be a sequence. The corresponding

- ▶ **generating function** (**Erzeugendenfunktion**) is

$$F(z) := \sum_{n \geq 0} a_n z^n ;$$

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- ▶ **exponential generating function** (**exponentielle Erzeugendenfunktion**) is

$$F(z) := \sum_{n \geq 0} \frac{a_n}{n!} z^n .$$

6.4 Generating Functions

Example 9

1. The generating function of the sequence $(1, 0, 0, \dots)$ is

$$F(z) = 1.$$

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- ▶ **Multiplication:** $f \cdot g := \sum_{n \geq 0} c_n z^n$ with $c_n = \sum_{p=0}^n a_p b_{n-p}$.

There are no convergence issues here.

6.4 Generating Functions

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We view a power series as a function $f : \mathbb{C} \rightarrow \mathbb{C}$.

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Then, it is important to think about convergence/convergence radius etc.

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It means that the power series $1 - z$ and the power series $\sum_{n \geq 0} z^n$ are invers, i.e.,

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This is well-defined.

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Hence, the generating function of the sequence $a_n = n + 1$ is $1/(1-z)^2$.

Formally the derivative of a formal power series $\sum_{n \geq 0} a_n z^n$ is defined as $\sum_{n \geq 0} n a_n z^{n-1}$.

The known rules for differentiation work for this definition. In particular, e.g. the derivative of $\frac{1}{1-z}$ is $\frac{1}{(1-z)^2}$.

Note that this requires a proof if we consider power series as algebraic objects. However, we did not prove this in the lecture.

6.4 Generating Functions

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Hence, the generating function of the sequence

$$a_n = (n+1)(n+2) \text{ is } \frac{2}{(1-z)^3} .$$

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Computing the k -th derivative of $\sum z^n$.

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Computing the k -th derivative of $\sum z^n$.

$$\begin{aligned}\sum_{n \geq k} n(n-1) \cdot \dots \cdot (n-k+1)z^{n-k} &= \sum_{n \geq 0} (n+k) \cdot \dots \cdot (n+1)z^n \\ &= \frac{k!}{(1-z)^{k+1}} \cdot\end{aligned}$$

6.4 Generating Functions

Computing the k -th derivative of $\sum z^n$.

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Hence:

$$\sum_{n \geq 0} \binom{n+k}{k} z^n = \frac{1}{(1-z)^{k+1}} \cdot$$

6.4 Generating Functions

Computing the k -th derivative of $\sum z^n$.

$$\begin{aligned}\sum_{n \geq k} n(n-1) \cdot \dots \cdot (n-k+1)z^{n-k} &= \sum_{n \geq 0} (n+k) \cdot \dots \cdot (n+1)z^n \\ &= \frac{k!}{(1-z)^{k+1}} \cdot\end{aligned}$$

Hence:

$$\sum_{n \geq 0} \binom{n+k}{k} z^n = \frac{1}{(1-z)^{k+1}} \cdot$$

The generating function of the sequence $a_n = \binom{n+k}{k}$ is $\frac{1}{(1-z)^{k+1}}$.

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The generating function of the sequence $f_n = a^n$ is $\frac{1}{1-az}$.

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$A(z)$

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Hence, $a_n = n + 1$.

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$\frac{1}{n!}$	e^z

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$$\begin{aligned} z^2 - z + 1 &= A(1 - z)^2 + B(1 - 3z)(1 - z) + C(1 - 3z) \\ &= A(1 - 2z + z^2) + B(1 - 4z + 3z^2) + C(1 - 3z) \\ &= (A + 3B)z^2 + (-2A - 4B - 3C)z + (A + B + C) \end{aligned}$$

Example: $a_n = 3a_{n-1} + n$, $a_0 = 1$

5. Write $f(z)$ as a formal power series:

This leads to the following conditions:

$$A + B + C = 1$$

$$2A + 4B + 3C = 1$$

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which gives

$$A = \frac{7}{4} \quad B = -\frac{1}{4} \quad C = -\frac{1}{2}$$

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6. This means $a_n = \frac{7}{4}3^n - \frac{1}{2}n - \frac{3}{4}$.

6.5 Transformation of the Recurrence

Example 10

$$f_0 = 1$$

$$f_1 = 2$$

$$f_n = f_{n-1} \cdot f_{n-2} \text{ for } n \geq 2 .$$

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6 Recurrences

Let $n = 2^k$:

$$g_k = 3^{k+1} - 2^{k+1}, \text{ hence}$$

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Part III

Data Structures

Abstract Data Type

An abstract data type (ADT) is defined by an interface of operations or methods that can be performed and that have a defined behavior.

The data types in this lecture all operate on objects that are represented by a `[key, value]` pair.

- ▶ The `key` comes from a totally ordered set, and we assume that there is an efficient comparison function.
- ▶ The `value` can be anything; it usually carries satellite information important for the application that uses the ADT.

Dynamic Set Operations

- ▶ **S .search(k)**: Returns pointer to object x from S with $\text{key}[x] = k$ or **null**.

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- ▶ **S.split(k, S')**:
 $S := \{x \in S \mid \text{key}[x] \leq k\}$, $S' := \{x \in S \mid \text{key}[x] > k\}$.

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 $S := \{x \in S \mid \text{key}[x] \leq k\}$, $S' := \{x \in S \mid \text{key}[x] > k\}$.
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Requires $\text{key}[S.\text{maximum}()] \leq \text{key}[S'.\text{minimum}()]$.
- ▶ **S. decrease-key(x, k):** Replace $\text{key}[x]$ by $k \leq \text{key}[x]$.

Examples of ADTs

Stack:

- ▶ **S .push(x)**: Insert an element.
- ▶ **S .pop()**: Return the element from S that was inserted most recently; delete it from S .
- ▶ **S .empty()**: Tell if S contains any object.

Examples of ADTs

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- ▶ **$S.$ empty()**: Tell if S contains any object.

Priority-Queue:

- ▶ **$S.$ insert(x)**: Insert an element.
- ▶ **$S.$ delete-min()**: Return the element with lowest key-value; delete it from S .

7 Dictionary

Dictionary:

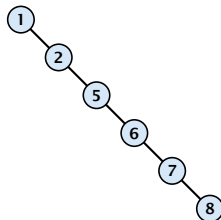
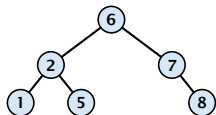
- ▶ **$S.insert(x)$** : Insert an element x .
- ▶ **$S.delete(x)$** : Delete the element pointed to by x .
- ▶ **$S.search(k)$** : Return a pointer to an element e with $key[e] = k$ in S if it exists; otherwise return **null**.

7.1 Binary Search Trees

An (**internal**) **binary search tree** stores the elements in a binary tree. Each tree-node corresponds to an element. All elements in the left sub-tree of a node v have a smaller key-value than $\text{key}[v]$ and elements in the right sub-tree have a larger-key value. We assume that all key-values are different.

(**External** Search Trees store objects only at leaf-vertices)

Examples:

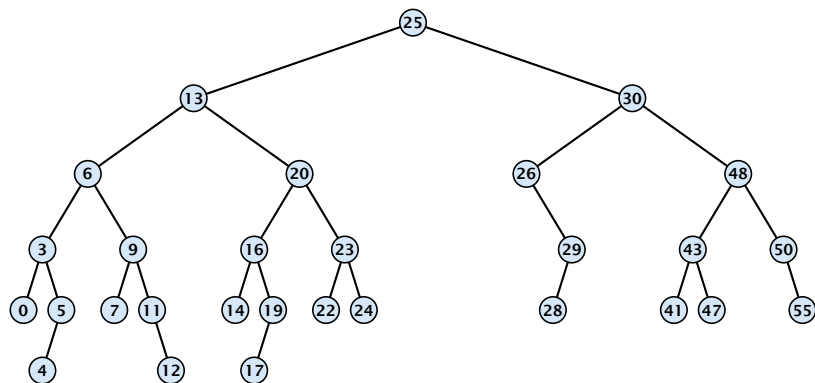


7.1 Binary Search Trees

We consider the following operations on binary search trees. Note that this is a super-set of the dictionary-operations.

- ▶ $T.\text{insert}(x)$
- ▶ $T.\text{delete}(x)$
- ▶ $T.\text{search}(k)$
- ▶ $T.\text{successor}(x)$
- ▶ $T.\text{predecessor}(x)$
- ▶ $T.\text{minimum}()$
- ▶ $T.\text{maximum}()$

Binary Search Trees: Searching

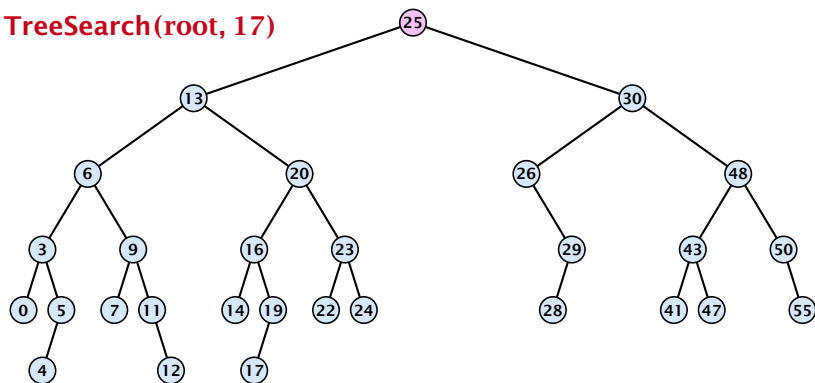


Algorithm 1 TreeSearch(x, k)

- 1: **if** $x = \text{null}$ **or** $k = \text{key}[x]$ **return** x
- 2: **if** $k < \text{key}[x]$ **return** TreeSearch(left[x], k)
- 3: **else return** TreeSearch(right[x], k)

Binary Search Trees: Searching

TreeSearch(root, 17)

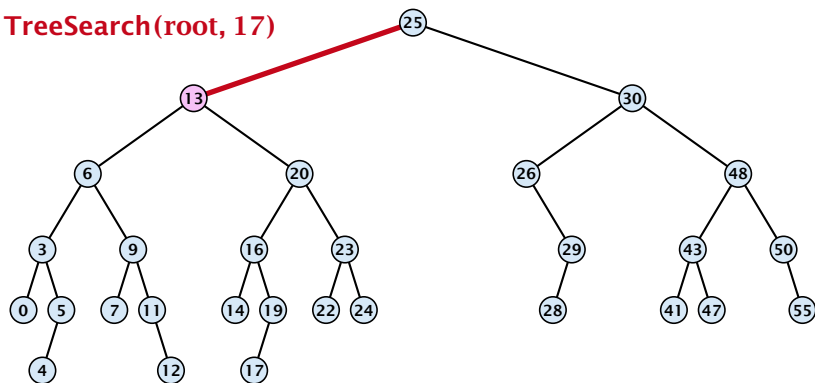


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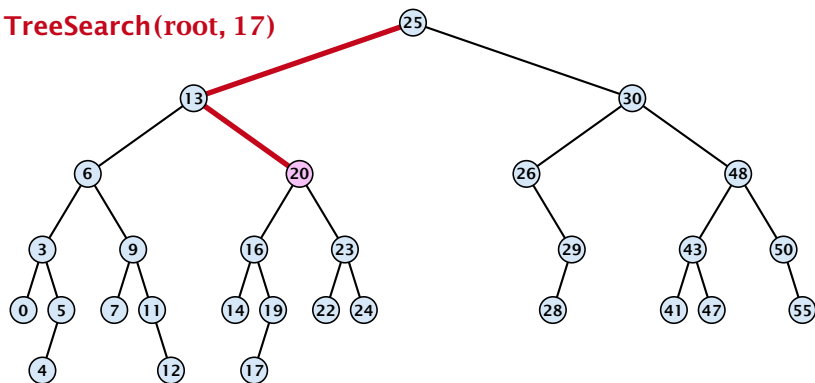


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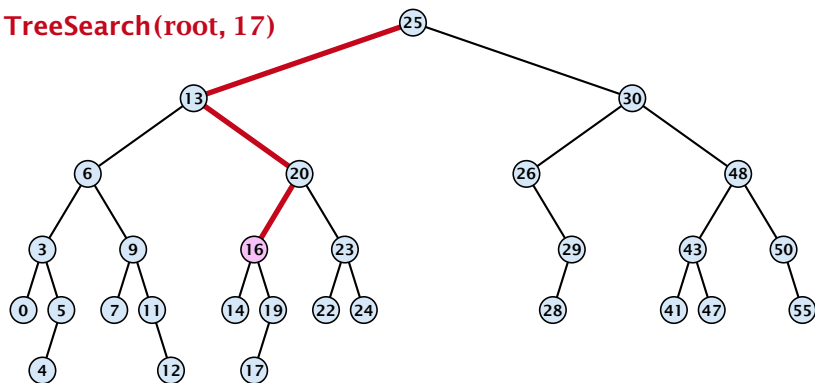


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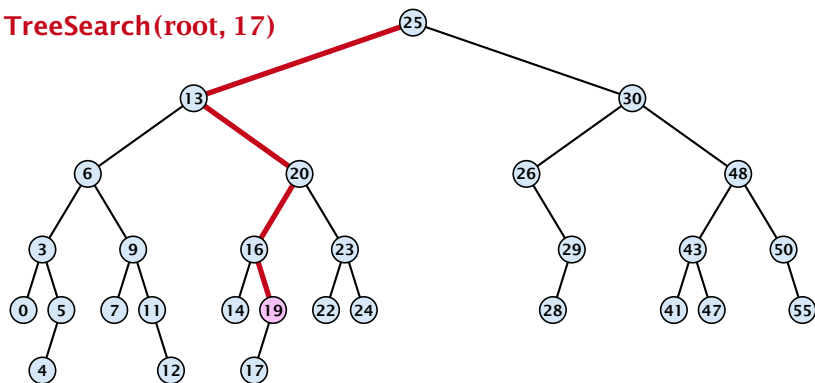


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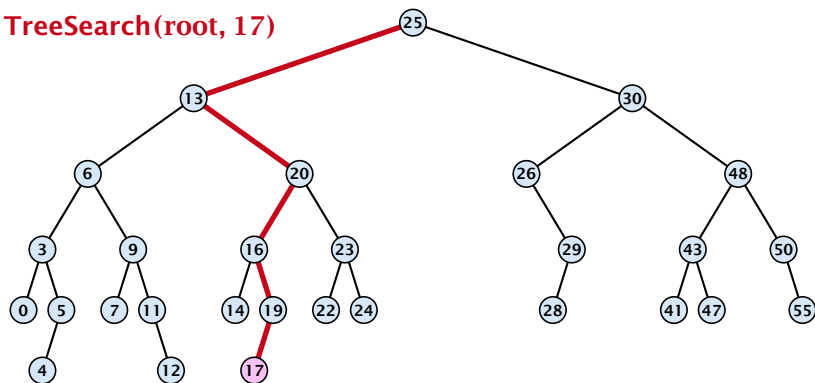


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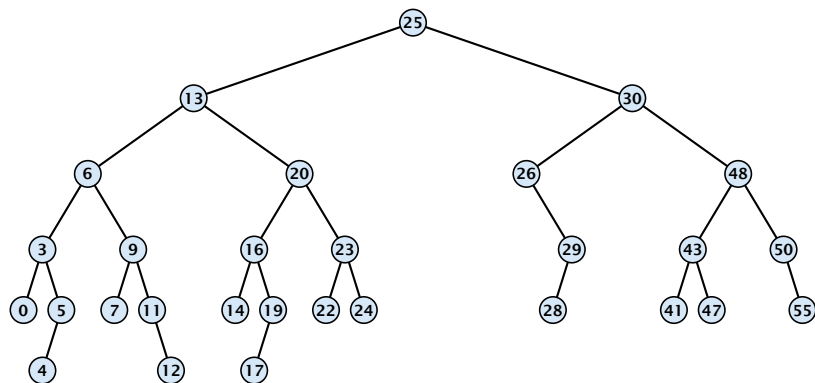
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Binary Search Trees: Searching

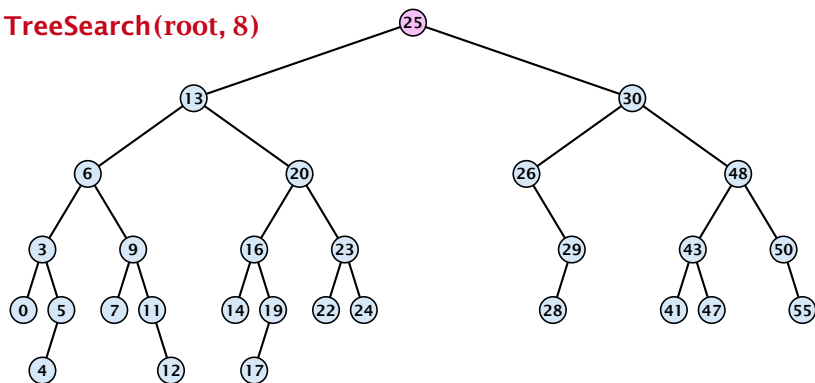


Algorithm 1 $\text{TreeSearch}(x, k)$

- 1: **if** $x = \text{null}$ **or** $k = \text{key}[x]$ **return** x
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Binary Search Trees: Searching

TreeSearch(root, 8)

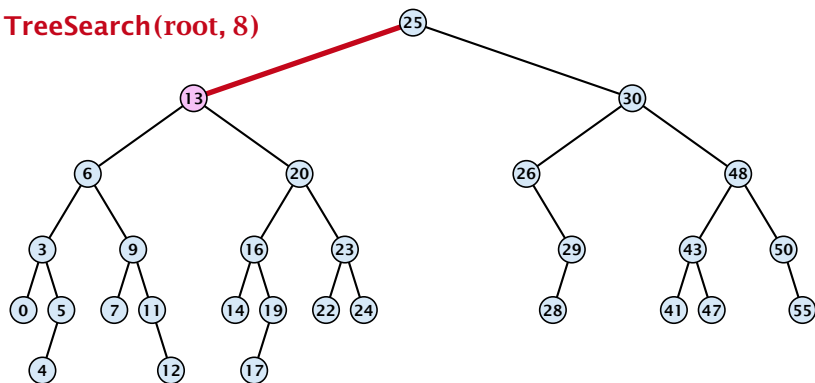


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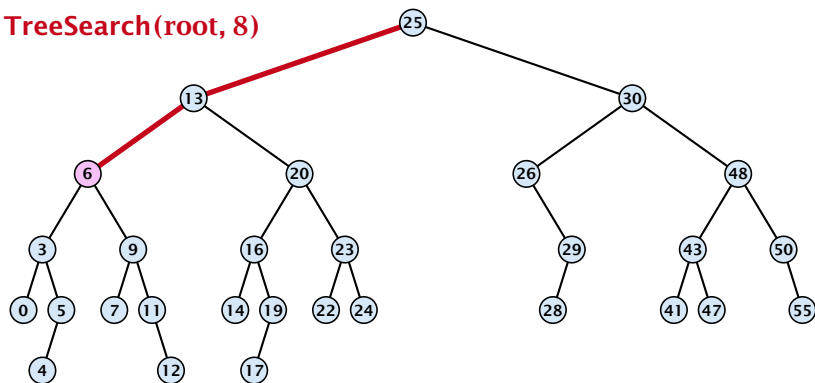


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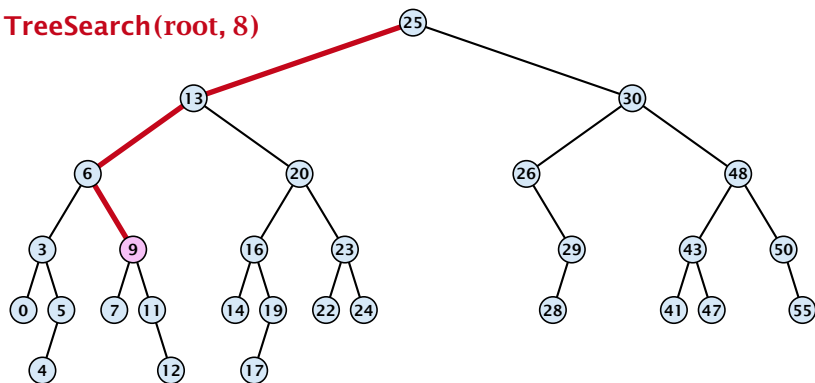


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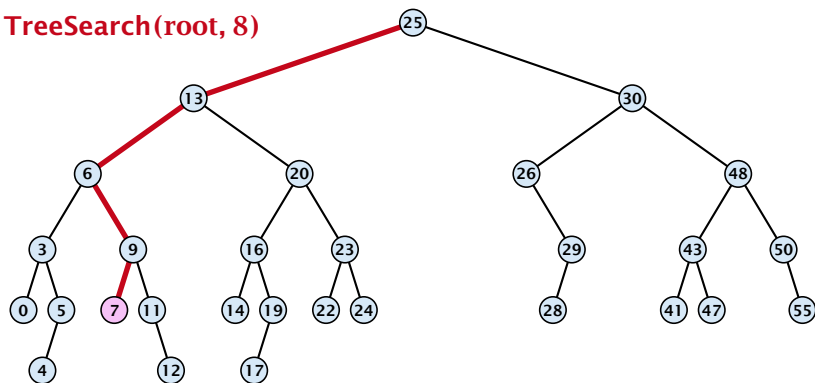


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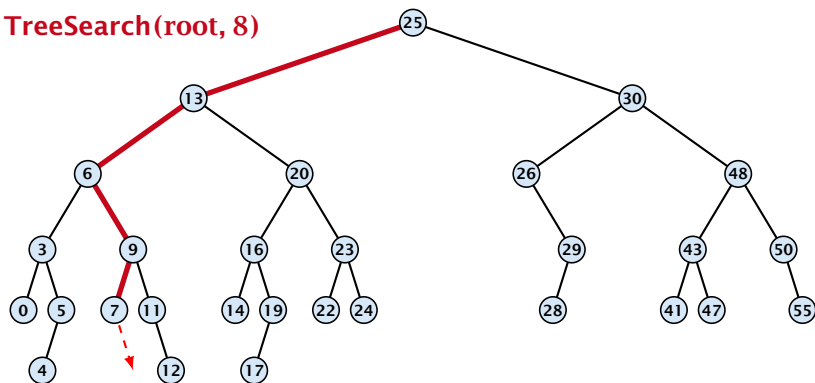


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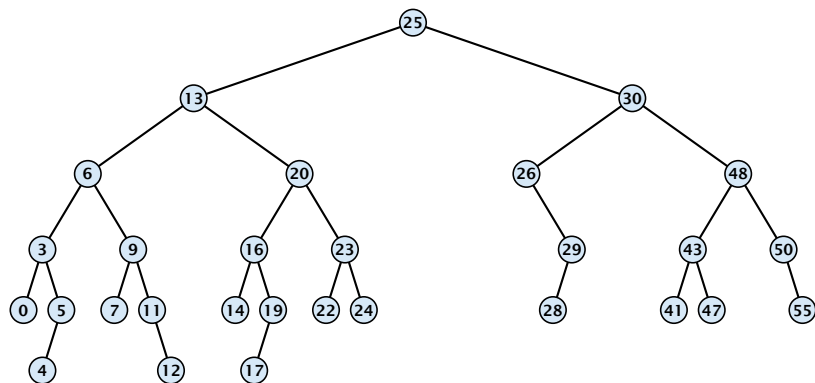
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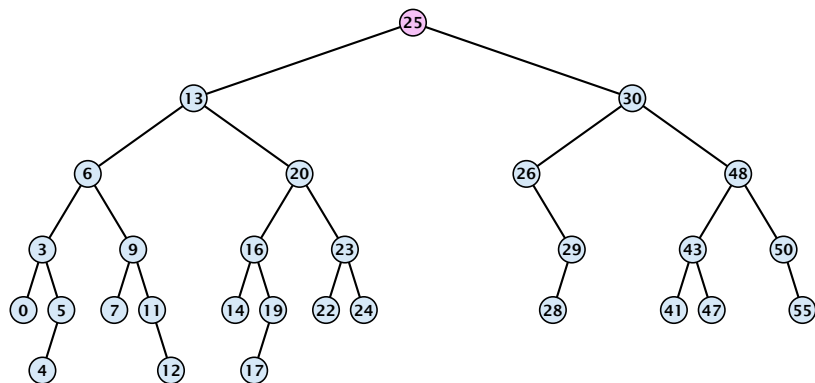
Binary Search Trees: Minimum



Algorithm 2 TreeMin(x)

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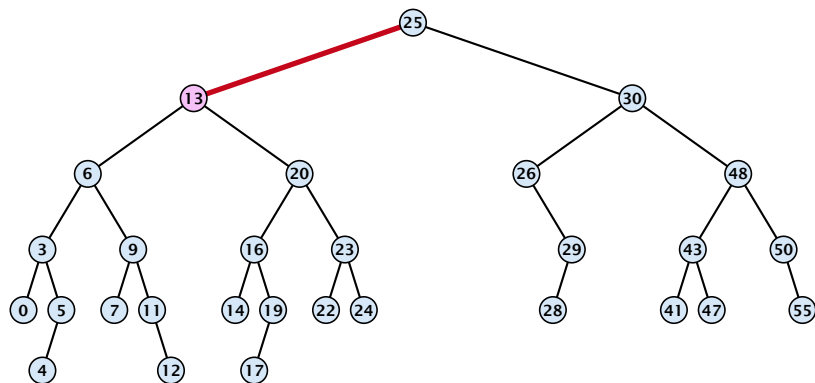
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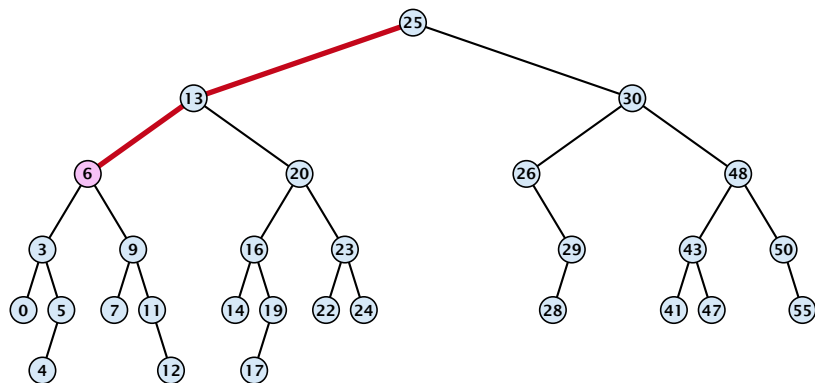
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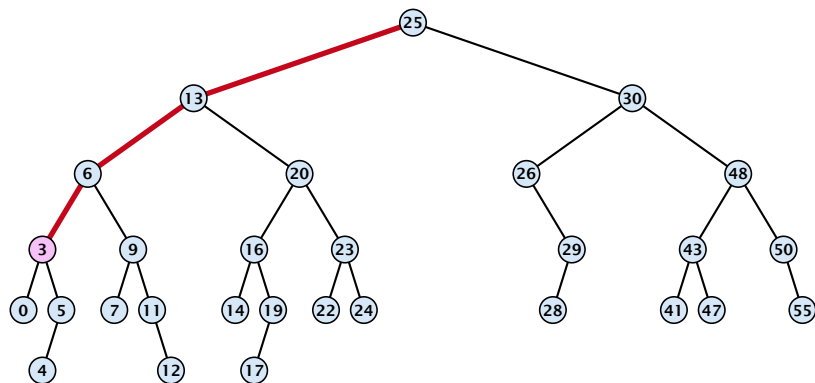
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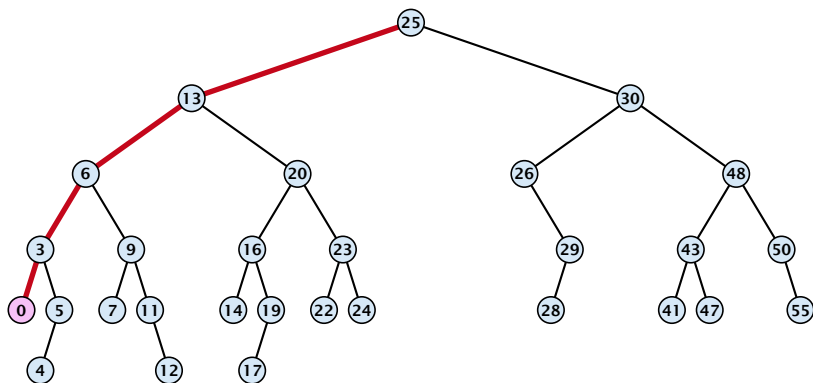
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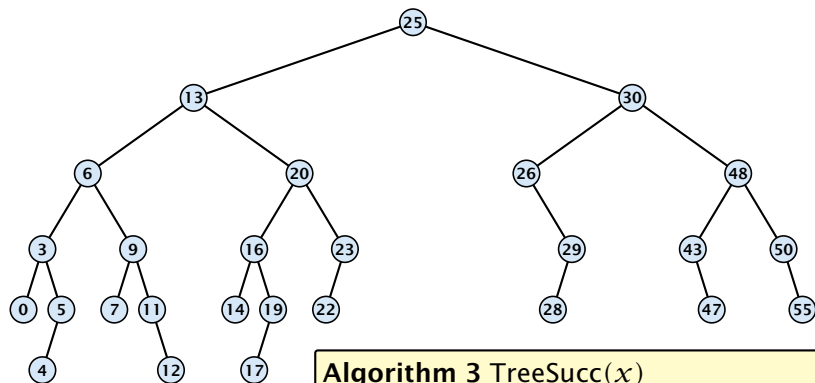
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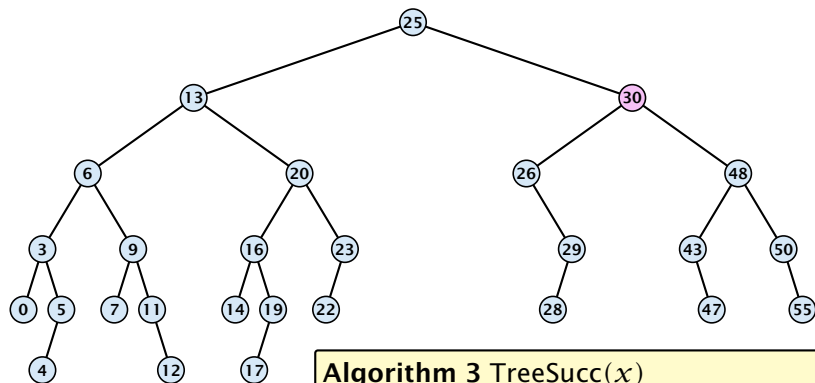
Binary Search Trees: Successor



Algorithm 3 TreeSucc(x)

- 1: **if** $\text{right}[x] \neq \text{null}$ **return** $\text{TreeMin}(\text{right}[x])$
- 2: $y \leftarrow \text{parent}[x]$
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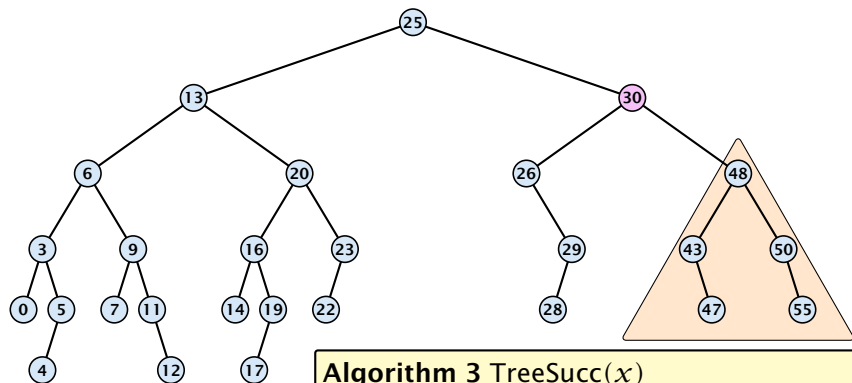
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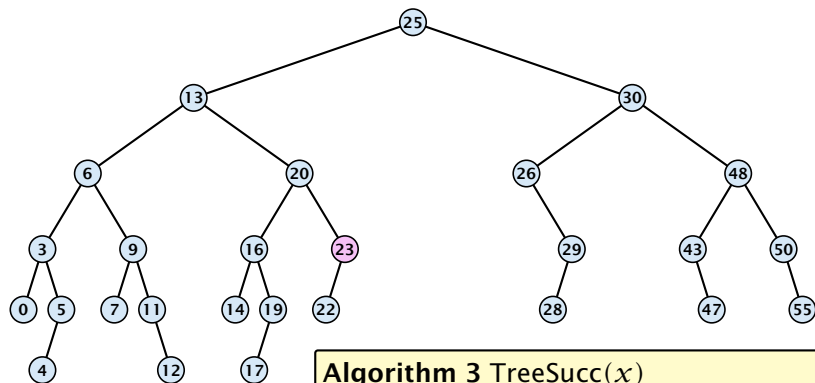
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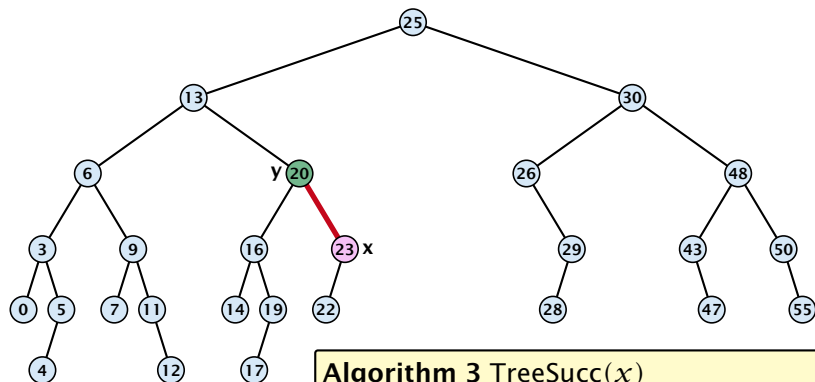
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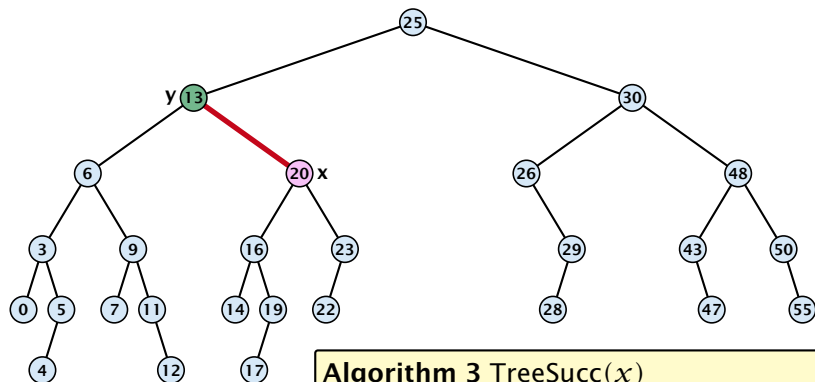
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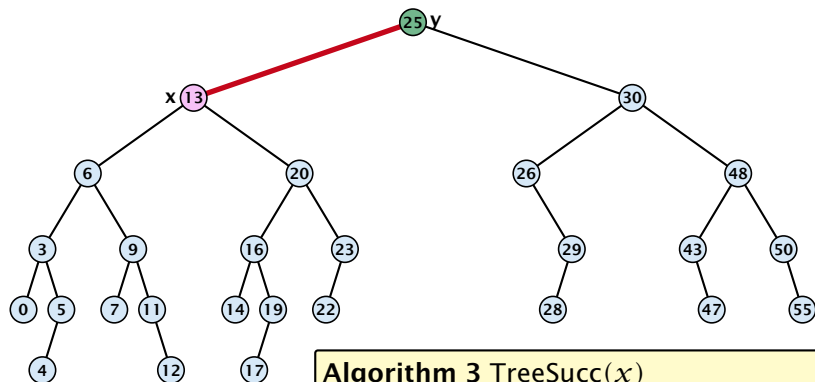
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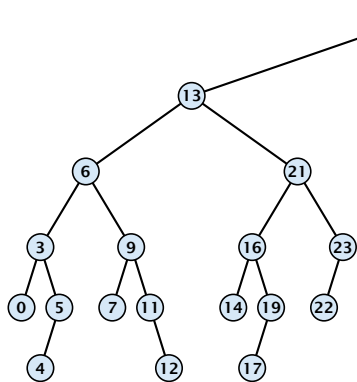
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Binary Search Trees: Insert

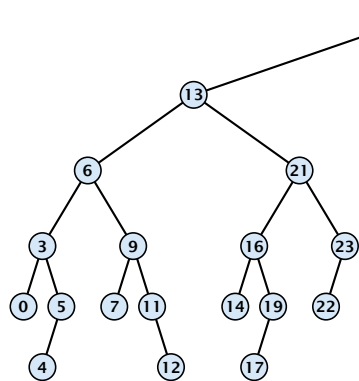


Algorithm 4 TreeInsert(x, z)

```
1: if  $x = \text{null}$  then
2:    $\text{root}[T] \leftarrow z$ ;  $\text{parent}[z] \leftarrow \text{null}$ ;
3:   return;
4: if  $\text{key}[x] > \text{key}[z]$  then
5:   if  $\text{left}[x] = \text{null}$  then
6:      $\text{left}[x] \leftarrow z$ ;  $\text{parent}[z] \leftarrow x$ ;
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Binary Search Trees: Insert

Insert element **not** in the tree.

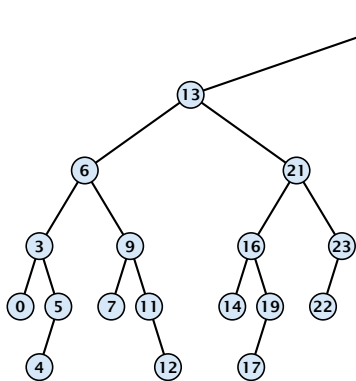


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Search for z . At some point the search stops at a null-pointer. This is the place to insert z .

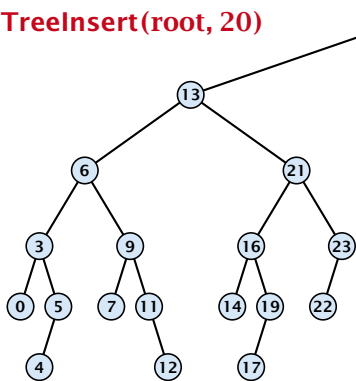
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TreeInsert(root, 20)



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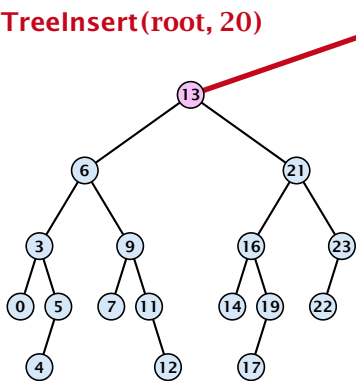
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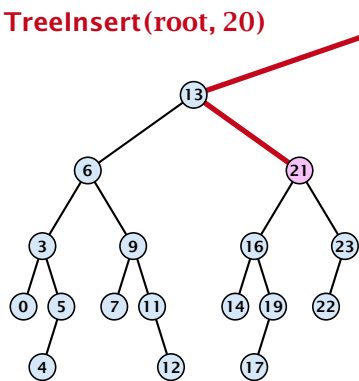
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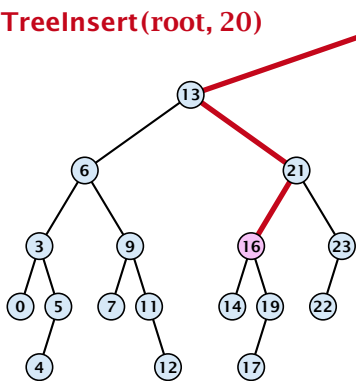
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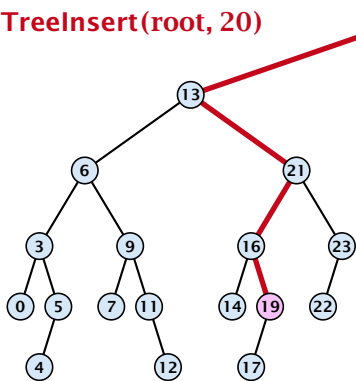
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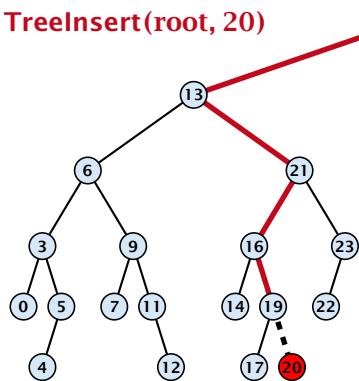
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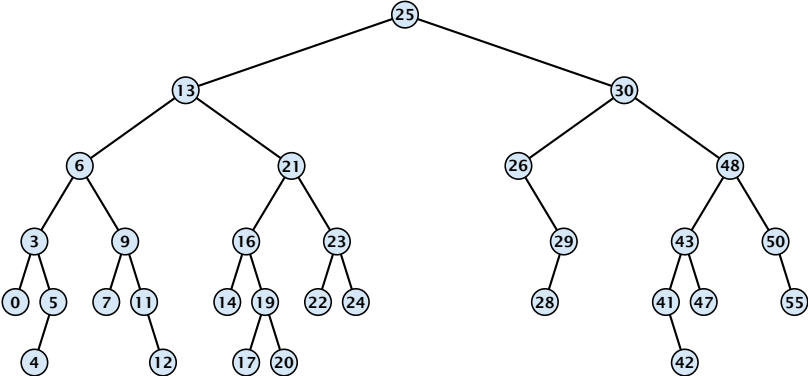


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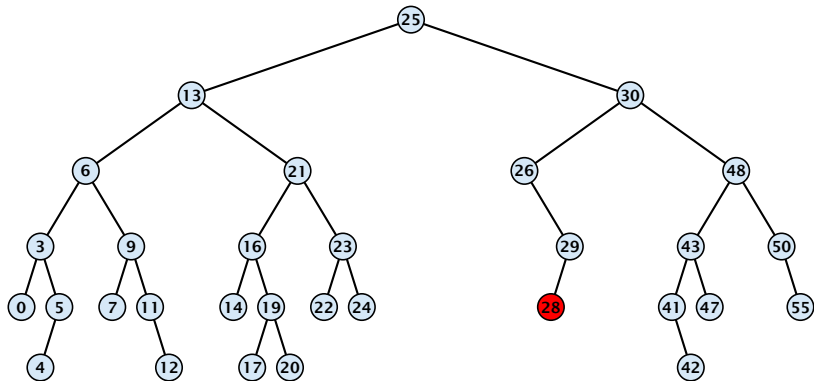
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Binary Search Trees: Delete



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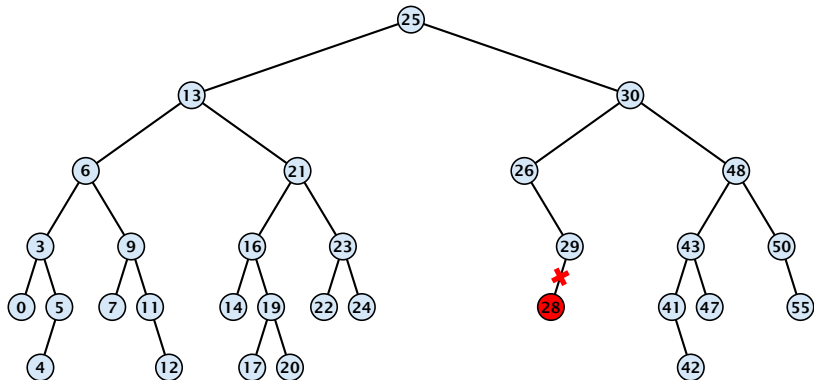


Case 1:

Element does not have any children

- ▶ Simply go to the parent and set the corresponding pointer to **null**.

Binary Search Trees: Delete

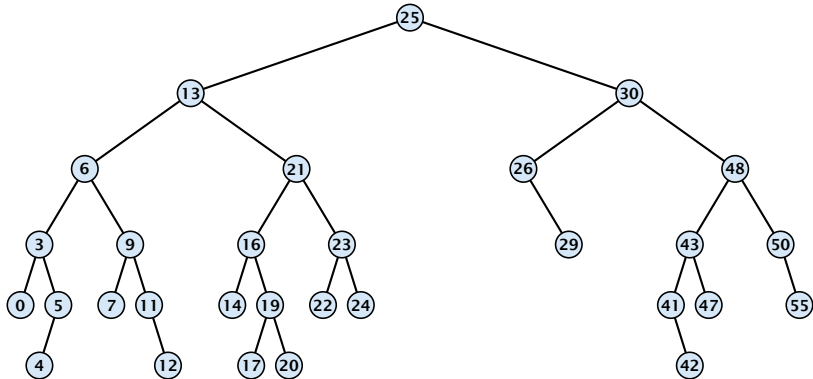


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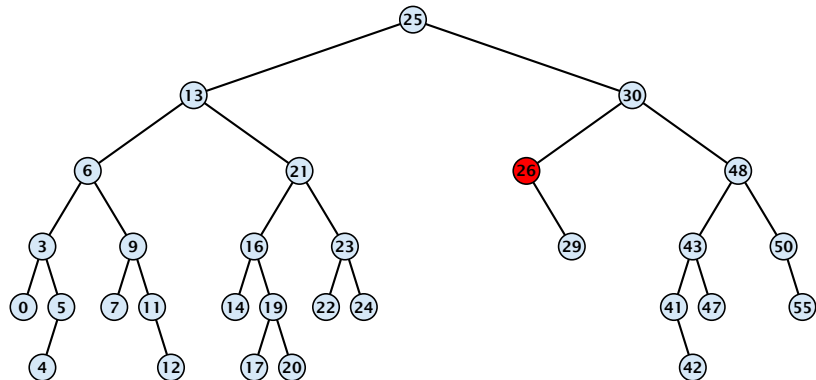


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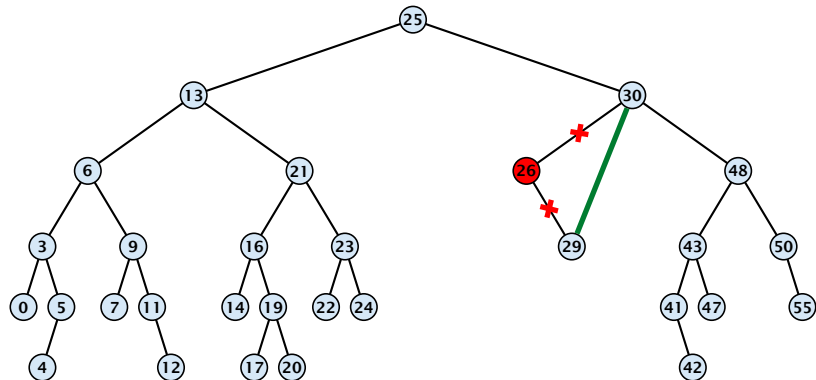


Case 2:

Element has exactly one child

- ▶ Splice the element out of the tree by connecting its parent to its successor.

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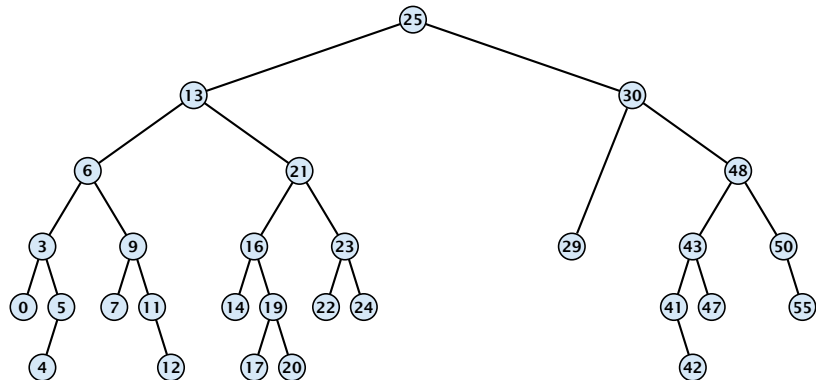


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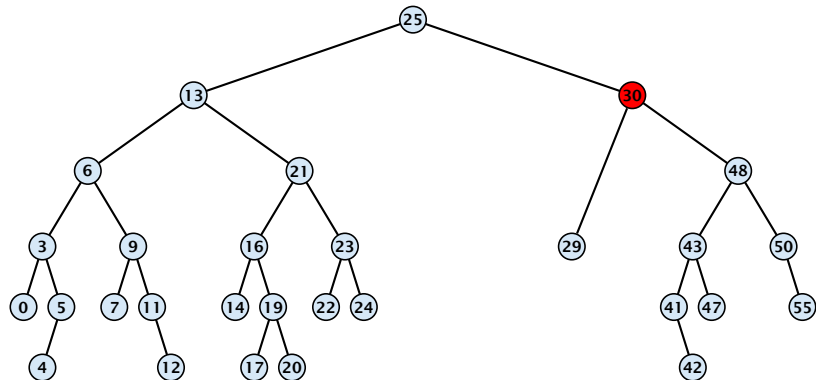


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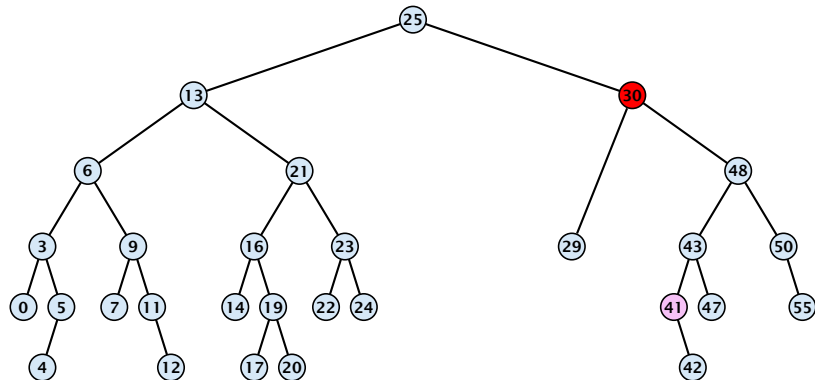


Case 3:

Element has two children

- ▶ Find the successor of the element
- ▶ Splice successor out of the tree
- ▶ Replace content of element by content of successor

Binary Search Trees: Delete

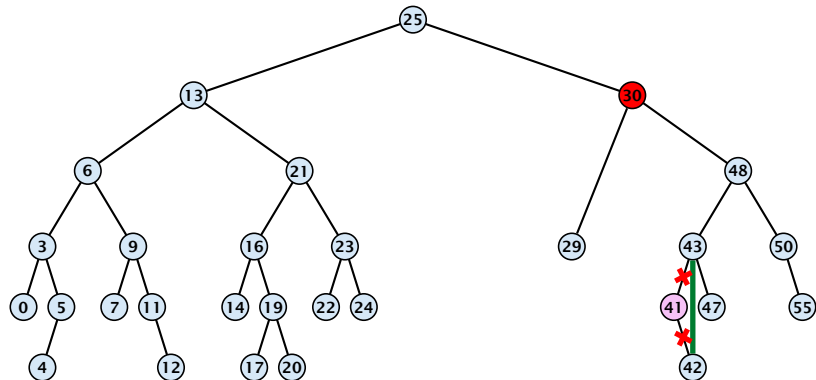


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Binary Search Trees: Delete

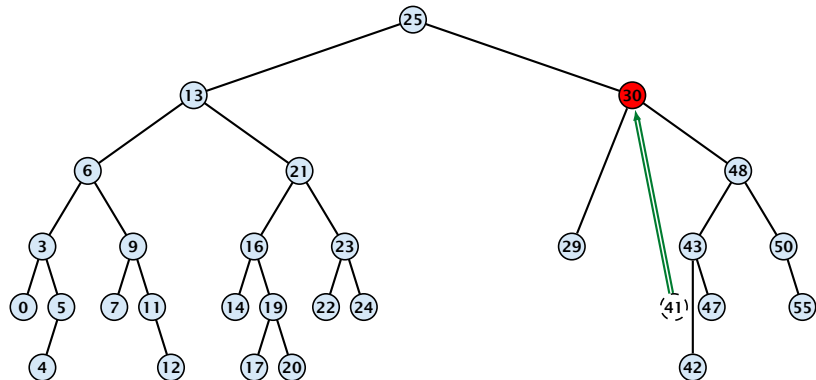


Case 3:

Element has two children

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- ▶ Splice successor out of the tree
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Binary Search Trees: Delete

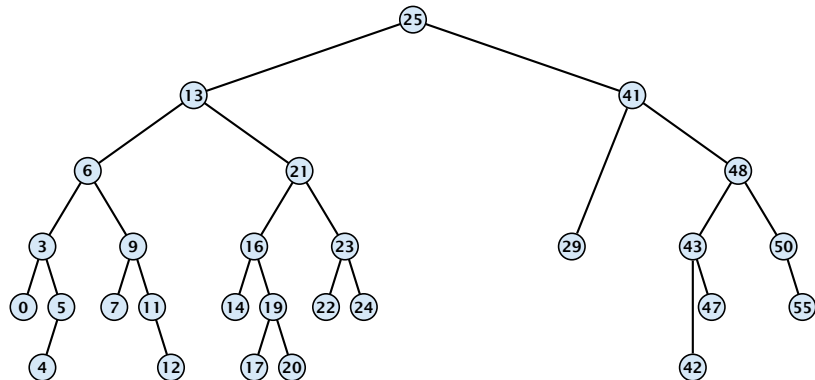


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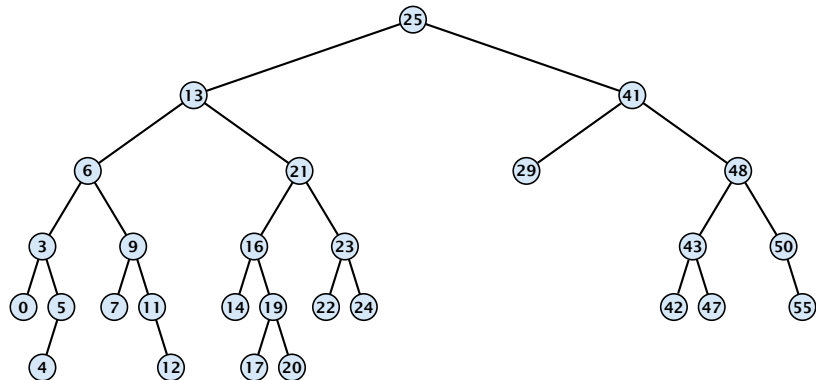


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Binary Search Trees: Delete

Algorithm 9 TreeDelete(z)

```
1: if left[ $z$ ] = null or right[ $z$ ] = null
2:   then  $y \leftarrow z$  else  $y \leftarrow \text{TreeSucc}(z)$ ;   select  $y$  to splice out
3:   if left[ $y$ ]  $\neq$  null
4:     then  $x \leftarrow \text{left}[y]$  else  $x \leftarrow \text{right}[y]$ ;  $x$  is child of  $y$  (or null)
5:   if  $x \neq \text{null}$  then parent[ $x$ ]  $\leftarrow$  parent[ $y$ ];   parent[ $x$ ] is correct
6:   if parent[ $y$ ] = null then
7:     root[ $T$ ]  $\leftarrow x$ 
8:   else
9:     if  $y = \text{left}[\text{parent}[y]]$  then
10:      left[parent[ $y$ ]]  $\leftarrow x$ 
11:    else
12:      right[parent[ $y$ ]]  $\leftarrow x$ 
13:   if  $y \neq z$  then copy  $y$ -data to  $z$ 
```

} fix pointer to x

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AVL-trees, Red-black trees, Scapegoat trees, 2-3 trees, B-trees, AA trees, Treaps

similar: SPLAY trees.

7.2 Red Black Trees

Definition 12

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7.2 Red Black Trees

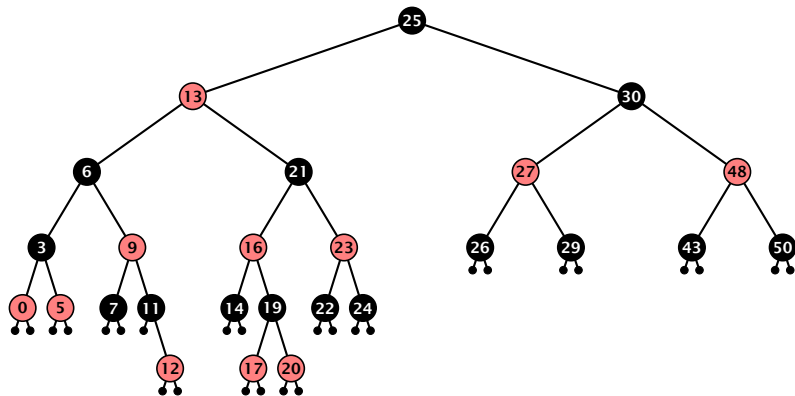
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Red Black Trees: Example



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A red-black tree with n internal nodes has height at most $\mathcal{O}(\log n)$.

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The **black height** $\text{bh}(v)$ of a node v in a red black tree is the number of black nodes on a path from v to a leaf vertex (not counting v).

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We first show:

Lemma 15

A sub-tree of black height $\text{bh}(v)$ in a red black tree contains at least $2^{\text{bh}(v)} - 1$ internal vertices.

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- ▶ The sub-tree rooted at v contains $0 = 2^{\text{bh}(v)} - 1$ inner vertices.

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- ▶ Then T_v contains at least $2(2^{\text{bh}(v)-1} - 1) + 1 \geq 2^{\text{bh}(v)} - 1$ vertices.



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Hence, $h \leq 2 \log(n + 1) = \mathcal{O}(\log n)$. □

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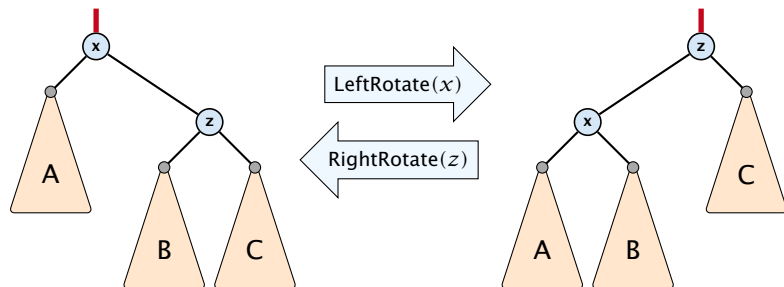
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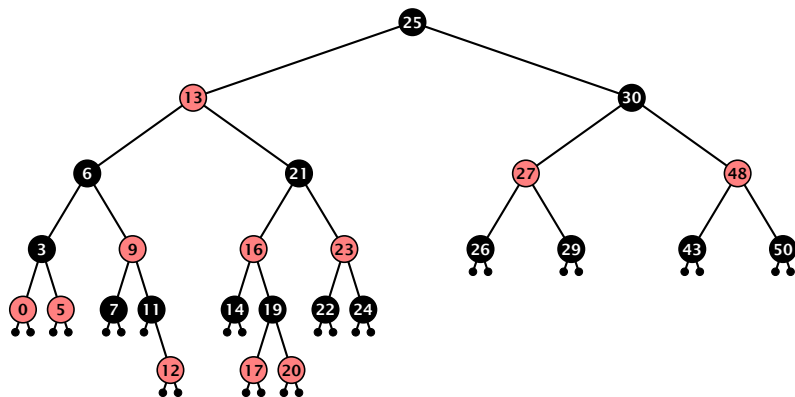
We need to adapt the insert and delete operations so that the red black properties are maintained.

Rotations

The properties will be maintained through rotations:



Red Black Trees: Insert

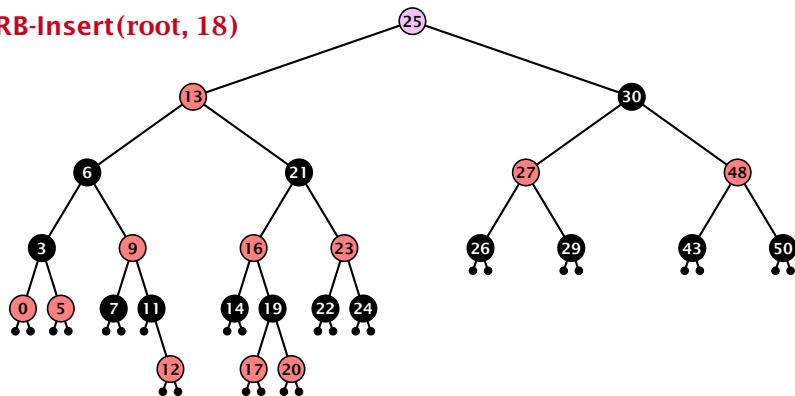


Insert:

- ▶ first make a normal insert into a binary search tree
- ▶ then fix red-black properties

Red Black Trees: Insert

RB-Insert(root, 18)

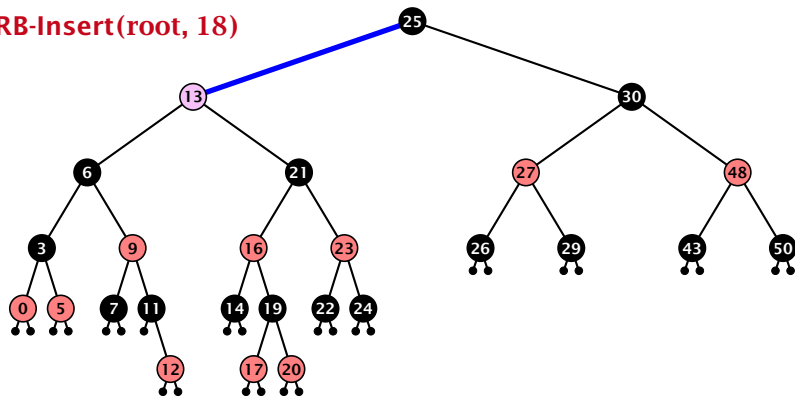


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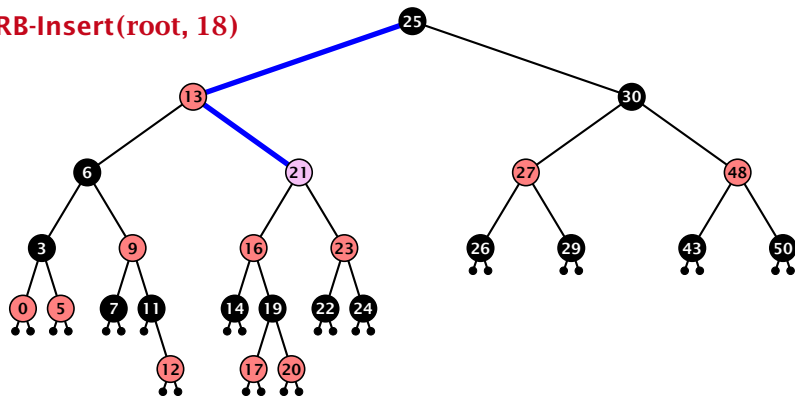


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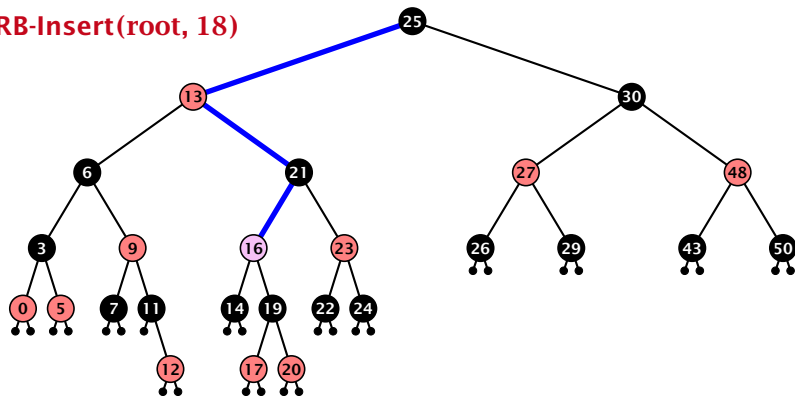


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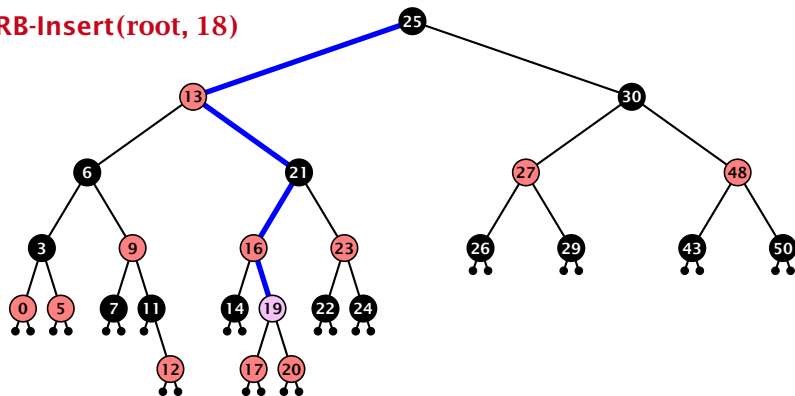


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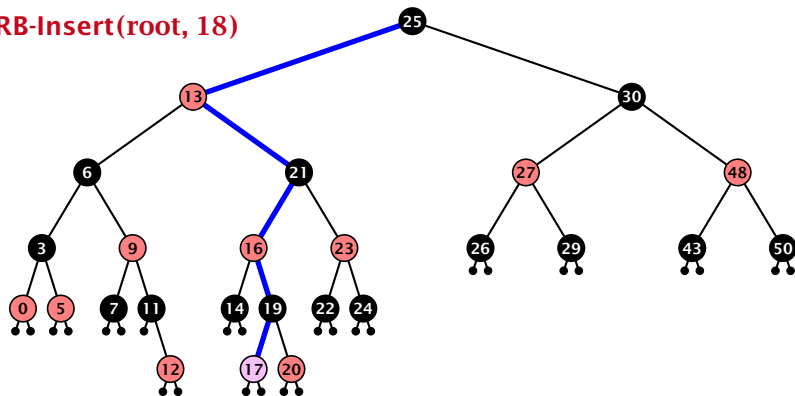


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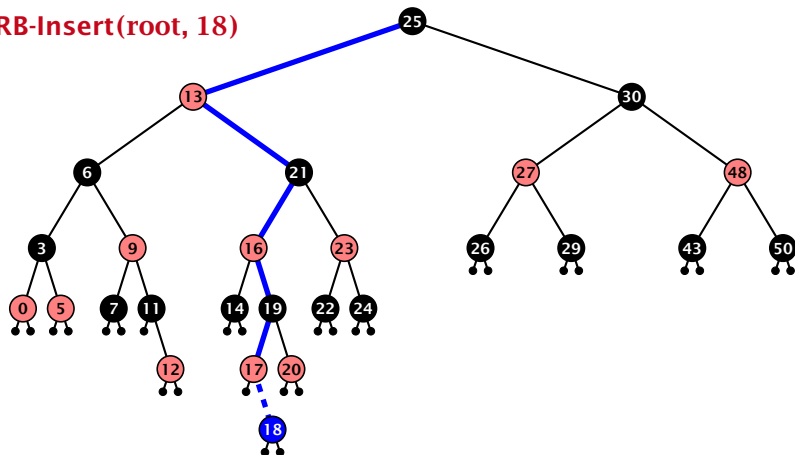


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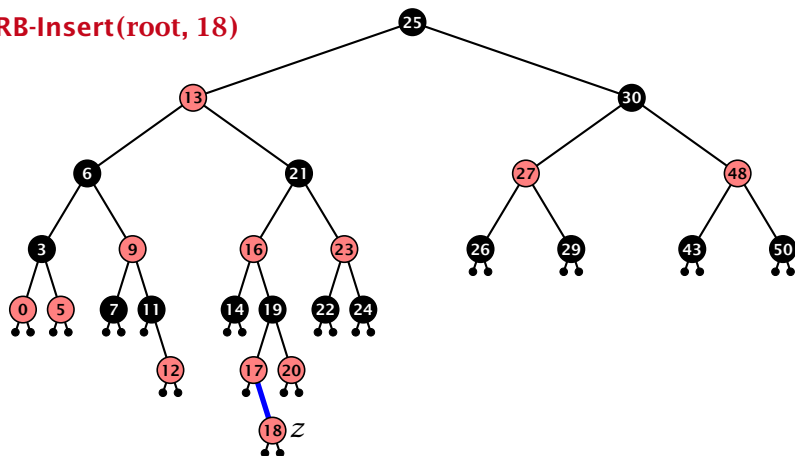


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If z has a parent but no grand-parent we could simply color the parent/root black; however this case never happens.

Red Black Trees: Insert

Algorithm 10 InsertFix(z)

```
1: while parent[ $z$ ]  $\neq$  null and col[parent[ $z$ ]] = red do
2:   if parent[ $z$ ] = left[gp[ $z$ ]] then
3:      $uncle \leftarrow$  right[grandparent[ $z$ ]]
4:     if col[ $uncle$ ] = red then
5:       col[p[ $z$ ]]  $\leftarrow$  black; col[ $u$ ]  $\leftarrow$  black;
6:       col[gp[ $z$ ]]  $\leftarrow$  red;  $z \leftarrow$  grandparent[ $z$ ];
7:     else
8:       if  $z$  = right[parent[ $z$ ]] then
9:          $z \leftarrow$  p[ $z$ ]; LeftRotate( $z$ );
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3:      $uncle \leftarrow$  right[grandparent[ $z$ ]]
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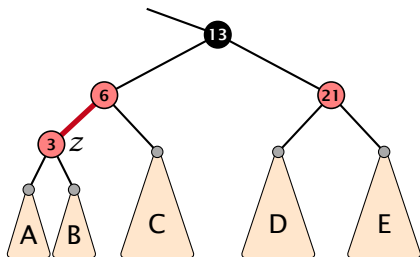
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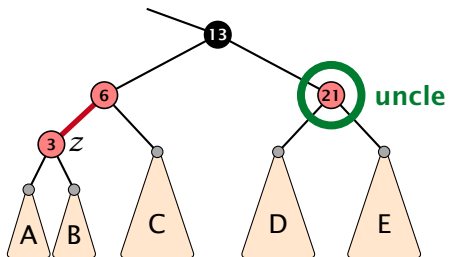
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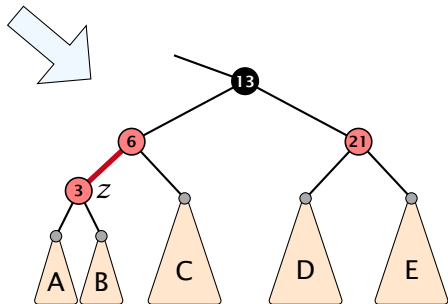
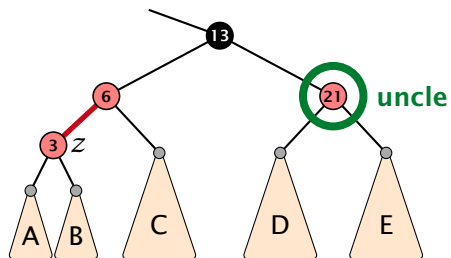
Case 1: Red Uncle



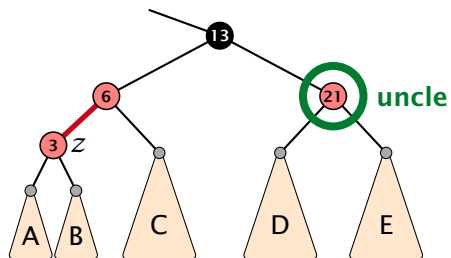
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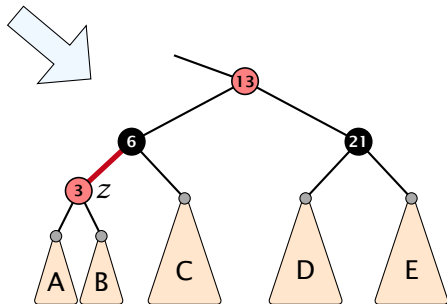
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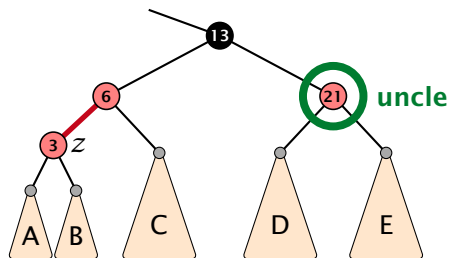
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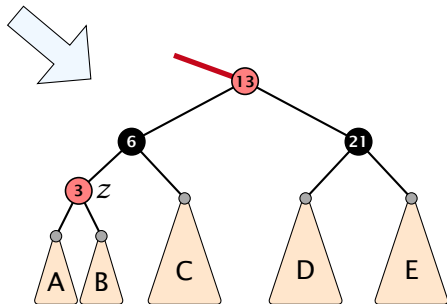
1. recolour



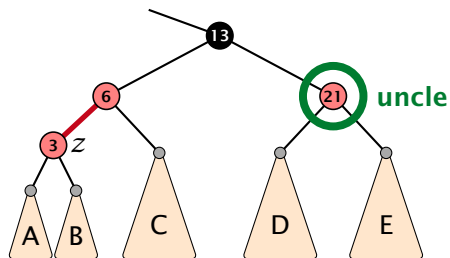
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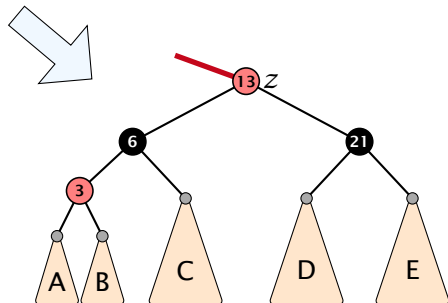
1. recolour



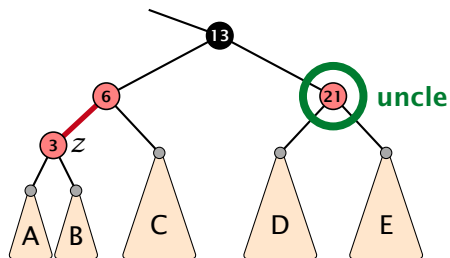
Case 1: Red Uncle



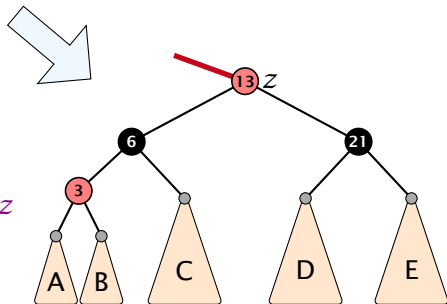
1. recolour
2. move z to grand-parent



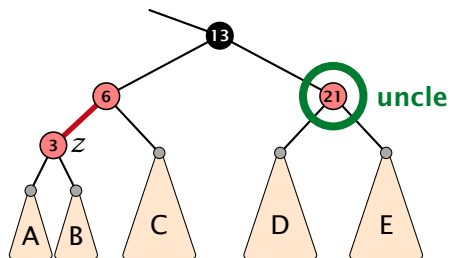
Case 1: Red Uncle



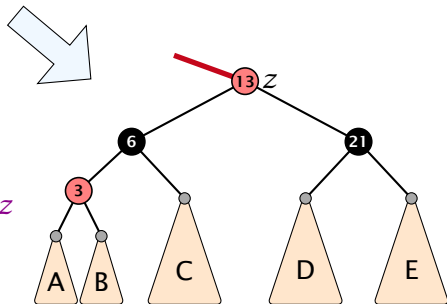
1. recolor
2. move z to grand-parent
3. invariant is fulfilled for new z



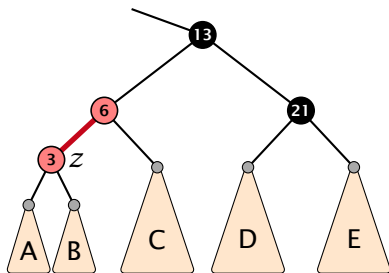
Case 1: Red Uncle



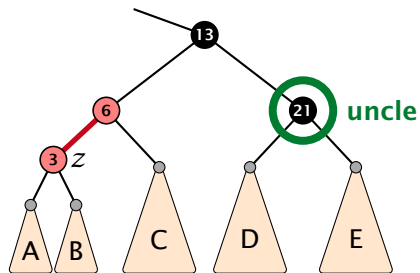
1. recolour
2. move z to grand-parent
3. invariant is fulfilled for new z
4. you made progress



Case 2b: Black uncle and z is left child

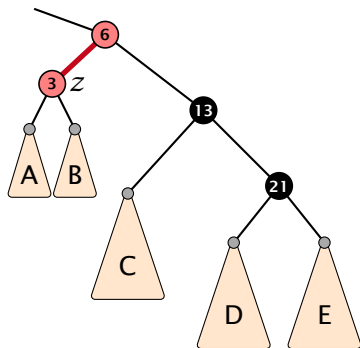
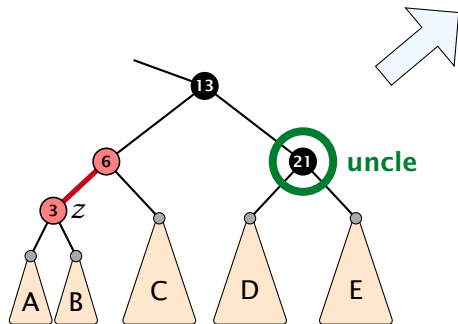


Case 2b: Black uncle and z is left child



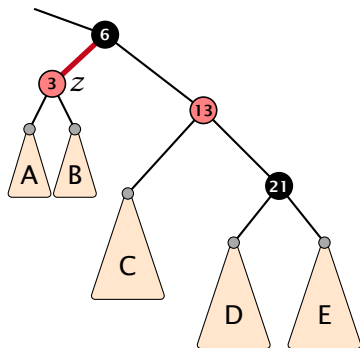
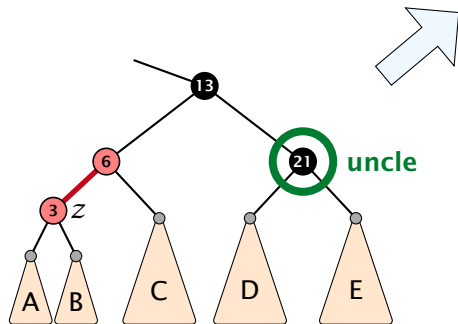
Case 2b: Black uncle and z is left child

1. rotate around grandparent



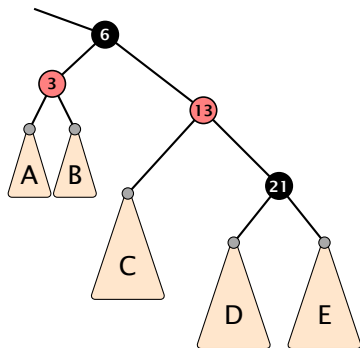
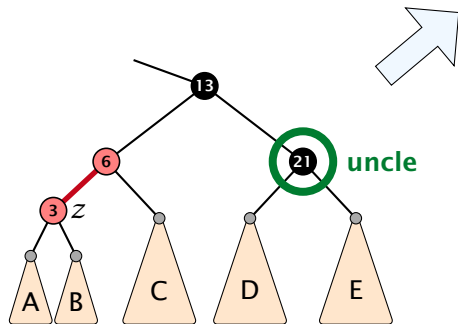
Case 2b: Black uncle and z is left child

1. rotate around grandparent
2. re-colour to ensure that black height property holds

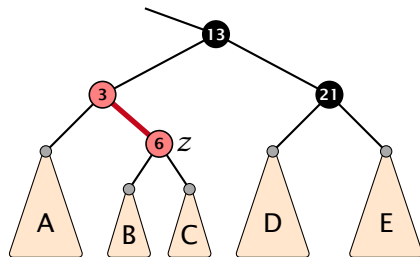


Case 2b: Black uncle and z is left child

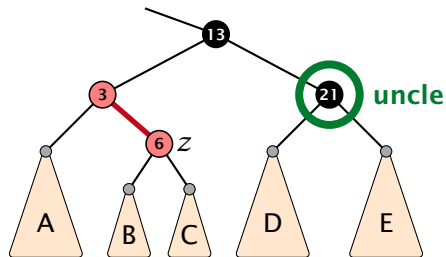
1. rotate around grandparent
2. re-colour to ensure that black height property holds
3. you have a red black tree



Case 2a: Black uncle and z is right child

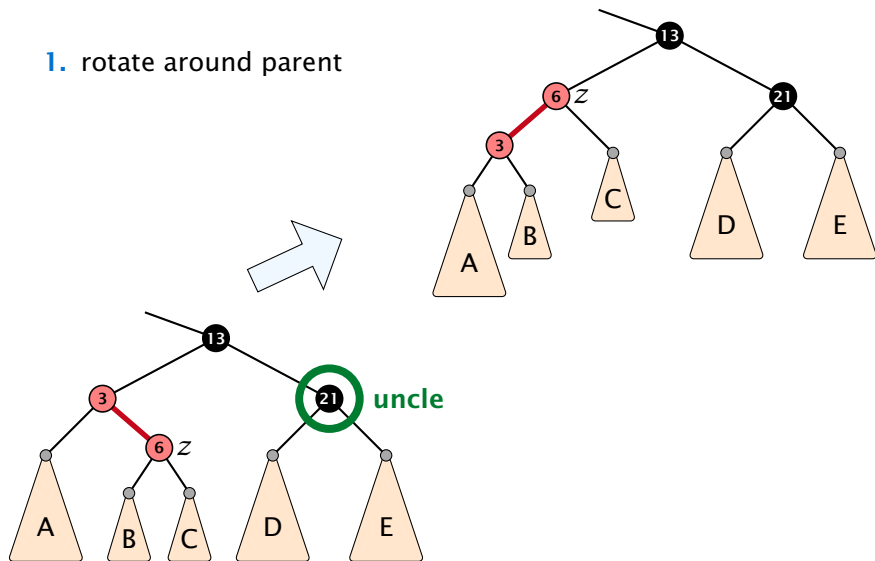


Case 2a: Black uncle and z is right child



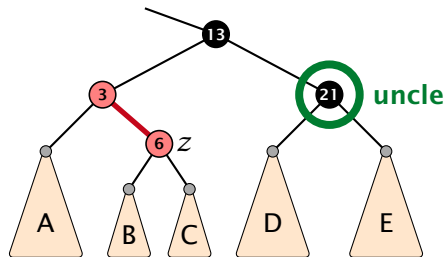
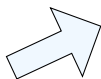
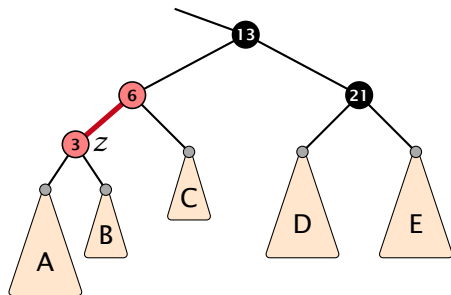
Case 2a: Black uncle and z is right child

1. rotate around parent



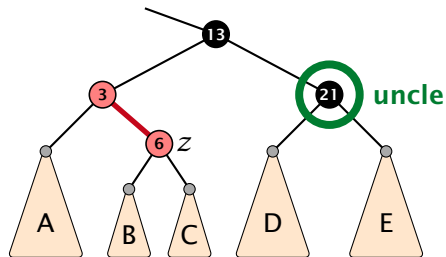
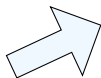
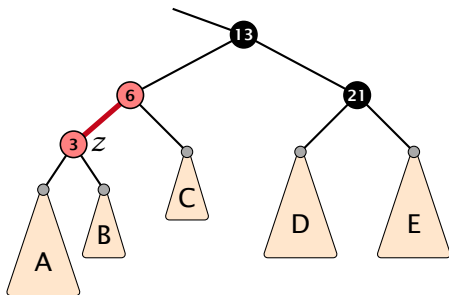
Case 2a: Black uncle and z is right child

1. rotate around parent
2. move z downwards



Case 2a: Black uncle and z is right child

1. rotate around parent
2. move z downwards
3. you have Case 2b.



Red Black Trees: Insert

Running time:

- ▶ Only Case 1 may repeat; but only $h/2$ many steps, where h is the height of the tree.

Red Black Trees: Insert

Running time:

- ▶ Only Case 1 may repeat; but only $h/2$ many steps, where h is the height of the tree.
- ▶ Case 2a \rightarrow Case 2b \rightarrow red-black tree

Red Black Trees: Insert

Running time:

- ▶ Only Case 1 may repeat; but only $h/2$ many steps, where h is the height of the tree.
- ▶ Case 2a → Case 2b → red-black tree
- ▶ Case 2b → red-black tree

Red Black Trees: Insert

Running time:

- ▶ Only Case 1 may repeat; but only $h/2$ many steps, where h is the height of the tree.
- ▶ Case 2a \rightarrow Case 2b \rightarrow red-black tree
- ▶ Case 2b \rightarrow red-black tree

Performing Case 1 at most $\mathcal{O}(\log n)$ times and every other case at most once, we get a red-black tree. Hence $\mathcal{O}(\log n)$ re-colorings and at most 2 rotations.

Red Black Trees: Delete

Red Black Trees: Delete

First do a standard delete.

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If the spliced out node x was red everything is fine.

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If it was black there may be the following problems.

Red Black Trees: Delete

First do a standard delete.

If the spliced out node x was red everything is fine.

If it was black there may be the following problems.

- ▶ Parent and child of x were red; two adjacent red vertices.

Red Black Trees: Delete

First do a standard delete.

If the spliced out node x was red everything is fine.

If it was black there may be the following problems.

- ▶ Parent and child of x were red; two adjacent red vertices.
- ▶ If you delete the root, the root may now be red.

Red Black Trees: Delete

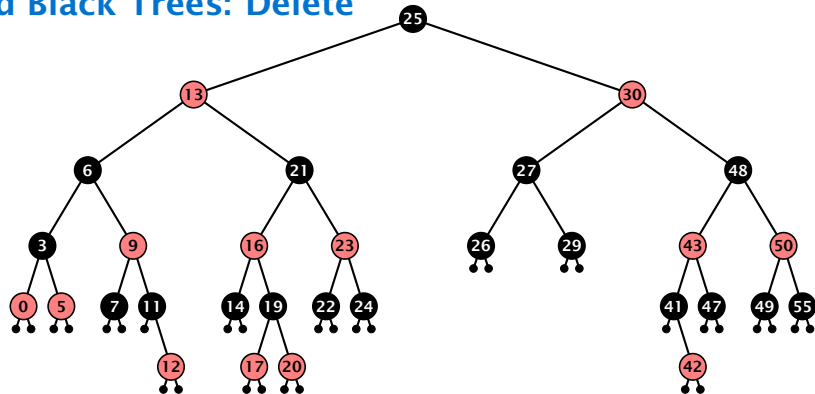
First do a standard delete.

If the spliced out node x was red everything is fine.

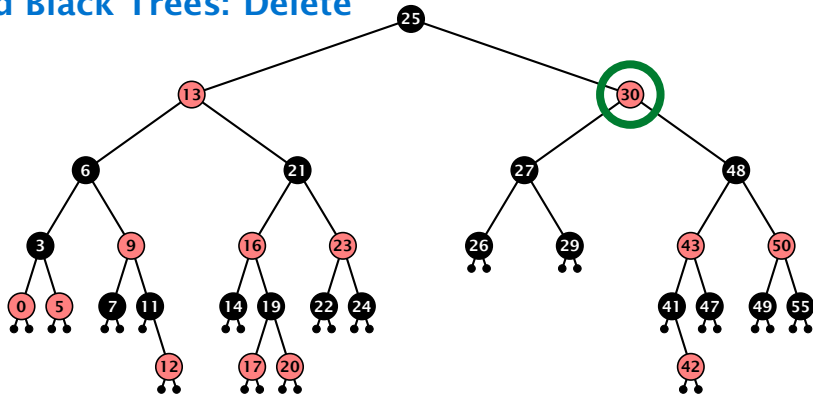
If it was black there may be the following problems.

- ▶ Parent and child of x were red; two adjacent red vertices.
- ▶ If you delete the root, the root may now be red.
- ▶ Every path from an ancestor of x to a descendant leaf of x changes the number of black nodes. Black height property might be violated.

Red Black Trees: Delete



Red Black Trees: Delete

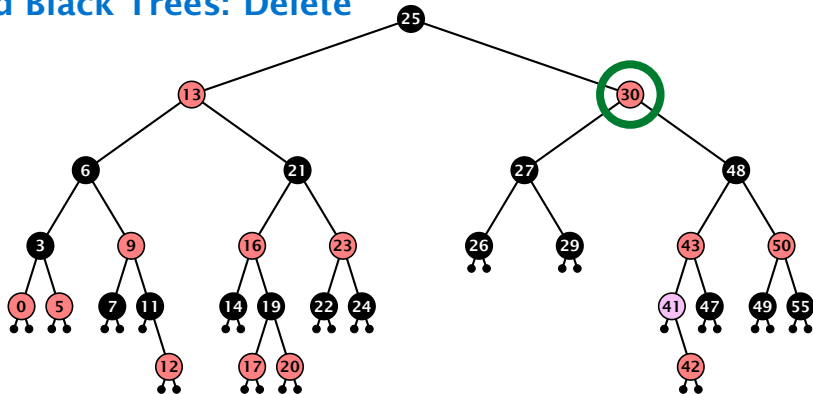


Case 3:

Element has two children

- ▶ do normal delete
- ▶ when replacing content by content of successor, don't change color of node

Red Black Trees: Delete

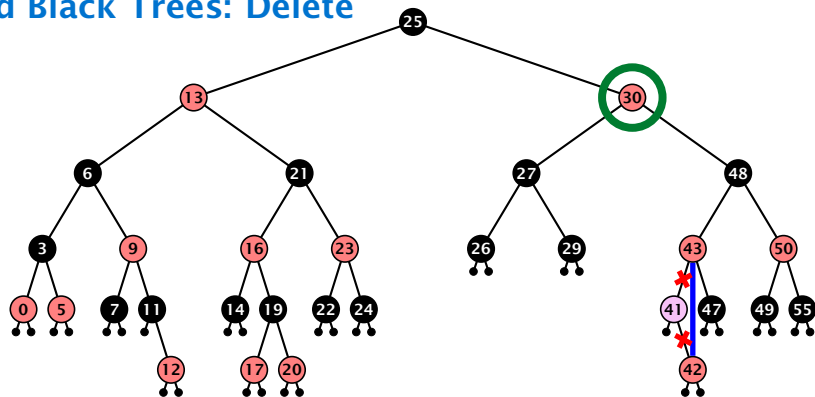


Case 3:

Element has two children

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Red Black Trees: Delete

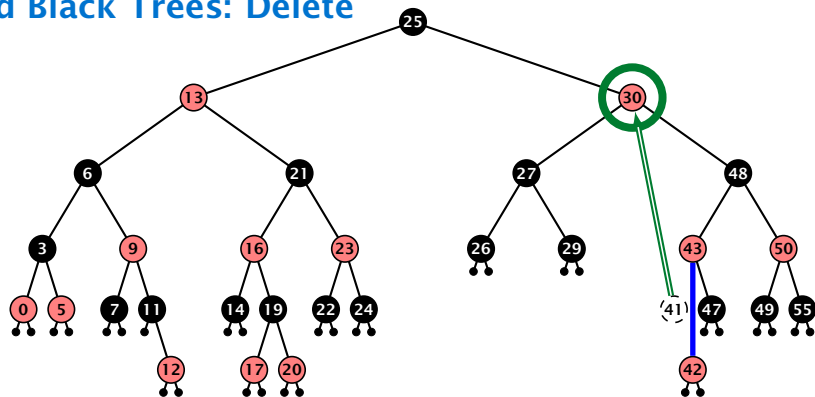


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Red Black Trees: Delete

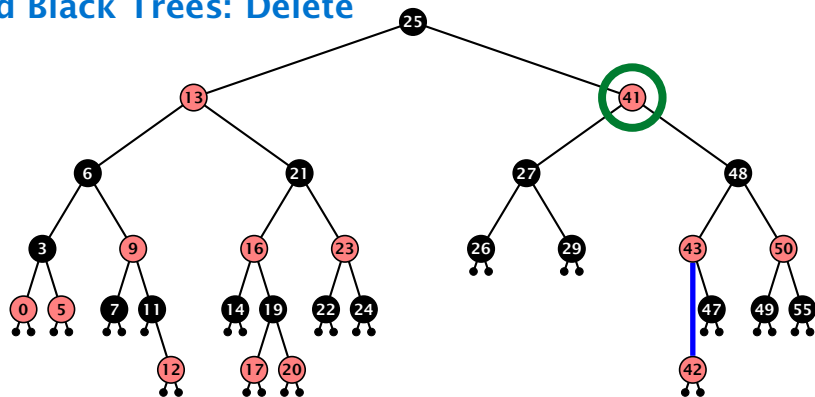


Case 3:

Element has two children

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Red Black Trees: Delete

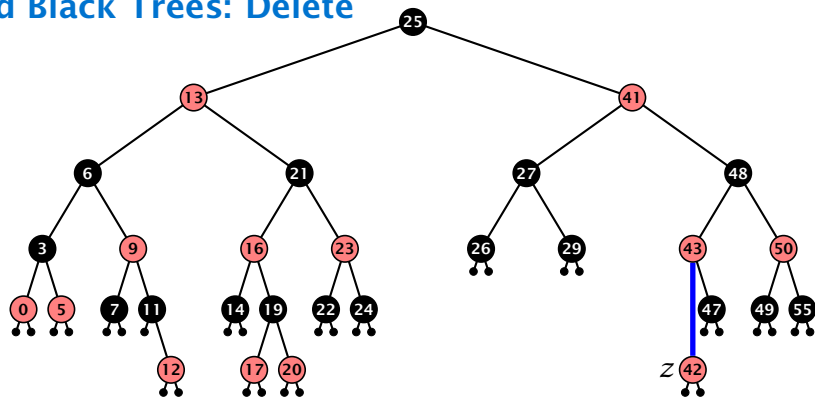


Case 3:

Element has two children

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- ▶ when replacing content by content of successor, don't change color of node

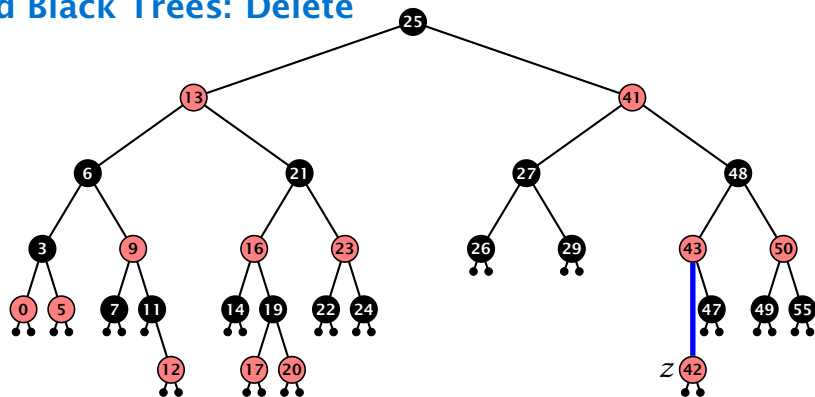
Red Black Trees: Delete



Delete:

- ▶ deleting black node messes up black-height property

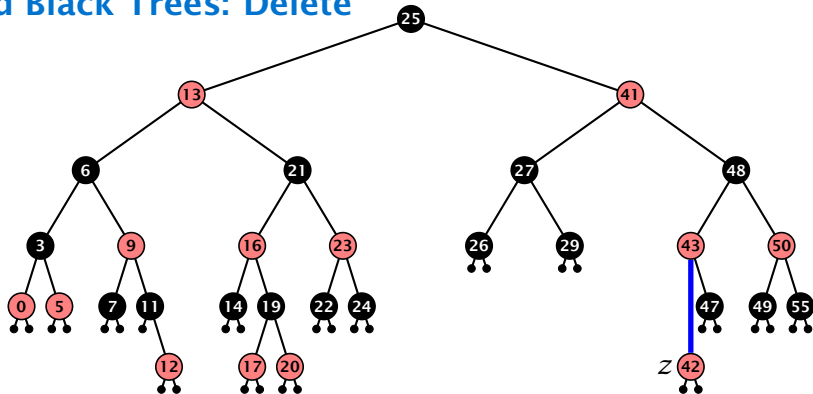
Red Black Trees: Delete



Delete:

- ▶ deleting black node messes up black-height property
- ▶ if z is red, we can simply color it black and everything is fine

Red Black Trees: Delete



Delete:

- ▶ deleting black node messes up black-height property
- ▶ if z is red, we can simply color it black and everything is fine
- ▶ the problem is if z is black (e.g. a dummy-leaf); we call a fix-up procedure to fix the problem.

Red Black Trees: Delete

Invariant of the fix-up algorithm

- ▶ the node z is black

Red Black Trees: Delete

Invariant of the fix-up algorithm

- ▶ the node z is black
- ▶ if we “assign” a fake black unit to the edge from z to its parent then the black-height property is fulfilled

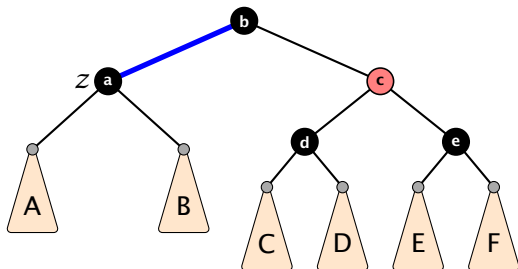
Red Black Trees: Delete

Invariant of the fix-up algorithm

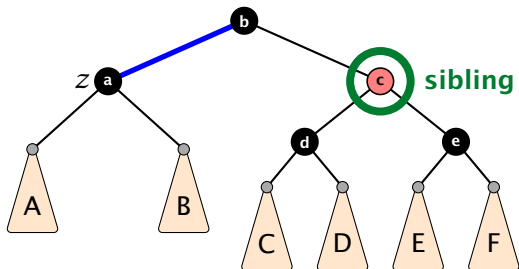
- ▶ the node z is black
- ▶ if we “assign” a fake black unit to the edge from z to its parent then the black-height property is fulfilled

Goal: make rotations in such a way that you at some point can remove the fake black unit from the edge.

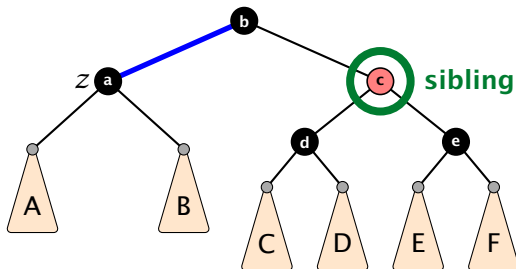
Case 1: Sibling of z is red



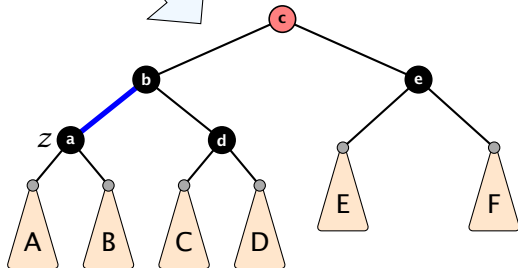
Case 1: Sibling of z is red



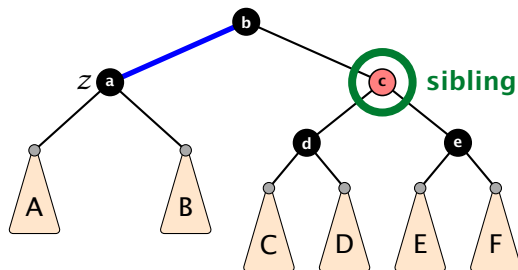
Case 1: Sibling of z is red



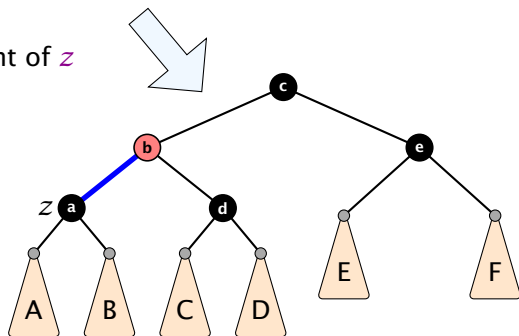
1. left-rotate around parent of z



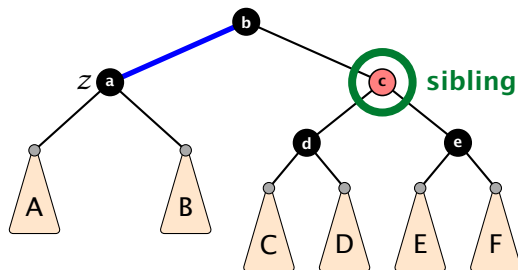
Case 1: Sibling of z is red



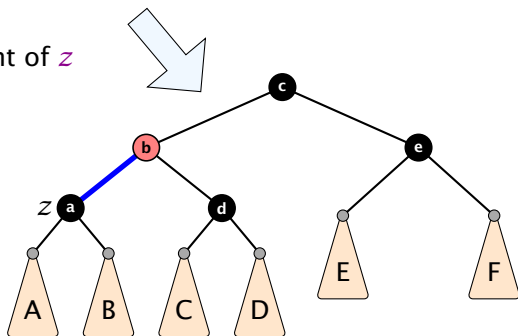
1. left-rotate around parent of z
2. recolor nodes b and c



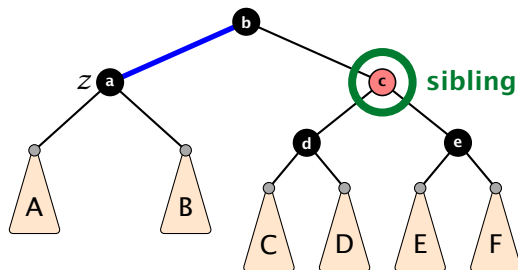
Case 1: Sibling of z is red



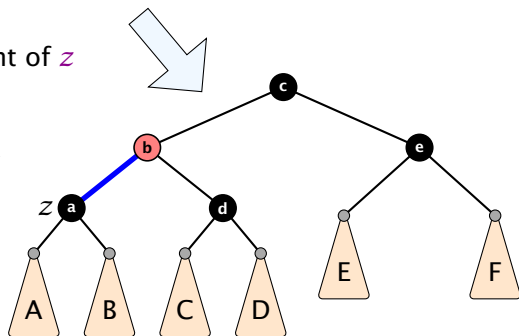
1. left-rotate around parent of z
2. recolor nodes b and c
3. the new sibling is black (and parent of z is red)



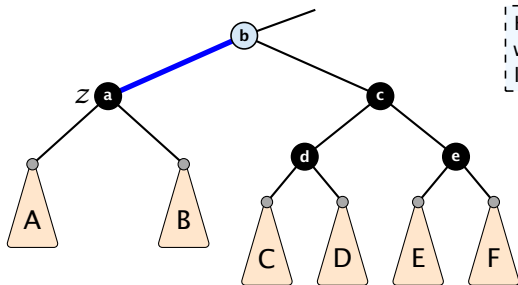
Case 1: Sibling of z is red



1. left-rotate around parent of z
2. recolor nodes b and c
3. the new sibling is black (and parent of z is red)
4. Case 2 (special), or Case 3, or Case 4

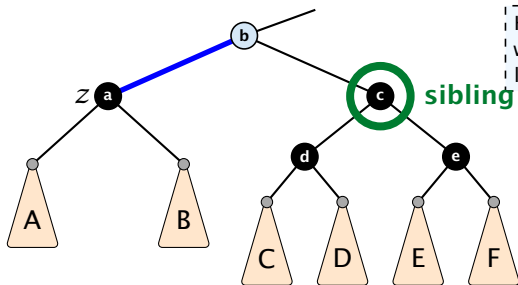


Case 2: Sibling is black with two black children



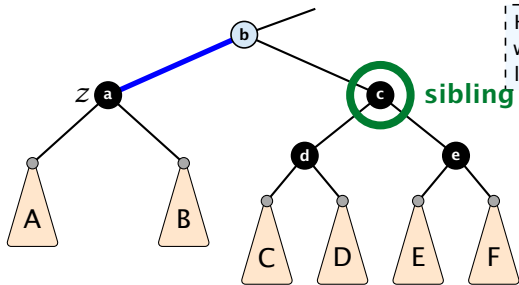
Here b is either black or red. If it is red we are in a special case that directly leads to a red-black tree.

Case 2: Sibling is black with two black children

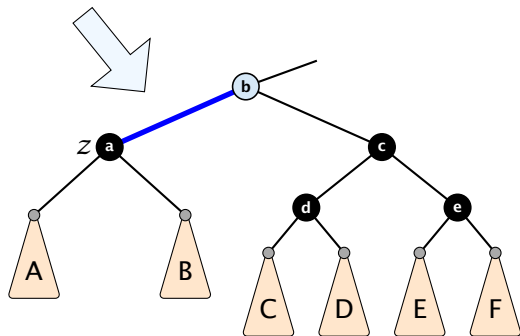


Here b is either black or red. If it is red we are in a special case that directly leads to a red-black tree.

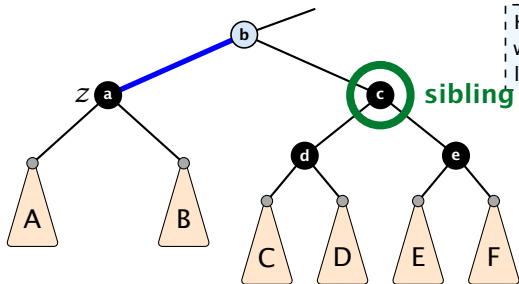
Case 2: Sibling is black with two black children



Here **b** is either black or red. If it is red we are in a special case that directly leads to a red-black tree.

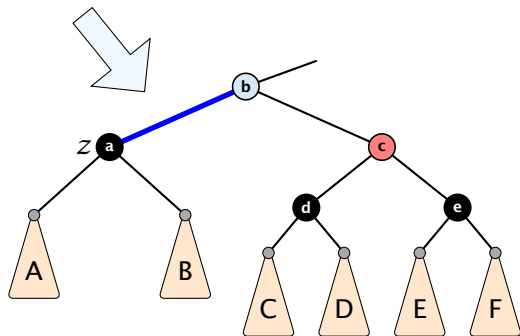


Case 2: Sibling is black with two black children

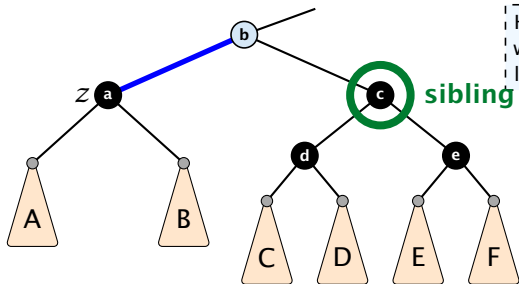


Here **b** is either black or red. If it is red we are in a special case that directly leads to a red-black tree.

1. re-color node **c**

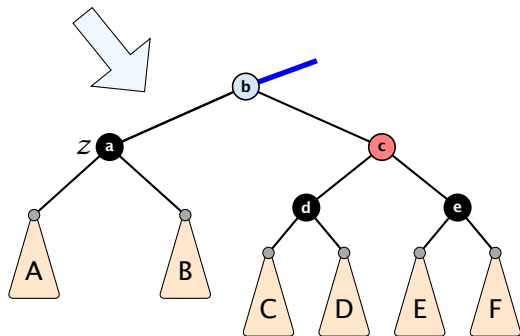


Case 2: Sibling is black with two black children

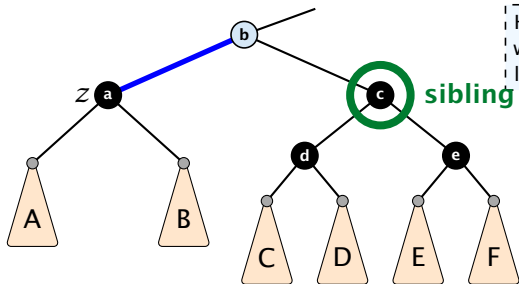


Here **b** is either black or red. If it is red we are in a special case that directly leads to a red-black tree.

1. re-color node **c**
2. move fake black unit upwards

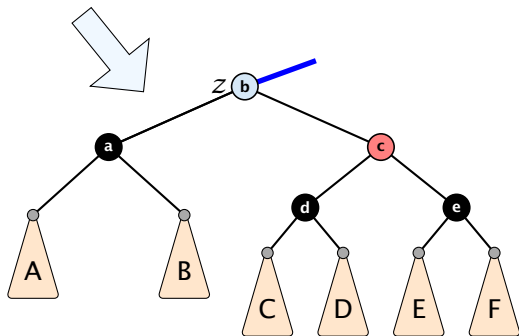


Case 2: Sibling is black with two black children

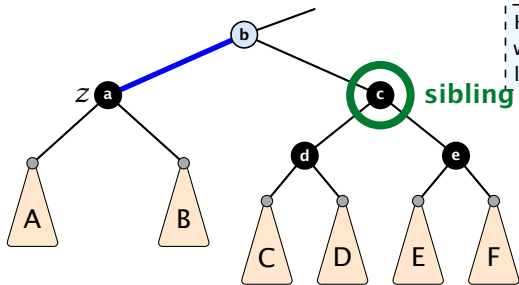


Here **b** is either black or red. If it is red we are in a special case that directly leads to a red-black tree.

1. re-color node **c**
2. move fake black unit upwards
3. move **z** upwards

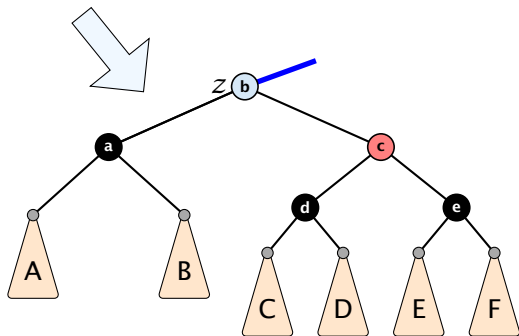


Case 2: Sibling is black with two black children

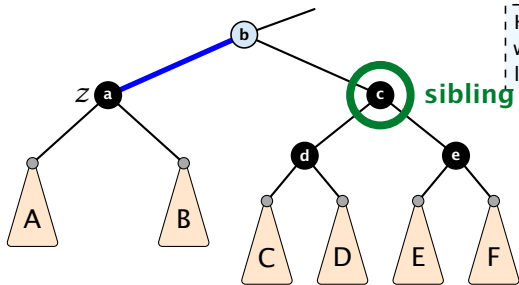


Here **b** is either black or red. If it is red we are in a special case that directly leads to a red-black tree.

1. re-color node **c**
2. move fake black unit upwards
3. move **z** upwards
4. we made progress

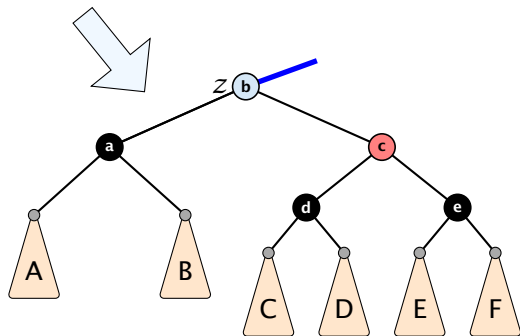


Case 2: Sibling is black with two black children

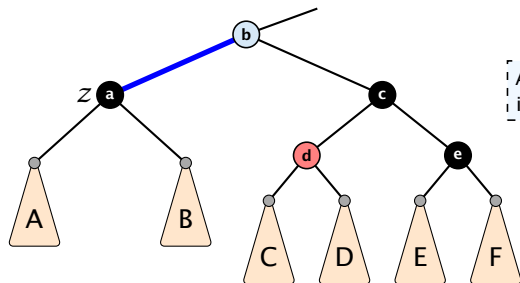


Here b is either black or red. If it is red we are in a special case that directly leads to a red-black tree.

1. re-color node c
2. move fake black unit upwards
3. move z upwards
4. we made progress
5. if b is red we color it black and are done

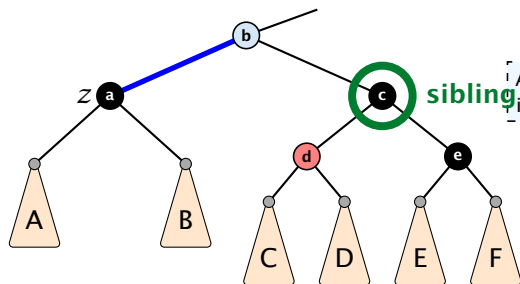


Case 3: Sibling black with one black child to the right



Again the blue color of b indicates that it can either be black or red.

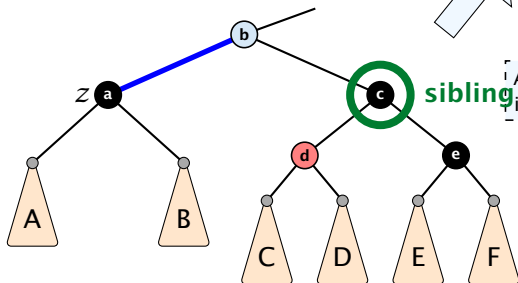
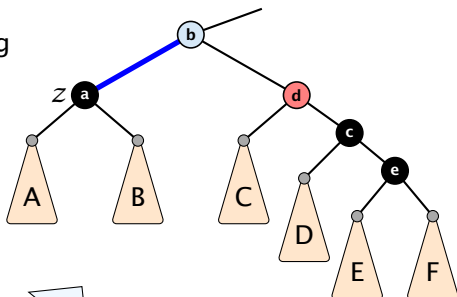
Case 3: Sibling black with one black child to the right



Again the blue color of b indicates that it can either be black or red.

Case 3: Sibling black with one black child to the right

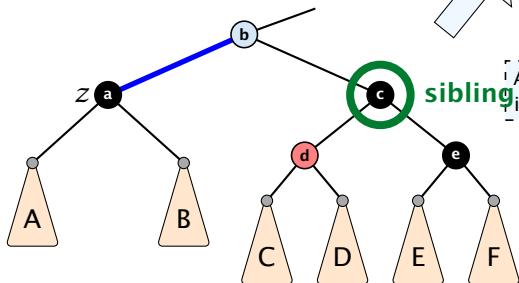
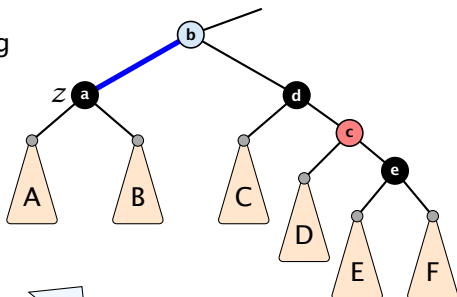
1. do a right-rotation at sibling



Again the blue color of b indicates that it can either be black or red.

Case 3: Sibling black with one black child to the right

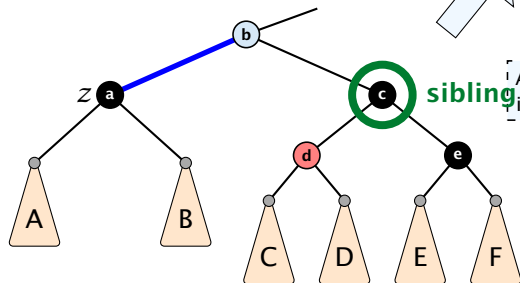
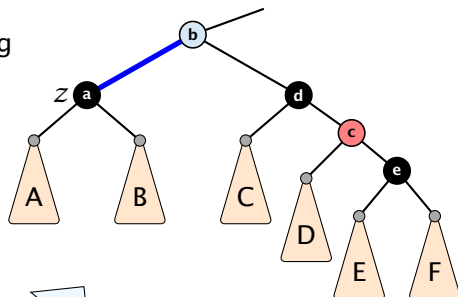
1. do a right-rotation at sibling
2. recolor c and d



Again the blue color of b indicates that it can either be black or red.

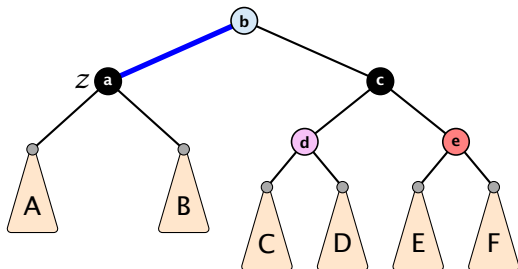
Case 3: Sibling black with one black child to the right

1. do a right-rotation at sibling
2. recolor c and d
3. new sibling is black with red right child (Case 4)



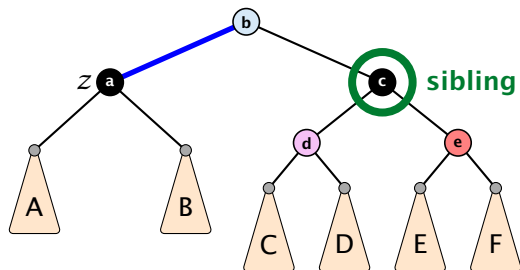
Again the blue color of b indicates that it can either be black or red.

Case 4: Sibling is black with red right child



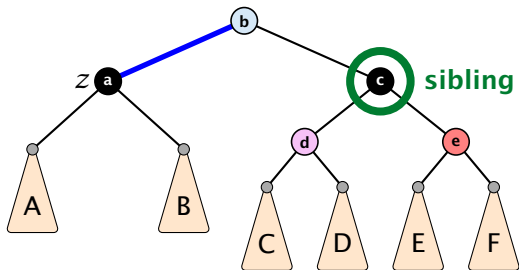
- Here b and d are either red or black but have possibly different colors.
- We recolor c by giving it the color of b.

Case 4: Sibling is black with red right child



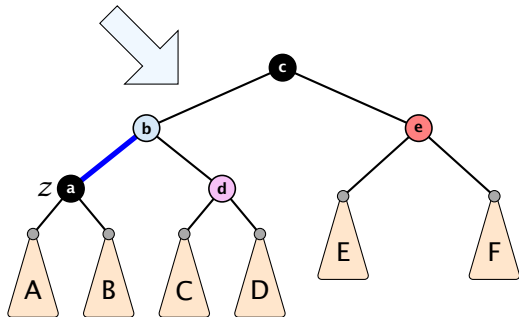
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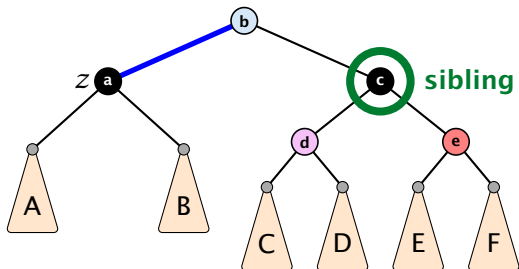


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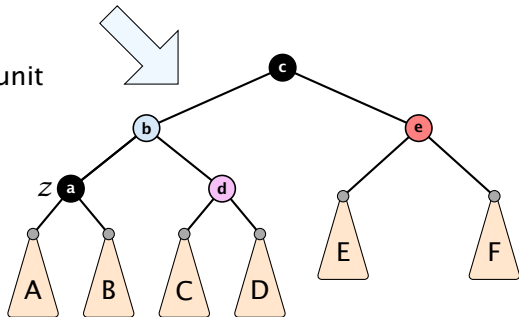


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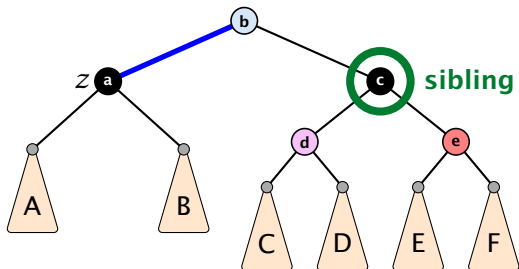


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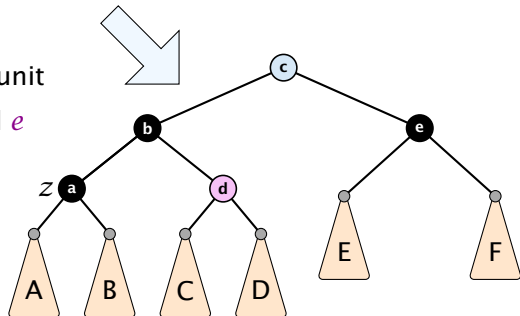


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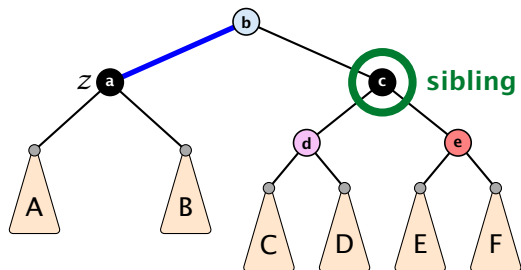


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3. recolor nodes **b**, **c**, and **e**

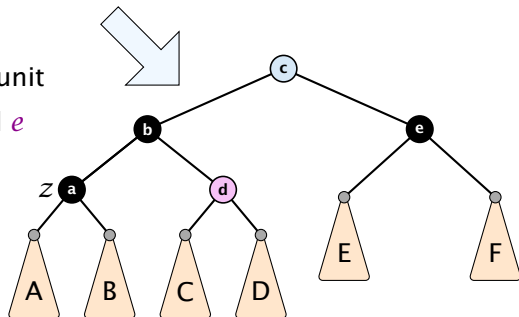


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- We recolor **c** by giving it the color of **b**.

1. left-rotate around **b**
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3. recolor nodes **b**, **c**, and **e**
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- ▶ only Case 2 can repeat; but only h many steps, where h is the height of the tree

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Performing Case 2 at most $\mathcal{O}(\log n)$ times and every other step at most once, we get a red black tree. Hence, $\mathcal{O}(\log n)$ re-colorings and at most 3 rotations.

Disadvantage of balanced search trees:

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Splay Trees

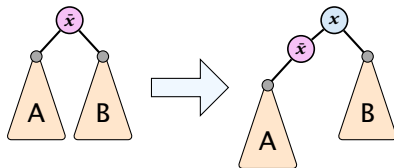
find(x)

- ▶ search for x according to a search tree
- ▶ let \tilde{x} be last element on search-path
- ▶ $\text{splay}(\tilde{x})$

Splay Trees

insert(x)

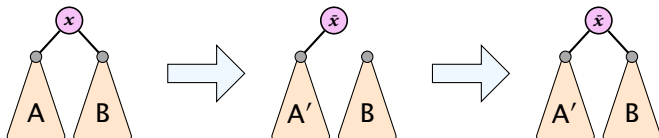
- ▶ search for x ; \bar{x} is last visited element during search (successor or predecessor of x)
- ▶ splay(\bar{x}) moves \bar{x} to the root
- ▶ insert x as new root



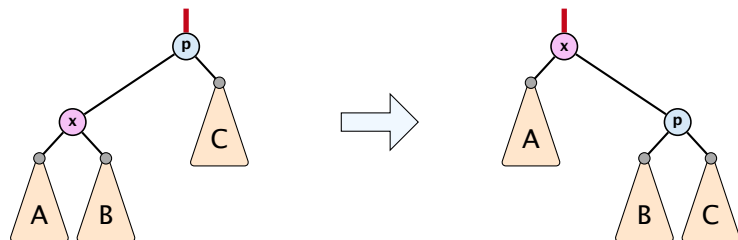
Splay Trees

delete(x)

- ▶ search for x ; splay(x); remove x
- ▶ search largest element \bar{x} in A
- ▶ splay(\bar{x}) (on subtree A)
- ▶ connect root of B as right child of \bar{x}



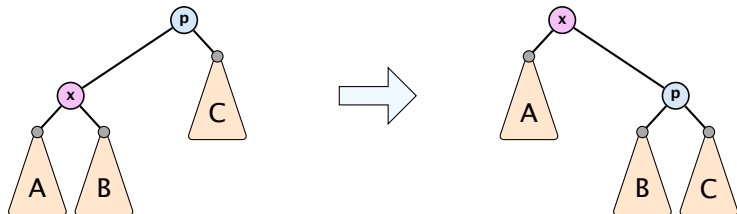
Move to Root



How to bring element to root?

- ▶ one (bad) option: `moveToRoot(x)`
- ▶ iteratively do rotation around parent of x until x is root
- ▶ if x is left child do right rotation otw. left rotation

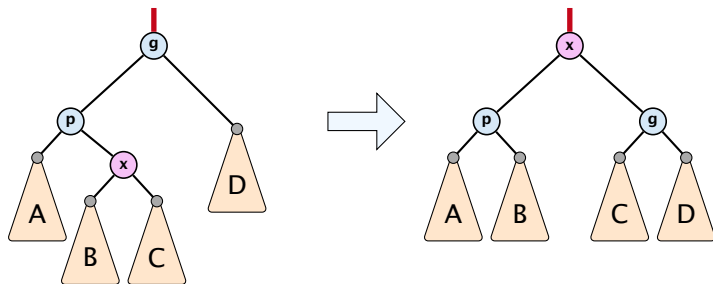
Splay: Zig Case



better option splay(x):

- ▶ zig case: if x is child of root do left rotation or right rotation around parent

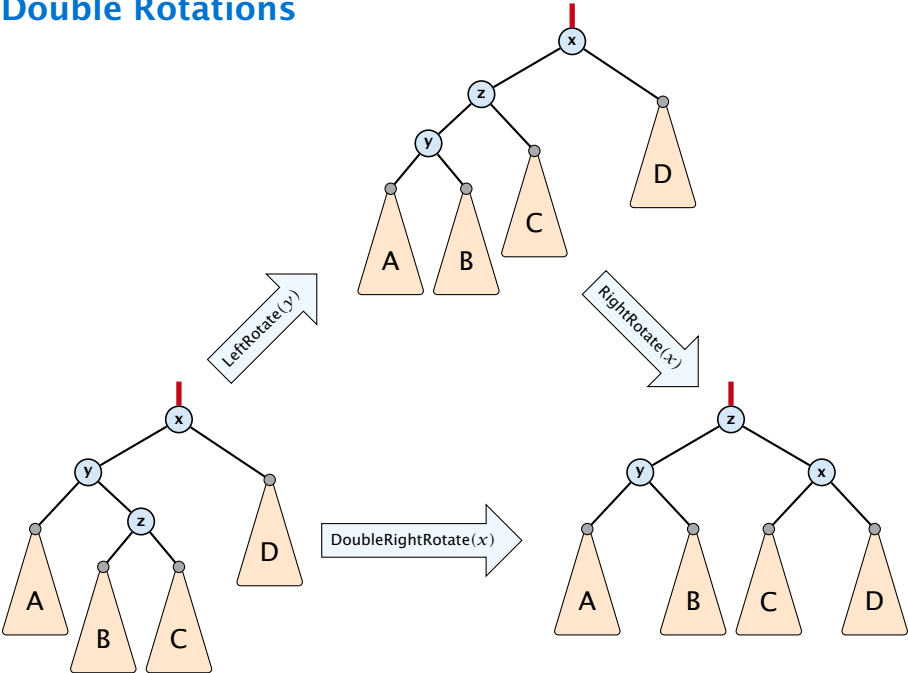
Splay: Zigzag Case



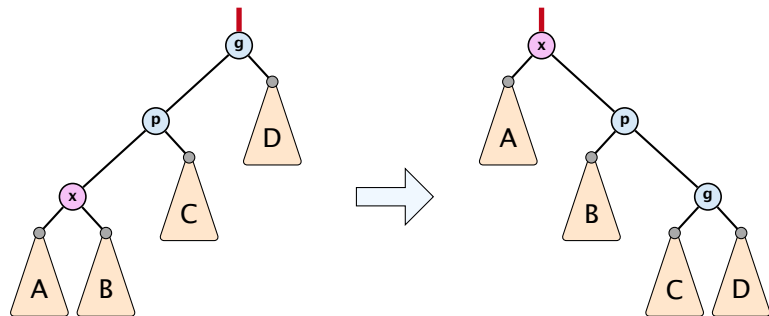
better option $\text{splay}(x)$:

- ▶ zigzag case: if x is right child and parent of x is left child (or x left child parent of x right child)
- ▶ do double right rotation around grand-parent (resp. double left rotation)

Double Rotations



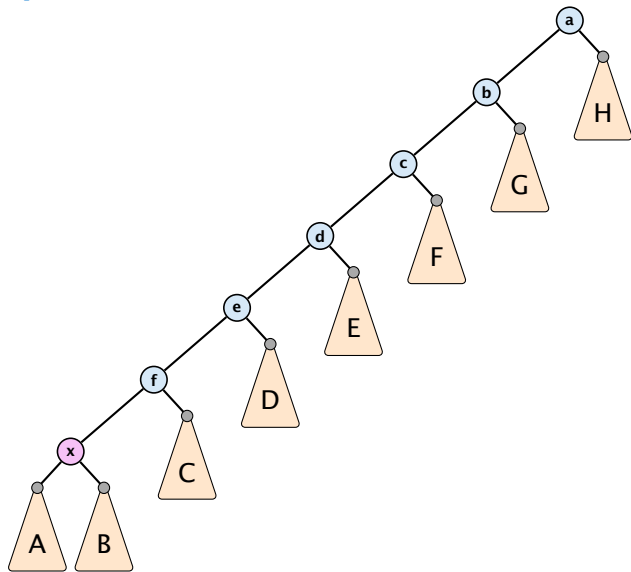
Splay: Zigzig Case



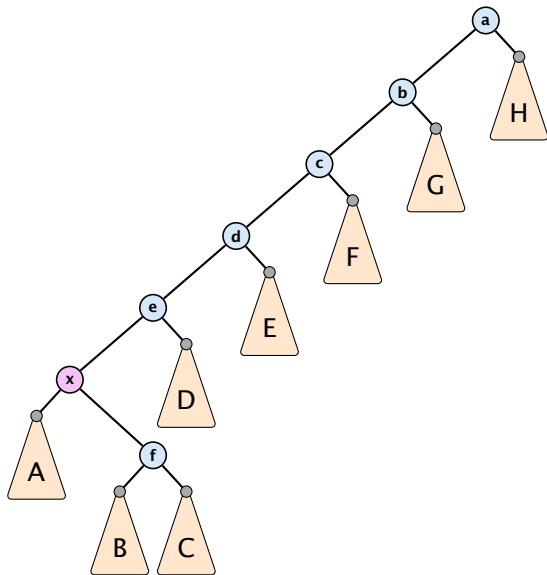
better option $\text{splay}(x)$:

- ▶ zigzig case: if x is left child and parent of x is left child (or x right child, parent of x right child)
- ▶ do right rotation around grand-parent followed by right rotation around parent (resp. left rotations)

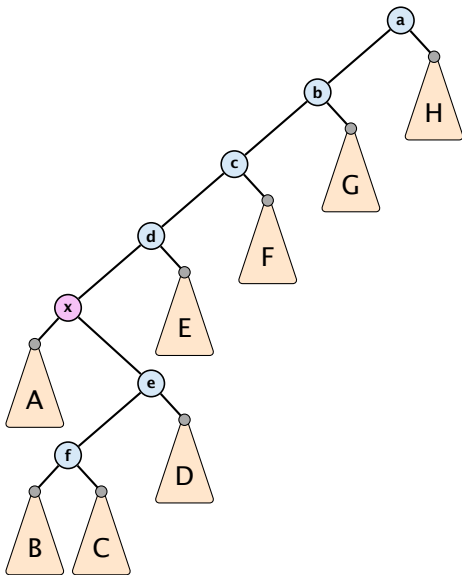
Splay vs. Move to Root



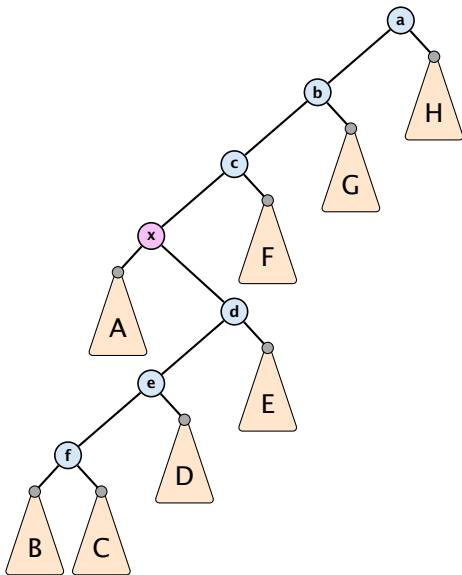
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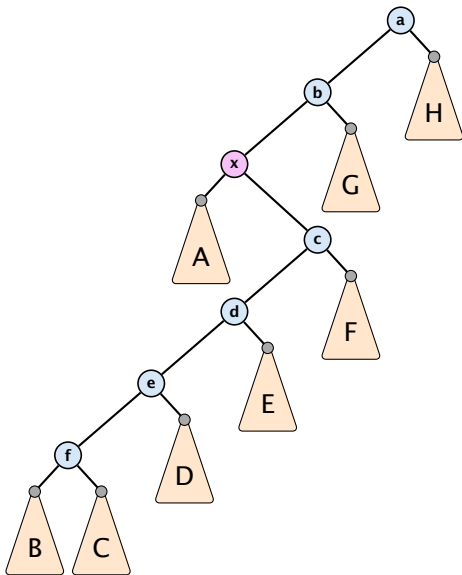
Splay vs. Move to Root



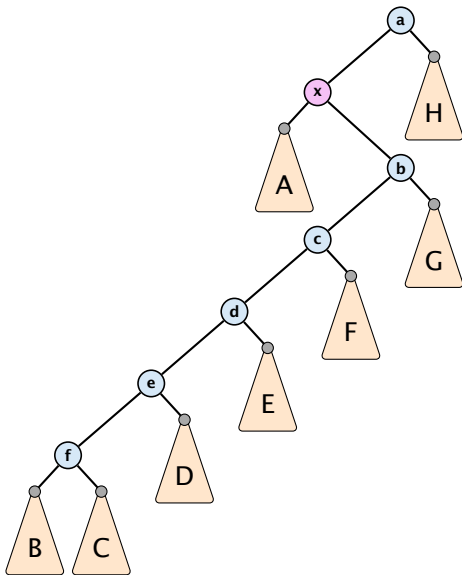
Splay vs. Move to Root



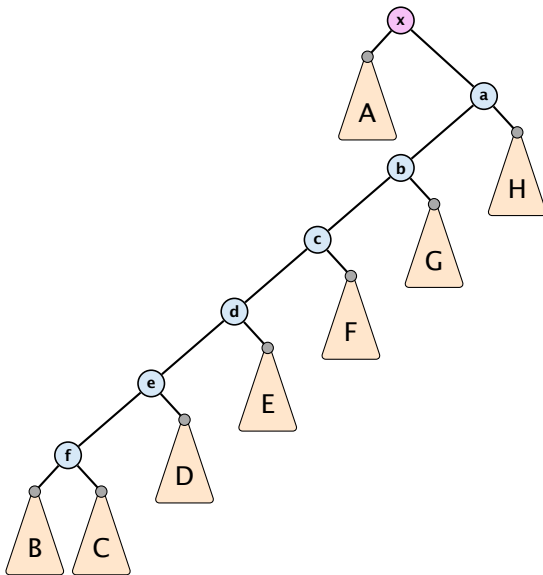
Splay vs. Move to Root



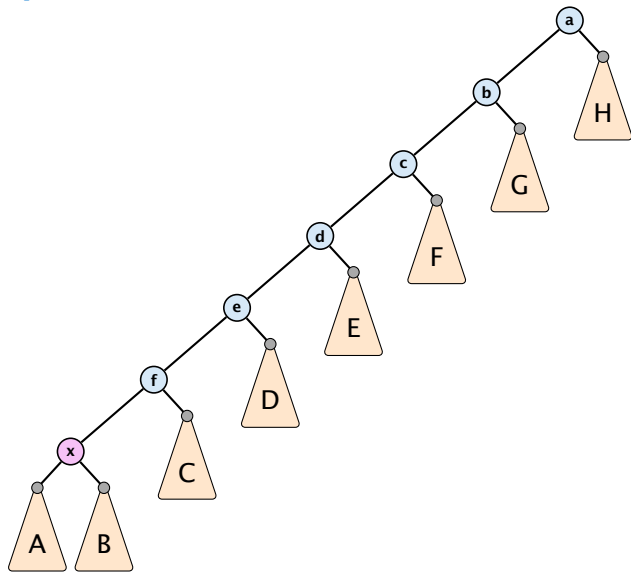
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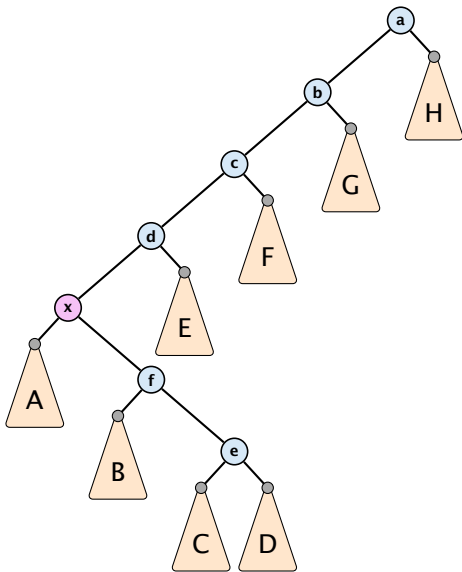
Splay vs. Move to Root



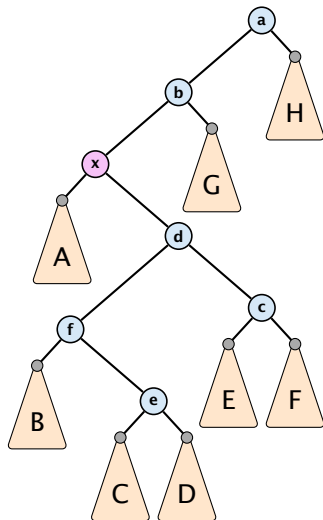
Splay vs. Move to Root



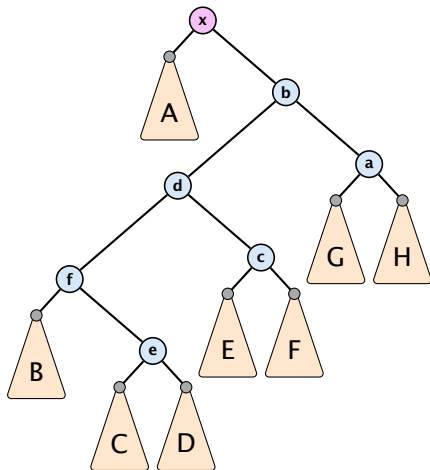
Splay vs. Move to Root



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Splay vs. Move to Root



Static Optimality

Suppose we have a sequence of m find-operations. $\text{find}(x)$ appears h_x times in this sequence.

The cost of a **static** search tree T is:

$$\text{cost}(T) = m + \sum_x h_x \text{depth}_T(x)$$

The total cost for processing the sequence on a splay-tree is $\mathcal{O}(\text{cost}(T_{\min}))$, where T_{\min} is an **optimal static search tree**.

Dynamic Optimality

Let S be a sequence with m find-operations.

Let A be a data-structure based on a search tree:

- ▶ the cost for accessing element x is $1 + \text{depth}(x)$;
- ▶ after accessing x the tree may be re-arranged through rotations;

Conjecture:

A splay tree that only contains elements from S has cost $\mathcal{O}(\text{cost}(A, S))$, for processing S .

Lemma 16

*Splay Trees have an **amortized** running time of $\mathcal{O}(\log n)$ for all operations.*

Amortized Analysis

Definition 17

A data structure with operations $\text{op}_1(), \dots, \text{op}_k()$ has amortized running times t_1, \dots, t_k for these operations if the following holds.

Suppose you are given a sequence of operations (**starting with an empty data-structure**) that operate on at most n elements, and let k_i denote the number of occurrences of $\text{op}_i()$ within this sequence. Then the actual running time must be at most $\sum_i k_i \cdot t_i(n)$.

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- ▶ Show that $\Phi(D_i) \geq \Phi(D_0)$.

Then

$$\sum_{i=1}^k c_i \leq \sum_{i=1}^k c_i + \Phi(D_k) - \Phi(D_0) = \sum_{i=1}^k \hat{c}_i$$

This means the amortized costs can be used to derive a bound on the total cost.

Example: Stack

Stack

- ▶ $S.$ push()
- ▶ $S.$ pop()
- ▶ $S.$ multipop(k): removes k items from the stack. If the stack currently contains less than k items it empties the stack.
- ▶ The user has to ensure that pop and multipop do not generate an underflow.

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- ▶ The user has to ensure that pop and multipop do not generate an underflow.

Actual cost:

- ▶ $S.$ push(): cost 1.
- ▶ $S.$ pop(): cost 1.
- ▶ $S.$ multipop(k): cost $\min\{\text{size}, k\} = k$.

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- ▶ **S . multipop(k):** cost

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Example: Binary Counter

Incrementing a binary counter:

Consider a computational model where each bit-operation costs one time-unit.

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Actual cost:

- ▶ Changing bit from 0 to 1: cost 1.
- ▶ Changing bit from 1 to 0: cost 1.
- ▶ Increment: cost is $k + 1$, where k is the number of consecutive ones in the least significant bit-positions (e.g., 001101 has $k = 1$).

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Amortized cost:

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- ▶ Changing bit from 1 to 0:

$$\hat{C}_{1 \rightarrow 0} = C_{1 \rightarrow 0} + \Delta\Phi = 1 - 1 \leq 0 .$$

- ▶ **Increment:** Let k denotes the number of consecutive ones in the least significant bit-positions. An increment involves k (1 \rightarrow 0)-operations, and one (0 \rightarrow 1)-operation.

Hence, the amortized cost is $k\hat{C}_{1 \rightarrow 0} + \hat{C}_{0 \rightarrow 1} \leq 2$.

Splay Trees

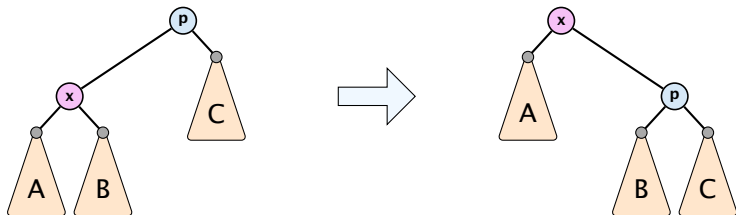
potential function for splay trees:

- ▶ size $s(x) = |T_x|$
- ▶ rank $r(x) = \log_2(s(x))$
- ▶ $\Phi(T) = \sum_{v \in T} r(v)$

amortized cost = real cost + potential change

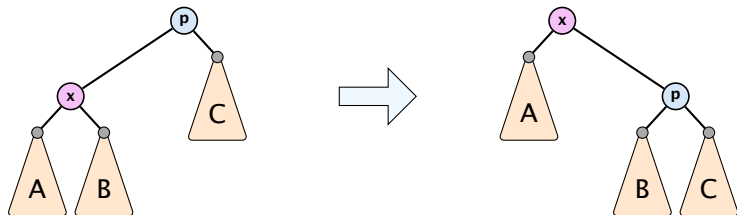
The cost is essentially the cost of the splay-operation, which is 1 plus the number of rotations.

Splay: Zig Case



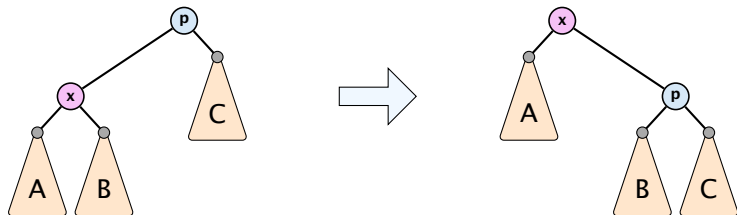
$$\Delta\Phi =$$

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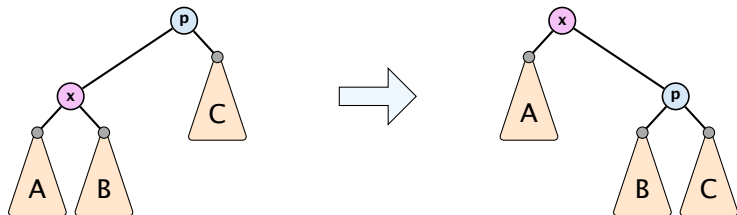
$$\Delta\Phi = r'(x) + r'(p) - r(x) - r(p)$$

Splay: Zig Case



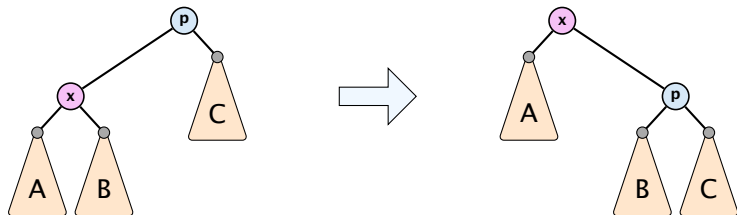
$$\begin{aligned}\Delta\Phi &= r'(x) + r'(p) - r(x) - r(p) \\ &= r'(p) - r(x)\end{aligned}$$

Splay: Zig Case



$$\begin{aligned}\Delta\Phi &= r'(x) + r'(p) - r(x) - r(p) \\ &= r'(p) - r(x) \\ &\leq r'(x) - r(x)\end{aligned}$$

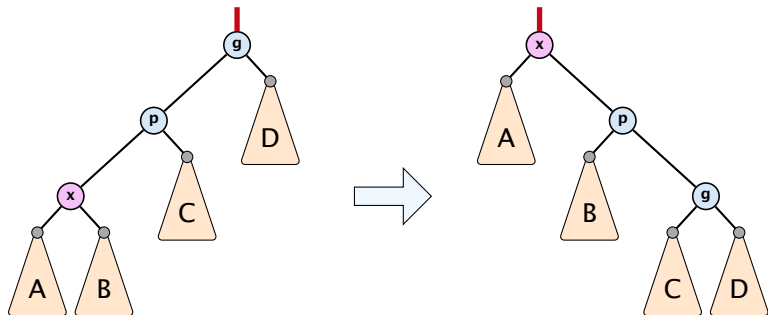
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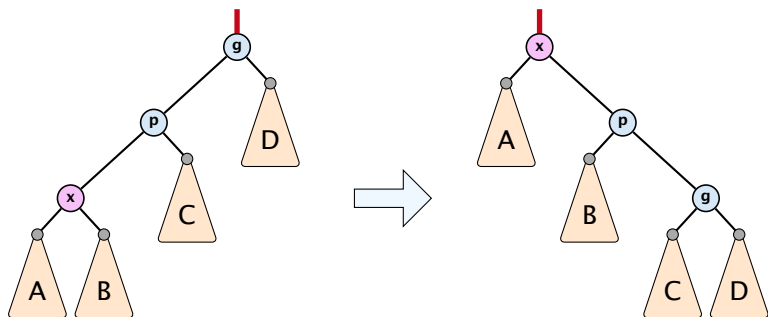
$$\text{cost}_{\text{zig}} \leq 1 + 3(r'(x) - r(x))$$

Splay: Zigzig Case



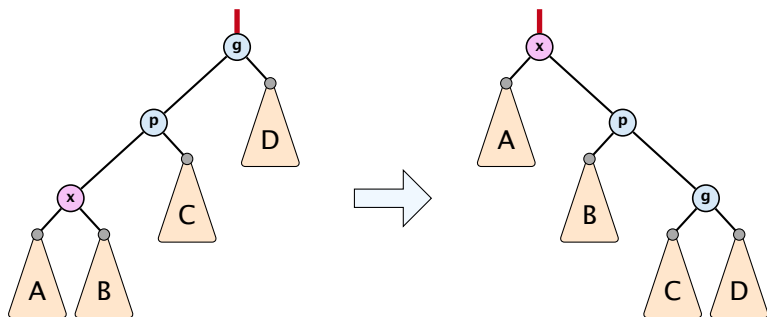
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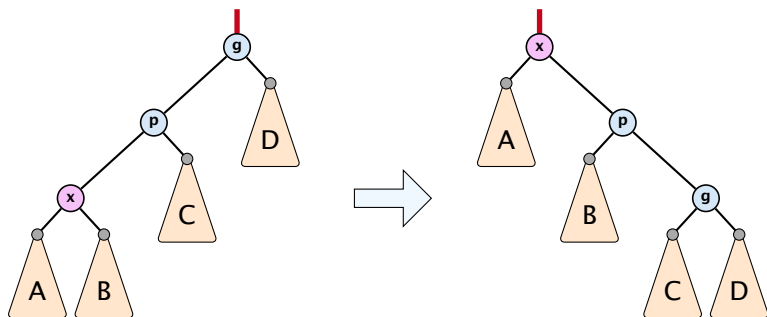
$$\Delta\Phi = r'(x) + r'(p) + r'(g) - r(x) - r(p) - r(g)$$

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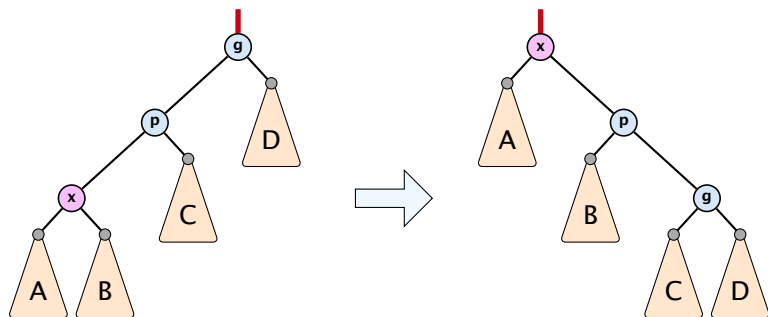
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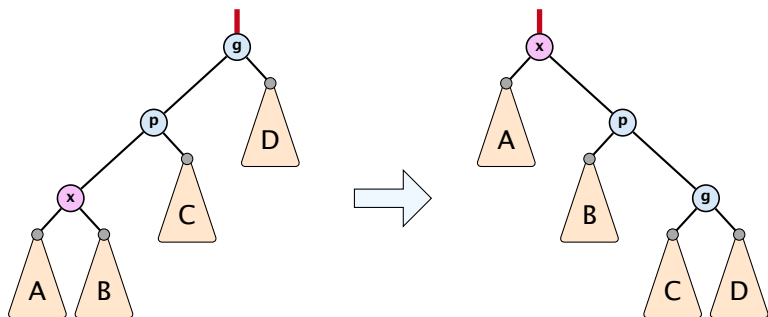
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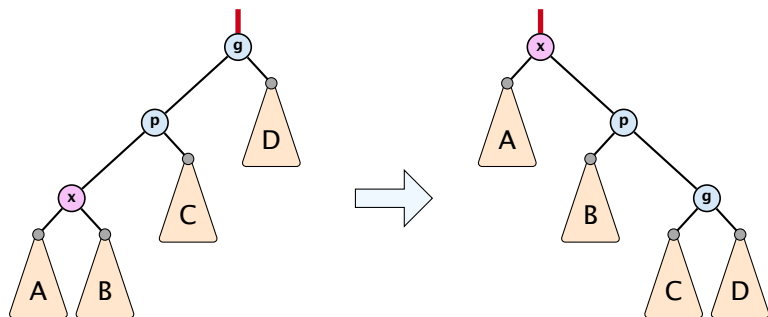
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Splay: Zigzig Case



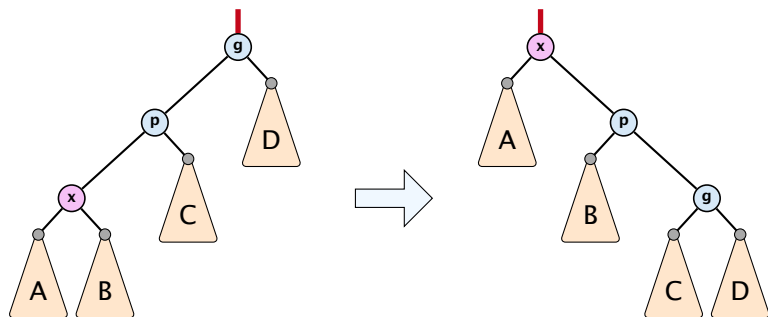
$$\begin{aligned}\Delta\Phi &= r'(x) + r'(p) + r'(g) - r(x) - r(p) - r(g) \\ &= r'(p) + r'(g) - r(x) - r(p) \\ &\leq r'(x) + r'(g) - r(x) - r(x) \\ &= r'(x) + r'(g) + r(x) - 3r'(x) + 3r'(x) - r(x) - 2r(x) \\ &= -2r'(x) + r'(g) + r(x) + 3(r'(x) - r(x))\end{aligned}$$

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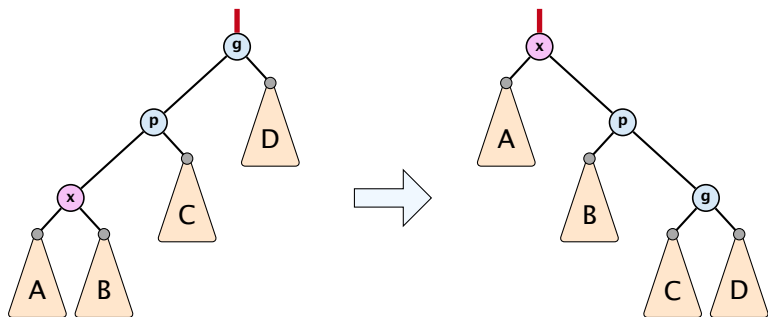
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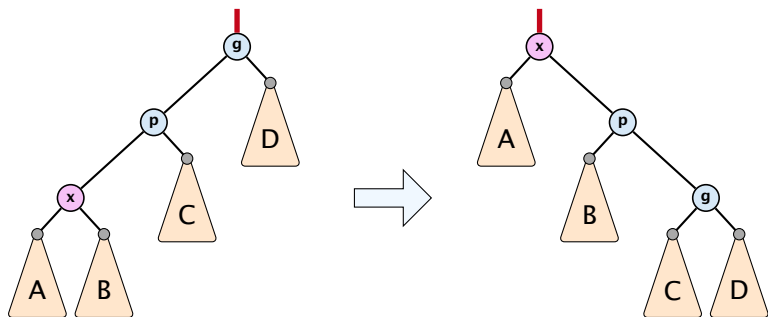
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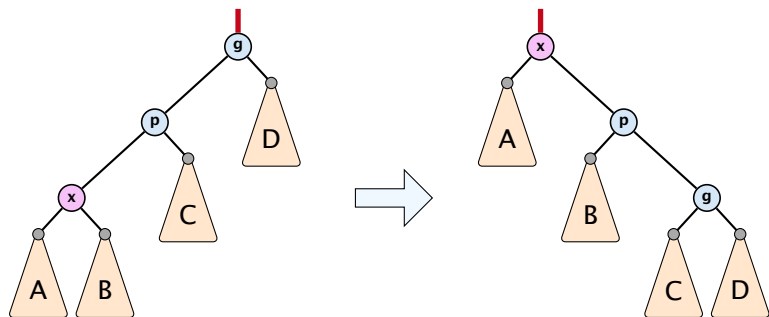
$$\frac{1}{2}(r(x) + r'(g) - 2r'(x))$$

Splay: Zigzig Case



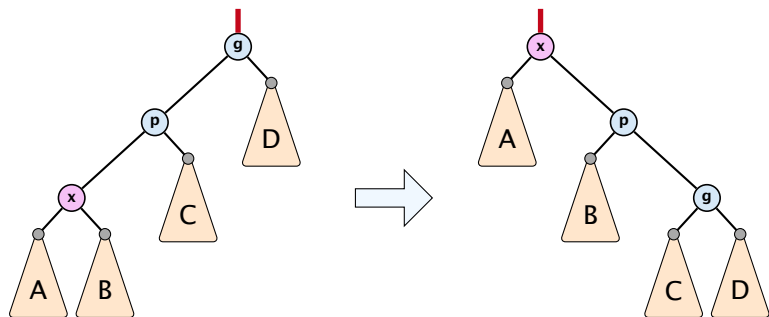
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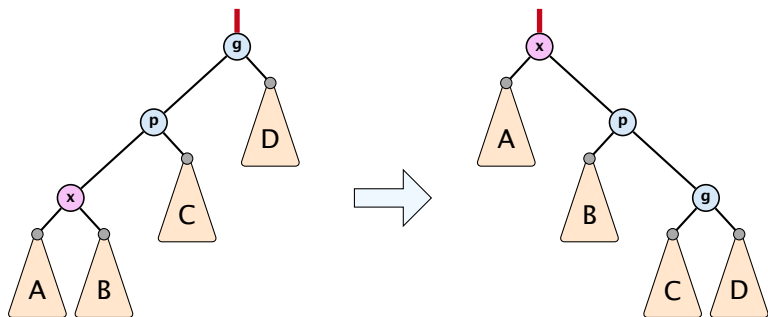
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Splay: Zigzig Case



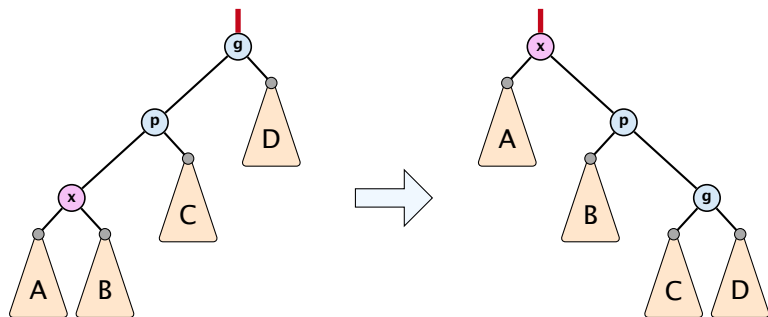
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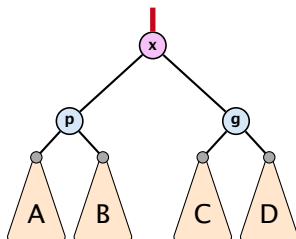
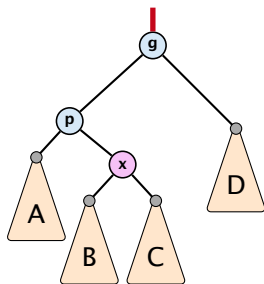
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Splay: Zigzig Case



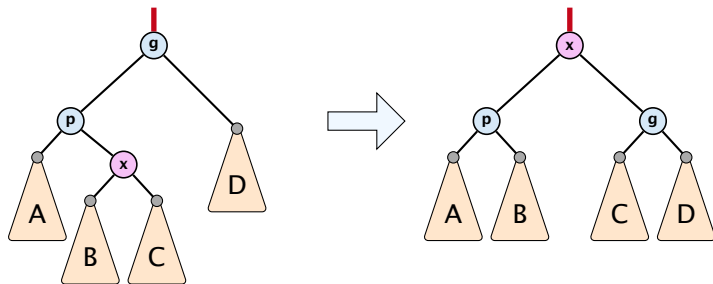
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Splay: Zigzag Case



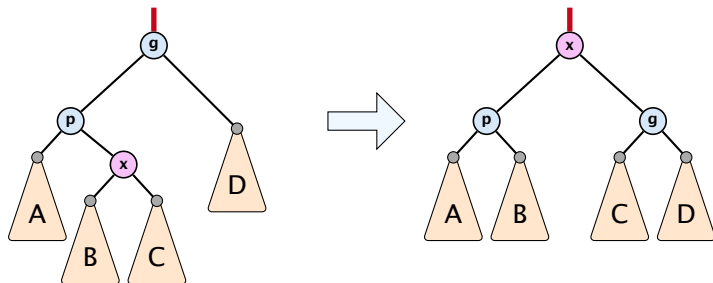
$\Delta\Phi =$

Splay: Zigzag Case



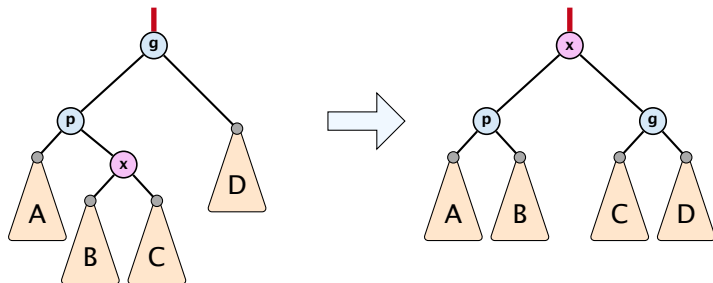
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Splay: Zigzag Case



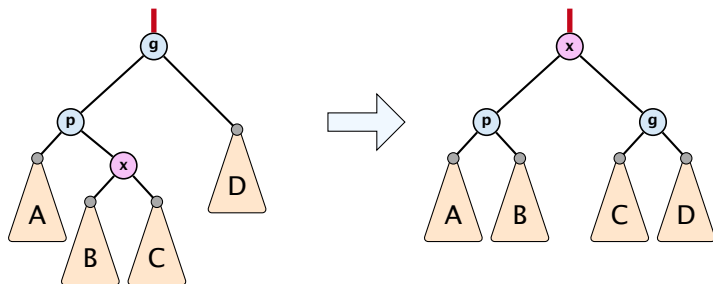
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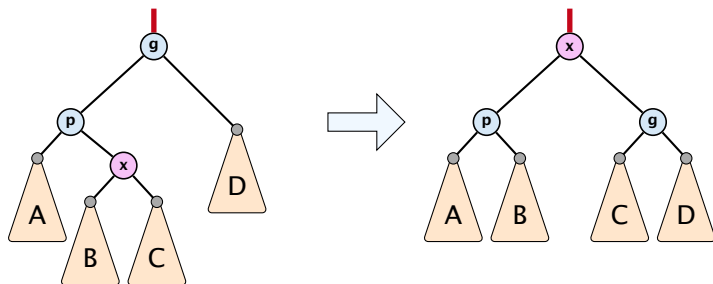
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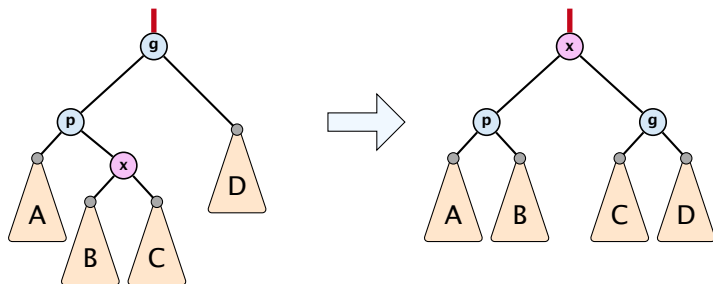
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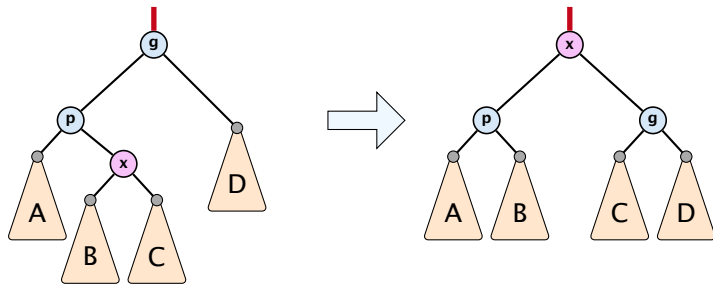
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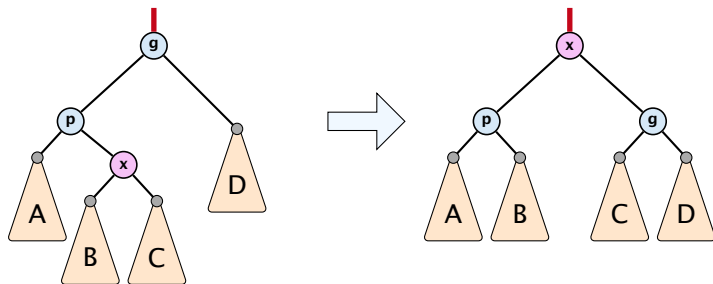
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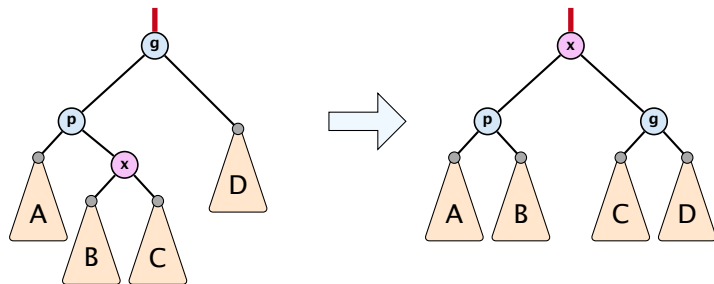
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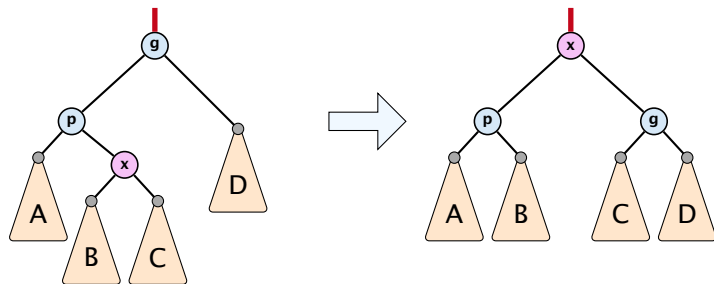
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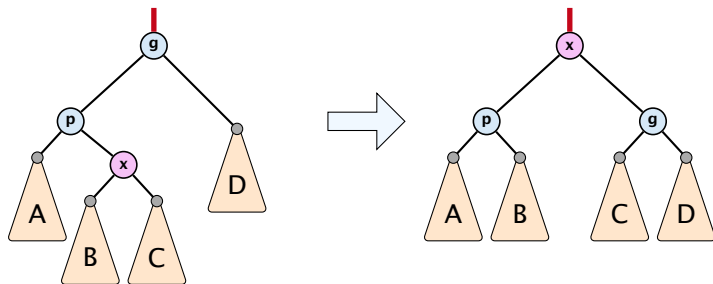
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Amortized cost of the whole splay operation:

$$\begin{aligned} &\leq 1 + 1 + \sum_{\text{steps } t} 3(r_t(x) - r_{t-1}(x)) \\ &= 2 + 3(r(\text{root}) - r_0(x)) \\ &\leq \mathcal{O}(\log n) \end{aligned}$$

7.4 Augmenting Data Structures

Suppose you want to develop a data structure with:

- ▶ **Insert(x)**: insert element x .
- ▶ **Search(k)**: search for element with key k .
- ▶ **Delete(x)**: delete element referenced by pointer x .
- ▶ **find-by-rank(ℓ)**: return the ℓ -th element; return “error” if the data-structure contains less than ℓ elements.

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Augment an existing data-structure instead of developing a new one.

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How to augment a data-structure

1. choose an underlying data-structure

- Of course, the above steps heavily depend on each other. For example it makes no sense to choose additional information to be stored (Step 2), and later realize that either the information cannot be maintained efficiently (Step 3) or is not sufficient to support the new operations (Step 4).
- However, the above outline is a good way to describe/document a new data-structure.

7.4 Augmenting Data Structures

How to augment a data-structure

1. choose an underlying data-structure
2. determine additional information to be stored in the underlying structure

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3. verify/show how the additional information can be maintained for the basic modifying operations on the underlying structure.

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Goal: Design a data-structure that supports insert, delete, search, and find-by-rank in time $\mathcal{O}(\log n)$.

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3. We need to be able to update the size-field in each node without asymptotically affecting the running time of insert, delete, and search. We come back to this step later...

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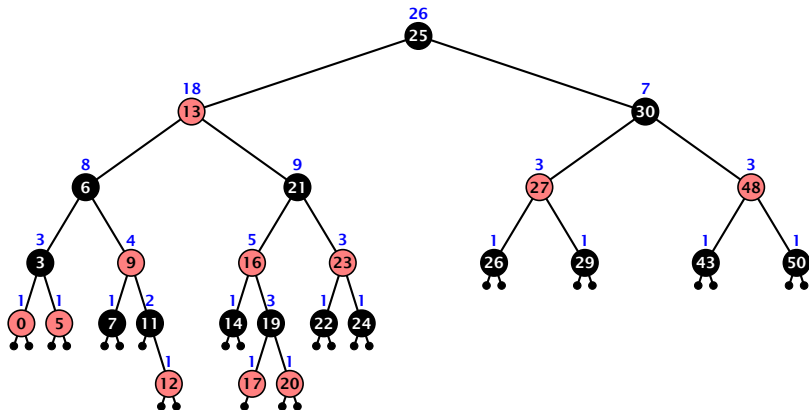
4. How does find-by-rank work?

Find-by-rank(k) := Select($root, k$) with

Algorithm 1 Select(x, i)

```
1: if  $x = \text{null}$  then return error
2: if left[ $x$ ]  $\neq$  null then  $r \leftarrow$  left[ $x$ ].size + 1 else  $r \leftarrow$  1
3: if  $i = r$  then return  $x$ 
4: if  $i < r$  then
5:     return Select(left[ $x$ ],  $i$ )
6: else
7:     return Select(right[ $x$ ],  $i - r$ )
```

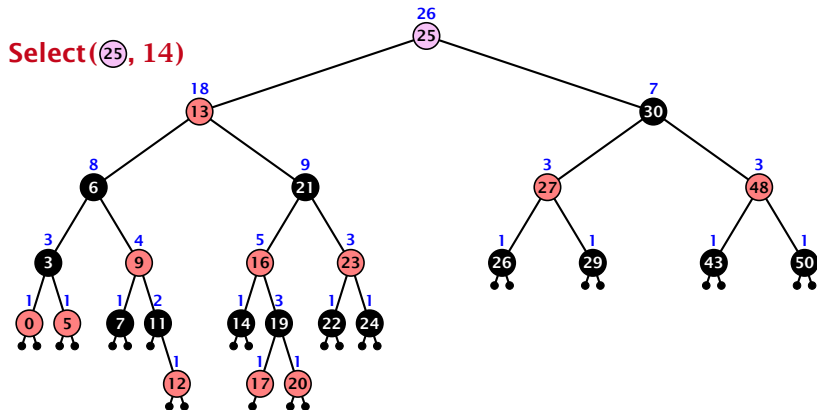
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Find-by-rank:

- ▶ decide whether you have to proceed into the left or right sub-tree
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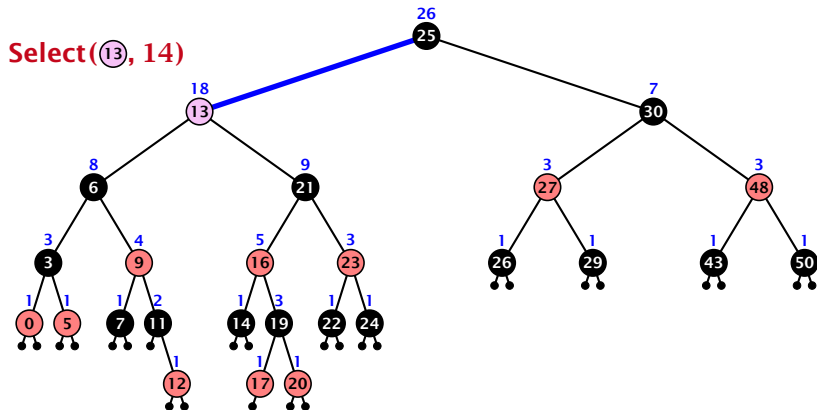
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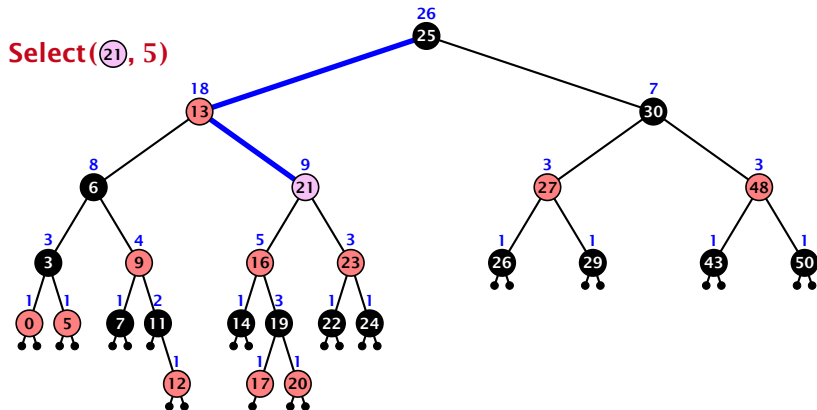
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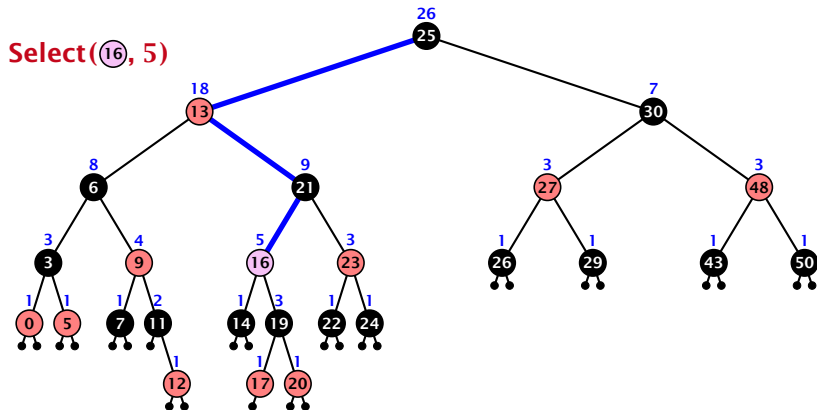
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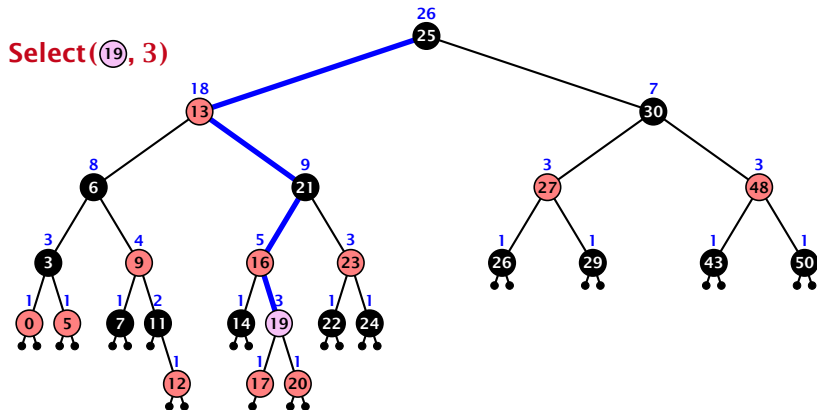
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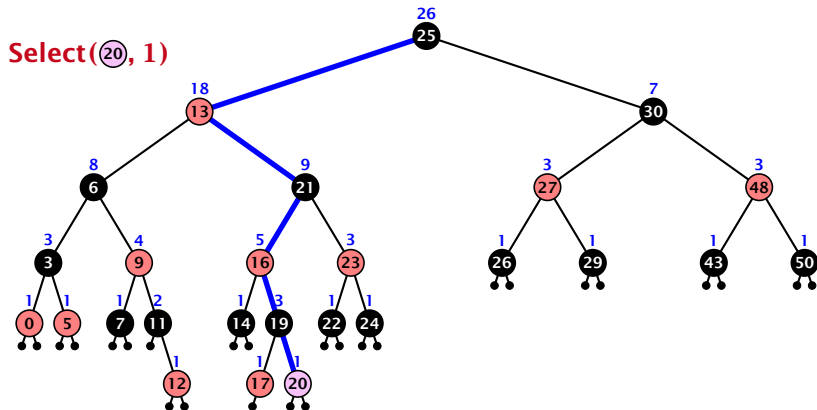
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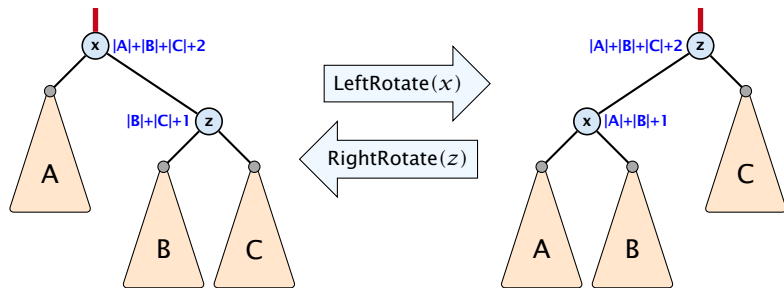
Search(k): Nothing to do.

Insert(x): When going down the search path increase the size field for each visited node. **Maintain the size field during rotations.**

Delete(x): Directly after splicing out a node traverse the path from the spliced out node upwards, and decrease the size counter on every node on this path. **Maintain the size field during rotations.**

Rotations

The only operation during the fix-up procedure that alters the tree and requires an update of the size-field:



The nodes x and z are the only nodes changing their size-fields.

The new size-fields can be computed **locally** from the size-fields of the children.

7.5 Skip Lists

Why do we not use a list for implementing the ADT Dynamic Set?

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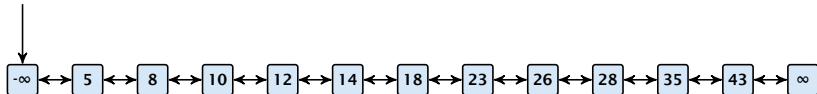
Why do we not use a list for implementing the ADT Dynamic Set?

- ▶ time for search $\Theta(n)$
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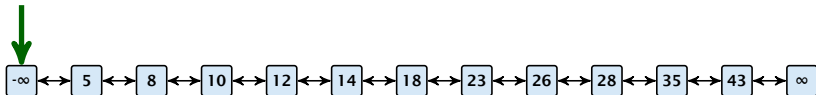
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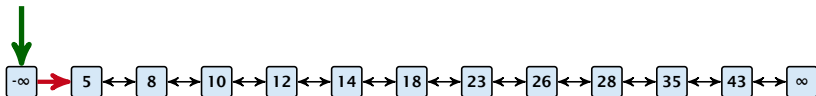
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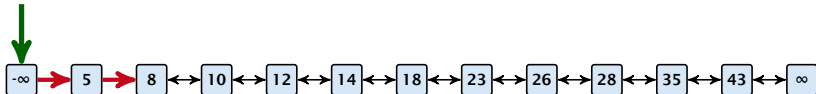
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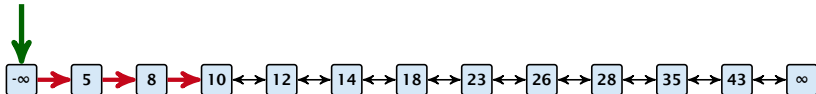
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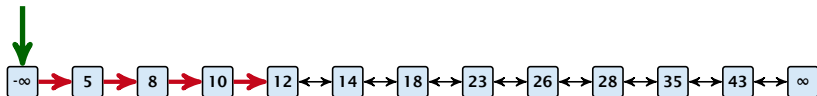
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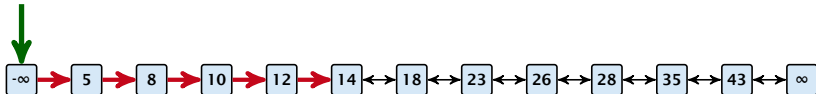
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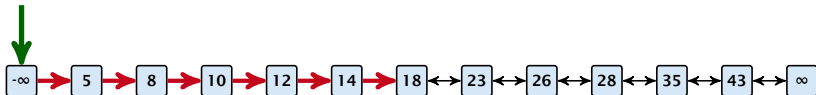
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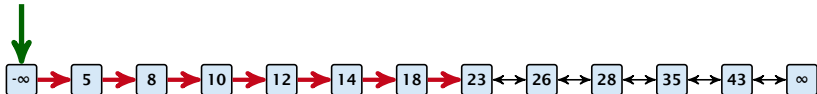
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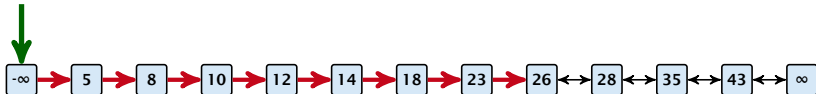
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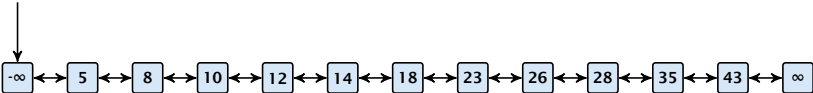
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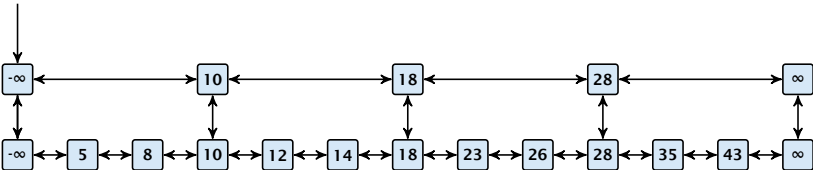
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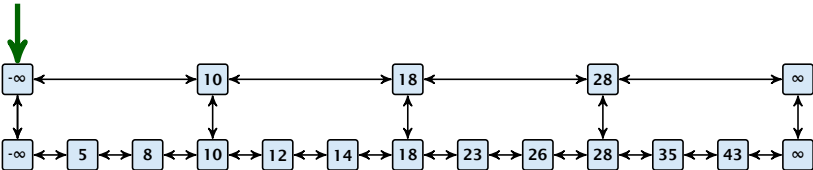
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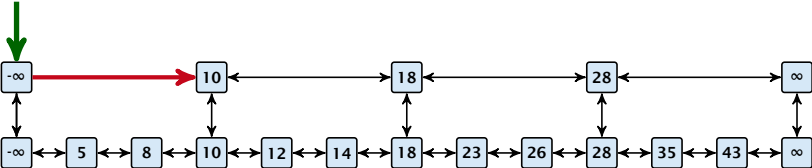
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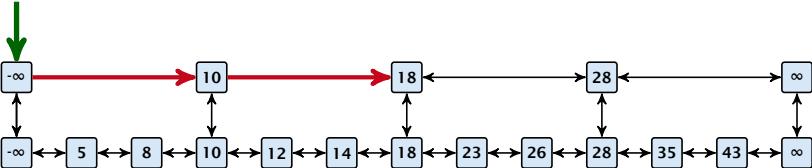
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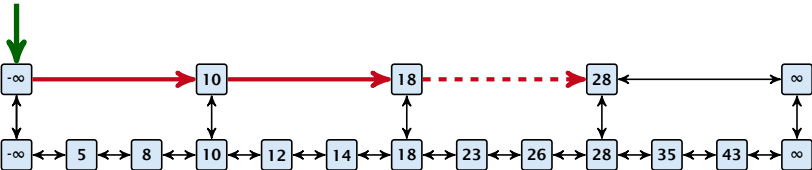
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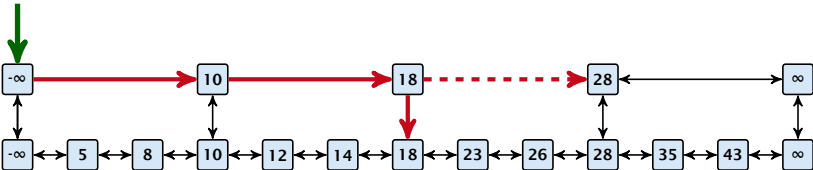
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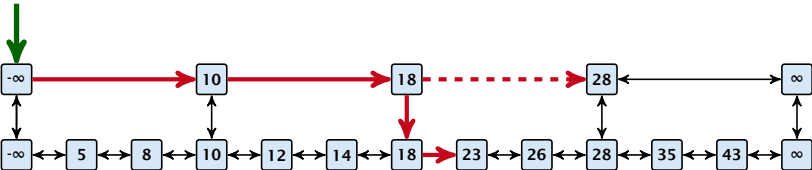
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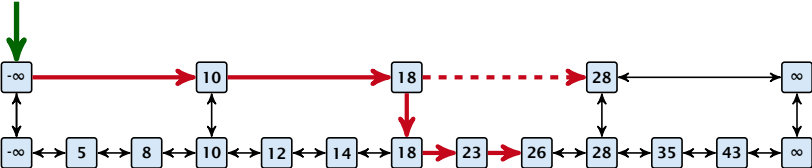
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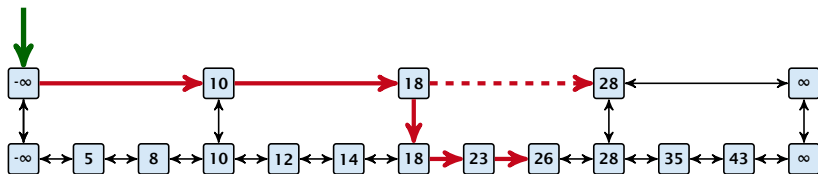
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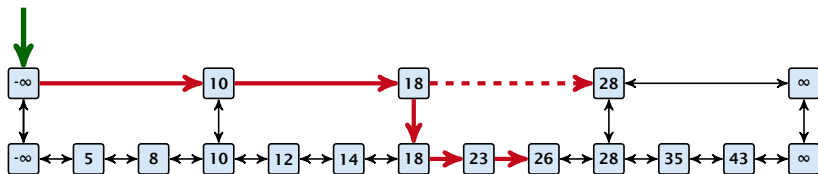


Let $|L_1|$ denote the number of elements in the “express lane”, and $|L_0| = n$ the number of all elements (ignoring dummy elements).

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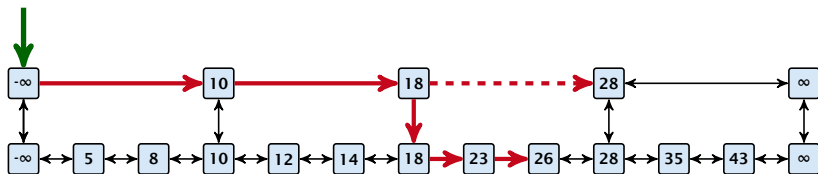
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Choose $|L_1| = \sqrt{n}$. Then search time $\Theta(\sqrt{n})$.

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- ▶ At most $|L_k| + \sum_{i=1}^k \frac{L_{i-1}}{L_i} + 3(k + 1)$ steps.

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Choosing $k = \Theta(\log n)$ gives a logarithmic running time.

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Use randomization instead!

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- ▶ A search operation gives you the insert position for element x in every list.
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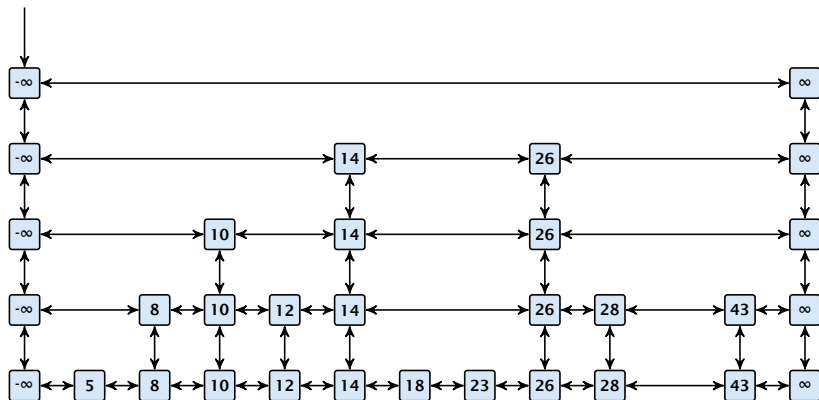
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The time for both operations is dominated by the search time.

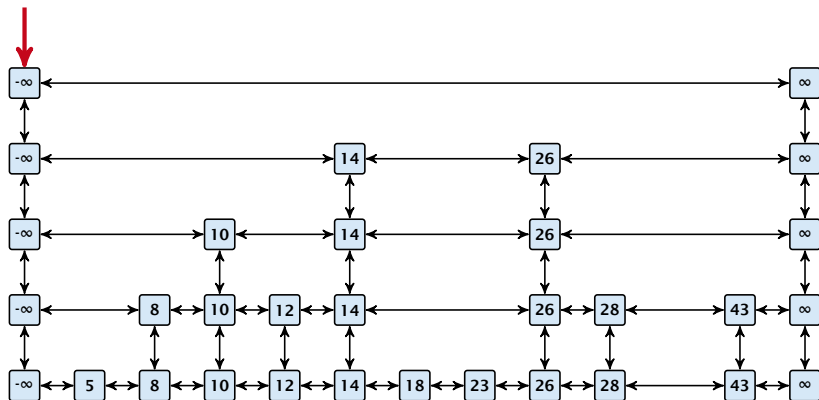
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Insert (35):



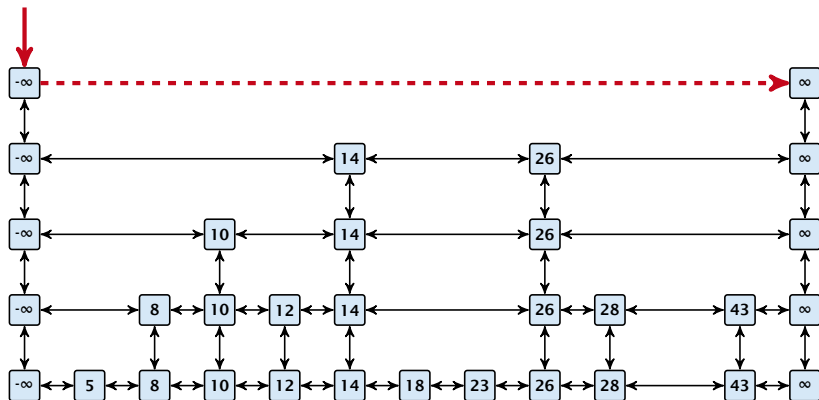
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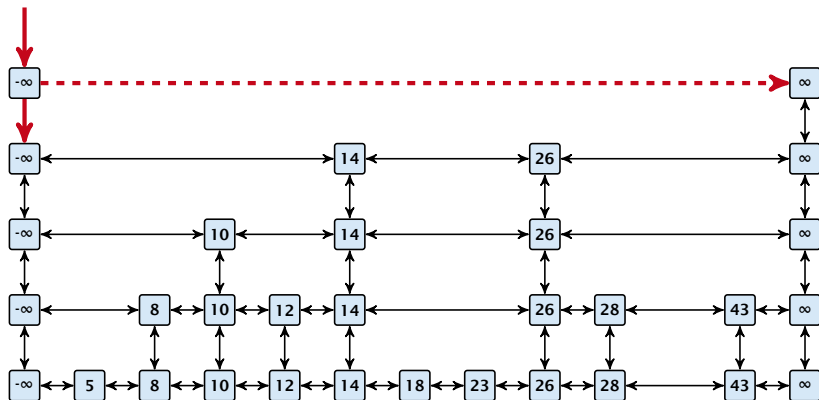
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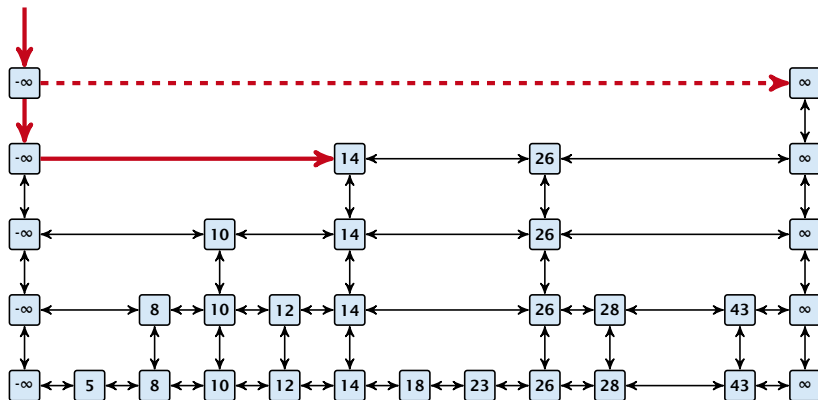
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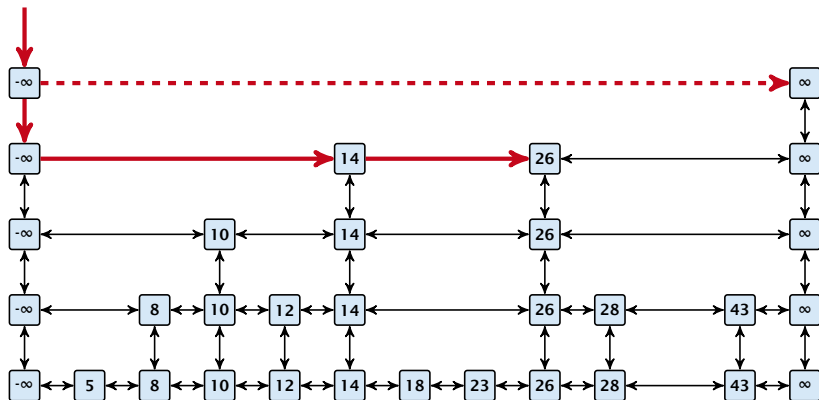
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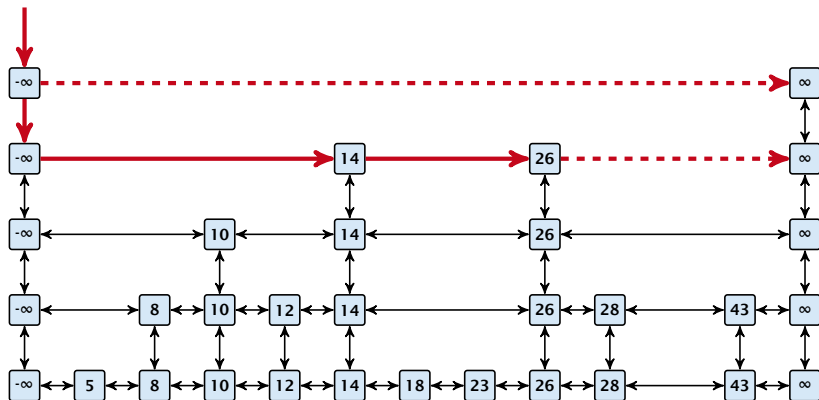
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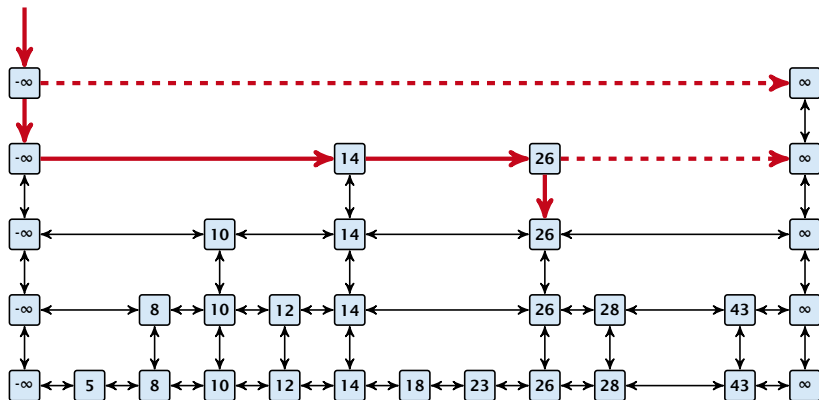
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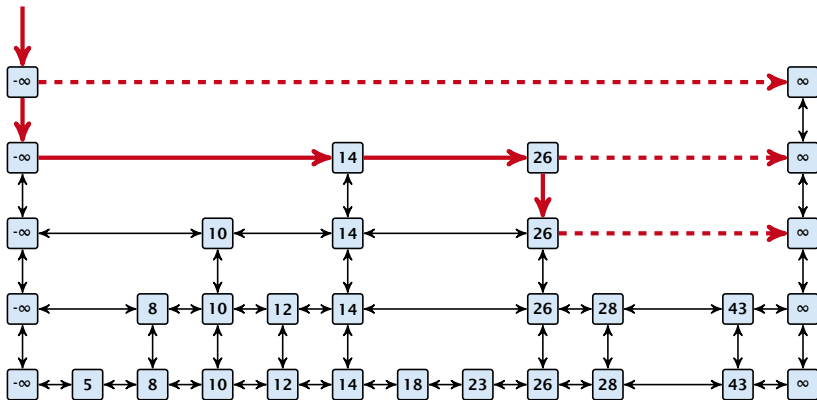
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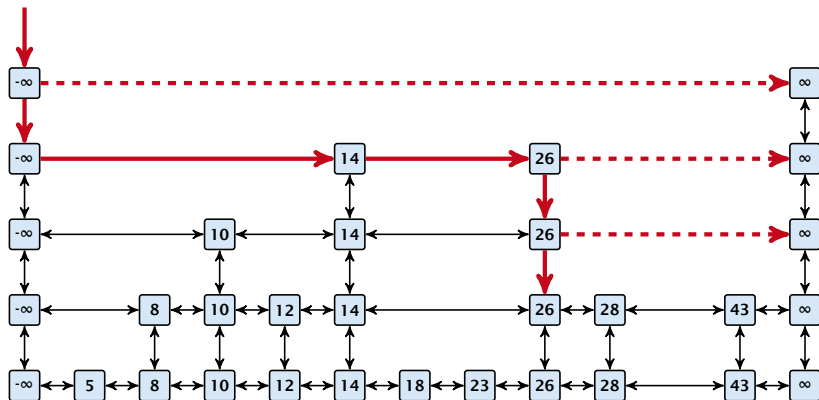
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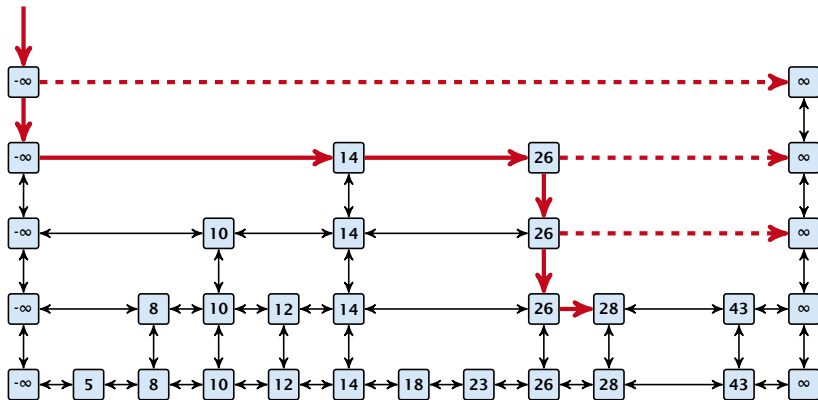
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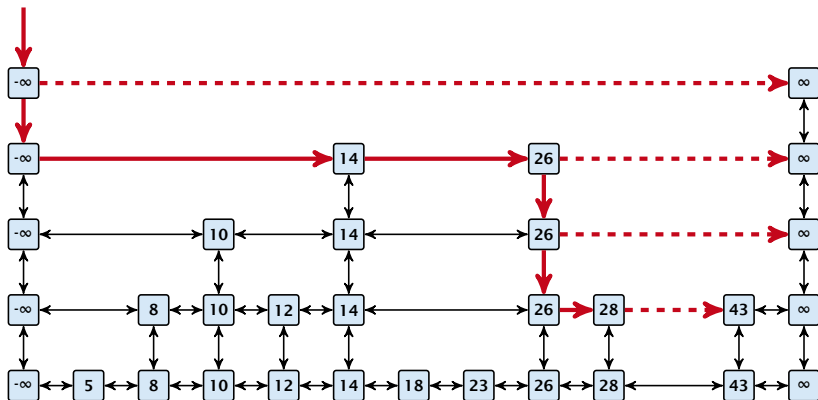
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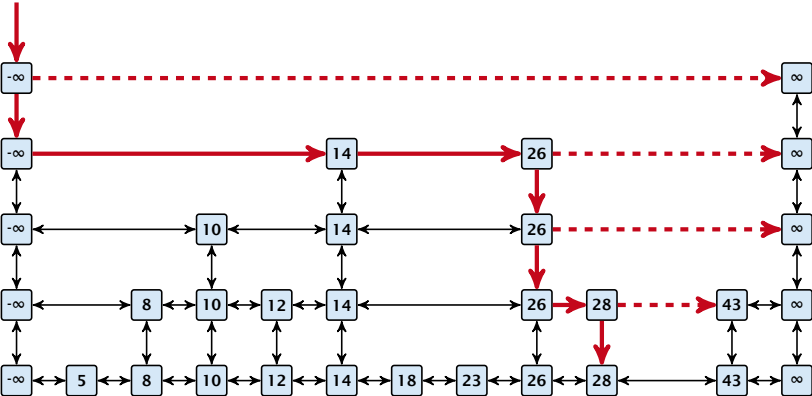
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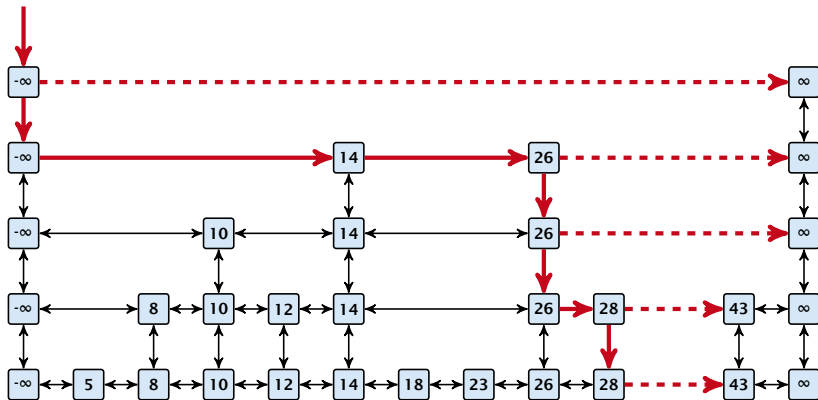
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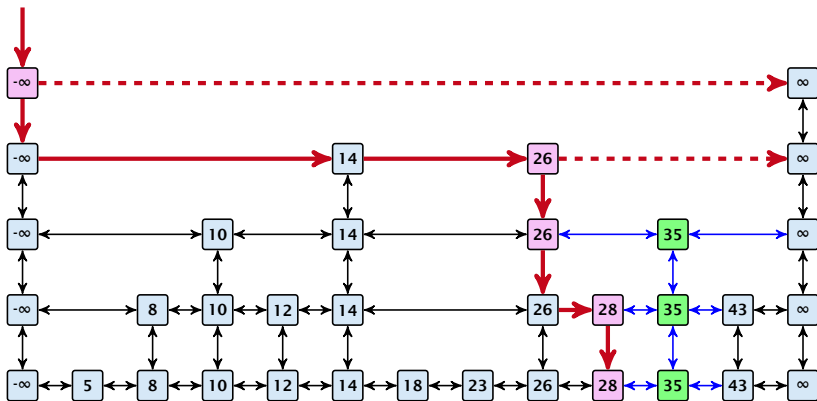
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High Probability

Definition 18 (High Probability)

We say a **randomized** algorithm has running time $\mathcal{O}(\log n)$ with **high probability** if for any constant α the running time is at most $\mathcal{O}(\log n)$ with probability at least $1 - \frac{1}{n^\alpha}$.

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Here the \mathcal{O} -notation hides a constant that may depend on α .

High Probability

Suppose there are **polynomially** many events E_1, E_2, \dots, E_ℓ , $\ell = n^c$ each holding with high probability (e.g. E_i may be the event that the i -th search in a skip list takes time at most $\mathcal{O}(\log n)$).

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Then the probability that all E_i hold is at least

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This means $E_1 \wedge \dots \wedge E_\ell$ holds with high probability.

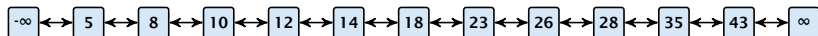
7.5 Skip Lists

Lemma 19

A search (and, hence, also insert and delete) in a skip list with n elements takes time $\mathcal{O}(\log n)$ with high probability (w. h. p.).

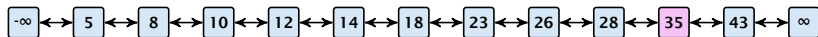
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Backward analysis:



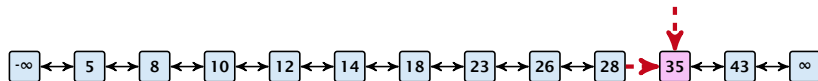
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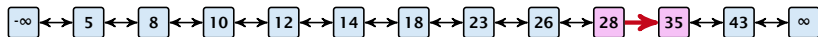
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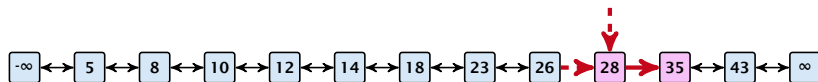
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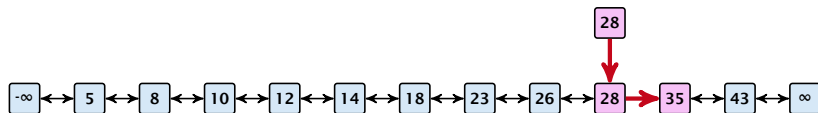
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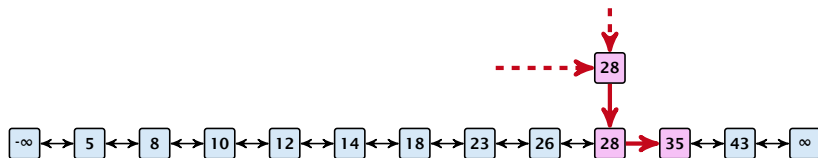
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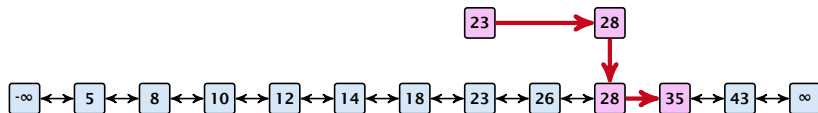
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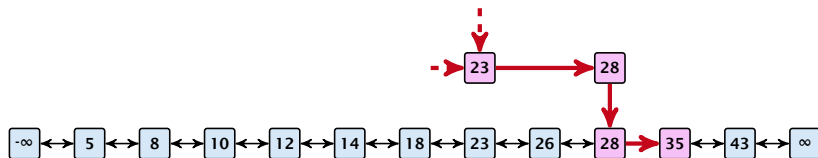
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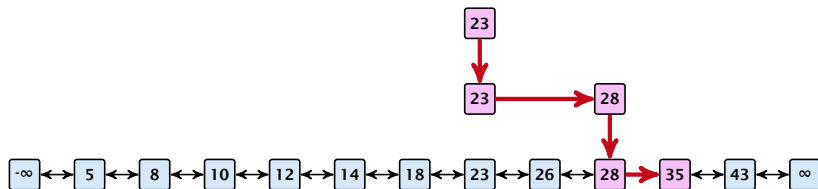
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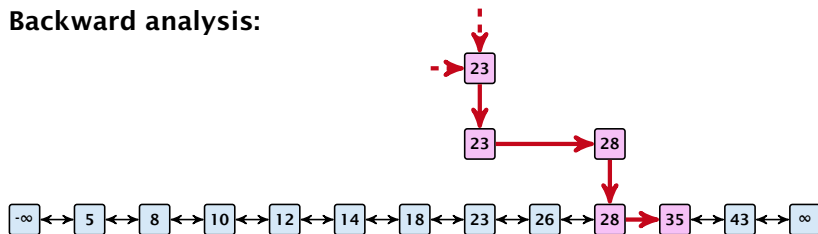
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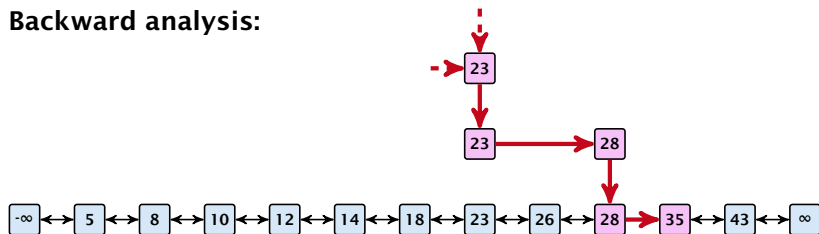
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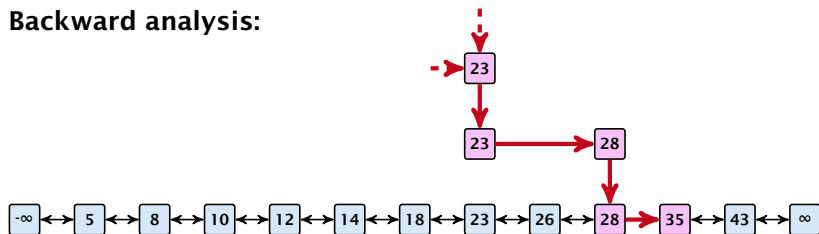
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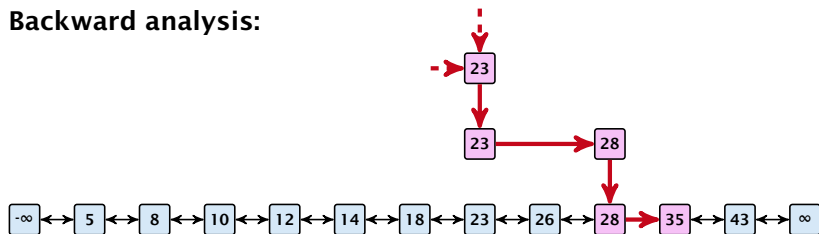
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We show that w.h.p:

- ▶ A “long” search path must also go very high.

7.5 Skip Lists

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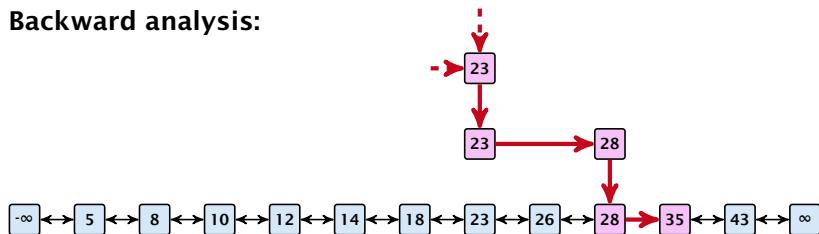
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We show that w.h.p:

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- ▶ There are no elements in high lists.

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Backward analysis:



At each point the path goes up with probability $1/2$ and left with probability $1/2$.

We show that w.h.p:

- ▶ A “long” search path must also go very high.
- ▶ There are no elements in high lists.

From this it follows that w.h.p. there are no long paths.

7.5 Skip Lists

Estimation for Binomial Coefficients

$$\left(\frac{n}{k}\right)^k \leq \binom{n}{k} \leq \left(\frac{en}{k}\right)^k$$

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In particular, this means that during the construction in the backward analysis we see at most k heads (i.e., coin flips that tell you to go up) in z trials.

7.5 Skip Lists

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$$\begin{aligned} \Pr[\text{search requires } z \text{ steps}] &\leq \Pr[E_{z,k}] + \Pr[A_{k+1}] \\ &\leq n^{-\alpha} + n^{-(\gamma-1)} \end{aligned}$$

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Hence,

$$\begin{aligned} \Pr[\text{search requires } z \text{ steps}] &\leq \Pr[E_{z,k}] + \Pr[A_{k+1}] \\ &\leq n^{-\alpha} + n^{-(\gamma-1)} \end{aligned}$$

This means, the search requires at most z steps, w. h. p.

7.6 van Emde Boas Trees

Dynamic Set Data Structure S :

- ▶ $S.insert(x)$
- ▶ $S.delete(x)$
- ▶ $S.search(x)$
- ▶ $S.min()$
- ▶ $S.max()$
- ▶ $S.succ(x)$
- ▶ $S.pred(x)$

7.6 van Emde Boas Trees

For this chapter we ignore the problem of storing satellite data:

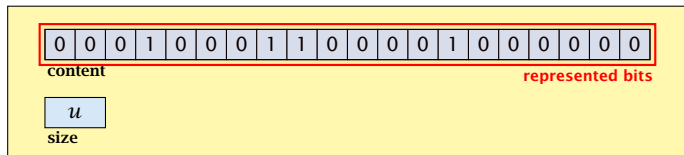
- ▶ **S . insert(x):** Inserts x into S .
- ▶ **S . delete(x):** Deletes x from S . Usually assumes that $x \in S$.
- ▶ **S . member(x):** Returns 1 if $x \in S$ and 0 otherwise.
- ▶ **S . min():** Returns the value of the minimum element in S .
- ▶ **S . max():** Returns the value of the maximum element in S .
- ▶ **S . succ(x):** Returns successor of x in S . Returns **null** if x is maximum or larger than any element in S . Note that x needs not to be in S .
- ▶ **S . pred(x):** Returns the predecessor of x in S . Returns **null** if x is minimum or smaller than any element in S . Note that x needs not to be in S .

7.6 van Emde Boas Trees

Can we improve the existing algorithms when the keys are from a restricted set?

In the following we assume that the keys are from $\{0, 1, \dots, u - 1\}$, where u denotes the size of the universe.

Implementation 1: Array



one array of u bits

Use an array that encodes the indicator function of the dynamic set.

Implementation 1: Array

Algorithm 1 `array.insert(x)`

1: `content[x] ← 1;`

Algorithm 2 `array.delete(x)`

1: `content[x] ← 0;`

Algorithm 3 `array.member(x)`

1: **return** `content[x];`

- ▶ Note that we assume that x is valid, i.e., it falls within the array boundaries.
- ▶ Obviously(?) the running time is constant.

Implementation 1: Array

Algorithm 4 `array.max()`

```
1: for ( $i = \text{size} - 1; i \geq 0; i--$ ) do  
2:     if content[i] = 1 then return  $i$ ;  
3: return null;
```

Implementation 1: Array

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Algorithm 5 `array.min()`

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1: for ( $i = 0; i < \text{size}; i++$ ) do  
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Implementation 1: Array

Algorithm 4 array.max()

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Algorithm 5 array.min()

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1: for ( $i = 0; i < \text{size}; i++$ ) do  
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```

- ▶ Running time is $\mathcal{O}(u)$ in the worst case.

Implementation 1: Array

Algorithm 6 `array.succ(x)`

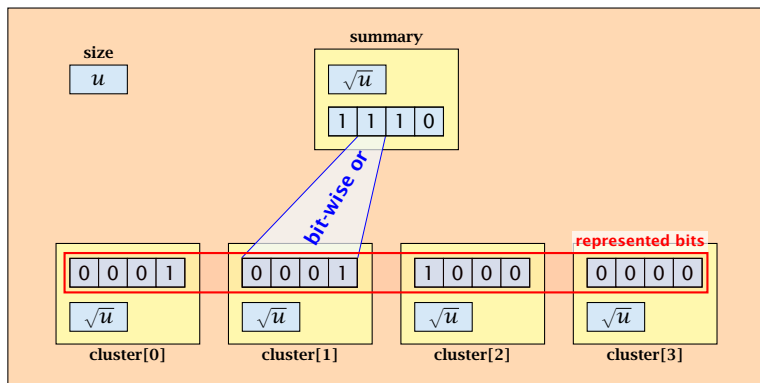
```
1: for ( $i = x + 1$ ;  $i < \text{size}$ ;  $i++$ ) do  
2:     if content[i] = 1 then return  $i$ ;  
3: return null;
```

Algorithm 7 `array.pred(x)`

```
1: for ( $i = x - 1$ ;  $i \geq 0$ ;  $i--$ ) do  
2:     if content[i] = 1 then return  $i$ ;  
3: return null;
```

- ▶ Running time is $\mathcal{O}(u)$ in the worst case.

Implementation 2: Summary Array



- ▶ \sqrt{u} cluster-arrays of \sqrt{u} bits.
- ▶ One summary-array of \sqrt{u} bits. The i -th bit in the summary array stores the bit-wise or of the bits in the i -th cluster.

Implementation 2: Summary Array

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The bit for a key x is contained in cluster number $\left\lfloor \frac{x}{\sqrt{u}} \right\rfloor$.

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The bit for a key x is contained in cluster number $\left\lfloor \frac{x}{\sqrt{u}} \right\rfloor$.

Within the cluster-array the bit is at position $x \bmod \sqrt{u}$.

Implementation 2: Summary Array

The bit for a key x is contained in cluster number $\lfloor \frac{x}{\sqrt{u}} \rfloor$.

Within the cluster-array the bit is at position $x \bmod \sqrt{u}$.

For simplicity we assume that $u = 2^{2k}$ for some $k \geq 1$. Then we can compute the cluster-number for an entry x as $\text{high}(x)$ (the upper half of the dual representation of x) and the position of x within its cluster as $\text{low}(x)$ (the lower half of the dual representation).

Implementation 2: Summary Array

Algorithm 8 $\text{member}(x)$

1: **return** $\text{cluster}[\text{high}(x)].\text{member}(\text{low}(x));$

Implementation 2: Summary Array

Algorithm 8 $\text{member}(x)$

1: **return** $\text{cluster}[\text{high}(x)].\text{member}(\text{low}(x));$

Algorithm 9 $\text{insert}(x)$

1: $\text{cluster}[\text{high}(x)].\text{insert}(\text{low}(x));$

2: $\text{summary}.\text{insert}(\text{high}(x));$

Implementation 2: Summary Array

Algorithm 8 $\text{member}(x)$

```
1: return cluster[high(x)].member(low(x));
```

Algorithm 9 $\text{insert}(x)$

```
1: cluster[high(x)].insert(low(x));  
2: summary.insert(high(x));
```

- ▶ The running times are constant, because the corresponding array-functions have constant running times.

Implementation 2: Summary Array

Algorithm 10 delete(x)

- 1: cluster[high(x)].delete(low(x));
- 2: **if** cluster[high(x)].min() = null **then**
- 3: summary.delete(high(x));

Implementation 2: Summary Array

Algorithm 10 delete(x)

```
1: cluster[high( $x$ )].delete(low( $x$ ));  
2: if cluster[high( $x$ )].min() = null then  
3:     summary.delete(high( $x$ ));
```

- ▶ The running time is dominated by the cost of a minimum computation on an array of size \sqrt{u} . Hence, $\mathcal{O}(\sqrt{u})$.

Implementation 2: Summary Array

Algorithm 11 $\text{max}()$

- 1: $\text{maxcluster} \leftarrow \text{summary}.\text{max}();$
- 2: **if** $\text{maxcluster} = \text{null}$ **return** $\text{null};$
- 3: $\text{offs} \leftarrow \text{cluster}[\text{maxcluster}].\text{max}();$
- 4: **return** $\text{maxcluster} \circ \text{offs};$

Implementation 2: Summary Array

Algorithm 11 $\text{max}()$

```
1:  $\text{maxcluster} \leftarrow \text{summary.max}();$   
2: if  $\text{maxcluster} = \text{null}$  return  $\text{null}$ ;  
3:  $\text{offs} \leftarrow \text{cluster}[\text{maxcluster}].\text{max}();$   
4: return  $\text{maxcluster} \circ \text{offs};$ 
```

Algorithm 12 $\text{min}()$

```
1:  $\text{mincluster} \leftarrow \text{summary.min}();$   
2: if  $\text{mincluster} = \text{null}$  return  $\text{null}$ ;  
3:  $\text{offs} \leftarrow \text{cluster}[\text{mincluster}].\text{min}();$   
4: return  $\text{mincluster} \circ \text{offs};$ 
```

Implementation 2: Summary Array

Algorithm 11 $\text{max}()$

```
1:  $\text{maxcluster} \leftarrow \text{summary.max}();$   
2: if  $\text{maxcluster} = \text{null}$  return  $\text{null}$ ;  
3:  $\text{offs} \leftarrow \text{cluster}[\text{maxcluster}].\text{max}();$   
4: return  $\text{maxcluster} \circ \text{offs}$ ;
```

Algorithm 12 $\text{min}()$

```
1:  $\text{mincluster} \leftarrow \text{summary.min}();$   
2: if  $\text{mincluster} = \text{null}$  return  $\text{null}$ ;  
3:  $\text{offs} \leftarrow \text{cluster}[\text{mincluster}].\text{min}();$   
4: return  $\text{mincluster} \circ \text{offs}$ ;
```

The operator \circ stands for the concatenation of two bitstrings.

This means if $x = 0111_2$ and $y = 0001_2$ then $x \circ y = 01110001_2$.

- ▶ Running time is roughly $2\sqrt{u} = \mathcal{O}(\sqrt{u})$ in the worst case.

Implementation 2: Summary Array

Algorithm 13 $\text{succ}(x)$

```
1:  $m \leftarrow \text{cluster}[\text{high}(x)].\text{succ}(\text{low}(x))$ 
2: if  $m \neq \text{null}$  then return  $\text{high}(x) \circ m$ ;
3:  $\text{succcluster} \leftarrow \text{summary}.\text{succ}(\text{high}(x))$ ;
4: if  $\text{succcluster} \neq \text{null}$  then
5:      $\text{offs} \leftarrow \text{cluster}[\text{succcluster}].\text{min}()$ ;
6:     return  $\text{succcluster} \circ \text{offs}$ ;
7: return  $\text{null}$ ;
```

Implementation 2: Summary Array

Algorithm 13 $\text{succ}(x)$

```
1:  $m \leftarrow \text{cluster}[\text{high}(x)].\text{succ}(\text{low}(x))$ 
2: if  $m \neq \text{null}$  then return  $\text{high}(x) \circ m$ ;
3:  $\text{succcluster} \leftarrow \text{summary}.\text{succ}(\text{high}(x))$ ;
4: if  $\text{succcluster} \neq \text{null}$  then
5:      $\text{offs} \leftarrow \text{cluster}[\text{succcluster}].\text{min}()$ ;
6:     return  $\text{succcluster} \circ \text{offs}$ ;
7: return  $\text{null}$ ;
```

- ▶ Running time is roughly $3\sqrt{u} = \mathcal{O}(\sqrt{u})$ in the worst case.

Implementation 2: Summary Array

Algorithm 14 $\text{pred}(x)$

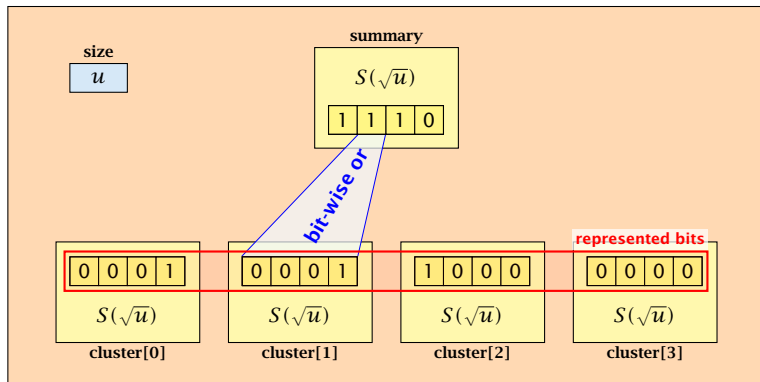
```
1:  $m \leftarrow \text{cluster}[\text{high}(x)].\text{pred}(\text{low}(x))$ 
2: if  $m \neq \text{null}$  then return  $\text{high}(x) \circ m$ ;
3:  $\text{predcluster} \leftarrow \text{summary}.\text{pred}(\text{high}(x))$ ;
4: if  $\text{predcluster} \neq \text{null}$  then
5:    $\text{offs} \leftarrow \text{cluster}[\text{predcluster}].\text{max}()$ ;
6:   return  $\text{predcluster} \circ \text{offs}$ ;
7: return  $\text{null}$ ;
```

- ▶ Running time is roughly $3\sqrt{u} = \mathcal{O}(\sqrt{u})$ in the worst case.

Implementation 3: Recursion

Instead of using sub-arrays, we build a recursive data-structure.

$S(u)$ is a dynamic set data-structure representing u bits:



Implementation 3: Recursion

We assume that $u = 2^{2^k}$ for some k .

The data-structure $S(2)$ is defined as an array of 2-bits (end of the recursion).

Implementation 3: Recursion

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The code from Implementation 2 can be used **unchanged**. We only need to redo the analysis of the running time.

Implementation 3: Recursion

The code from Implementation 2 can be used **unchanged**. We only need to redo the analysis of the running time.

Note that in the code we do not need to specifically address the non-recursive case. This is achieved by the fact that an $S(4)$ will contain $S(2)$'s as sub-datastructures, which are **arrays**. Hence, a call like `cluster[1].min()` from within the data-structure $S(4)$ is **not** a recursive call as it will call the function `array.min()`.

Implementation 3: Recursion

The code from Implementation 2 can be used **unchanged**. We only need to redo the analysis of the running time.

Note that in the code we do not need to specifically address the non-recursive case. This is achieved by the fact that an $S(4)$ will contain $S(2)$'s as sub-datastructures, which are **arrays**. Hence, a call like `cluster[1].min()` from within the data-structure $S(4)$ is **not** a recursive call as it will call the function `array.min()`.

This means that the non-recursive case is been dealt with while initializing the data-structure.

Implementation 3: Recursion

Algorithm 15 $\text{member}(x)$

1: **return** $\text{cluster}[\text{high}(x)].\text{member}(\text{low}(x));$

- ▶ $T_{\text{mem}}(u) = T_{\text{mem}}(\sqrt{u}) + 1.$

Implementation 3: Recursion

Algorithm 16 insert(x)

```
1: cluster[high( $x$ )].insert(low( $x$ ));  
2: summary.insert(high( $x$ ));
```

► $T_{\text{ins}}(u) = 2T_{\text{ins}}(\sqrt{u}) + 1.$

Implementation 3: Recursion

Algorithm 17 delete(x)

```
1: cluster[high( $x$ )].delete(low( $x$ ));  
2: if cluster[high( $x$ )].min() = null then  
3:     summary.delete(high( $x$ ));
```

► $T_{\text{del}}(u) = 2T_{\text{del}}(\sqrt{u}) + T_{\text{min}}(\sqrt{u}) + 1.$

Implementation 3: Recursion

Algorithm 18 $\text{min}()$

```
1: mincluster  $\leftarrow$  summary.min();  
2: if mincluster = null return null;  
3: offs  $\leftarrow$  cluster[mincluster].min();  
4: return mincluster  $\circ$  offs;
```

► $T_{\min}(u) = 2T_{\min}(\sqrt{u}) + 1.$

Implementation 3: Recursion

Algorithm 19 $\text{succ}(x)$

```
1:  $m \leftarrow \text{cluster}[\text{high}(x)].\text{succ}(\text{low}(x))$ 
2: if  $m \neq \text{null}$  then return  $\text{high}(x) \circ m$ ;
3:  $\text{succcluster} \leftarrow \text{summary}.\text{succ}(\text{high}(x))$ ;
4: if  $\text{succcluster} \neq \text{null}$  then
5:      $\text{offs} \leftarrow \text{cluster}[\text{succcluster}].\text{min}()$ ;
6:     return  $\text{succcluster} \circ \text{offs}$ ;
7: return  $\text{null}$ ;
```

► $T_{\text{succ}}(u) = 2T_{\text{succ}}(\sqrt{u}) + T_{\text{min}}(\sqrt{u}) + 1.$

Implementation 3: Recursion

$$T_{\text{mem}}(\mathbf{u}) = T_{\text{mem}}(\sqrt{\mathbf{u}}) + 1:$$

Implementation 3: Recursion

$$T_{\text{mem}}(\mathbf{u}) = T_{\text{mem}}(\sqrt{\mathbf{u}}) + \mathbf{1}:$$

Set $\ell := \log u$ and $X(\ell) := T_{\text{mem}}(2^\ell)$.

Implementation 3: Recursion

$$T_{\text{mem}}(\mathbf{u}) = T_{\text{mem}}(\sqrt{\mathbf{u}}) + \mathbf{1}:$$

Set $\ell := \log u$ and $X(\ell) := T_{\text{mem}}(2^\ell)$. Then

Implementation 3: Recursion

$$T_{\text{mem}}(\mathbf{u}) = T_{\text{mem}}(\sqrt{\mathbf{u}}) + 1:$$

Set $\ell := \log u$ and $X(\ell) := T_{\text{mem}}(2^\ell)$. Then

$$X(\ell)$$

Implementation 3: Recursion

$$T_{\text{mem}}(\mathbf{u}) = T_{\text{mem}}(\sqrt{\mathbf{u}}) + \mathbf{1}:$$

Set $\ell := \log u$ and $X(\ell) := T_{\text{mem}}(2^\ell)$. Then

$$X(\ell) = T_{\text{mem}}(2^\ell)$$

Implementation 3: Recursion

$$T_{\text{mem}}(\mathbf{u}) = T_{\text{mem}}(\sqrt{\mathbf{u}}) + \mathbf{1}:$$

Set $\ell := \log u$ and $X(\ell) := T_{\text{mem}}(2^\ell)$. Then

$$X(\ell) = T_{\text{mem}}(2^\ell) = T_{\text{mem}}(u)$$

Implementation 3: Recursion

$$T_{\text{mem}}(\mathbf{u}) = T_{\text{mem}}(\sqrt{\mathbf{u}}) + 1:$$

Set $\ell := \log u$ and $X(\ell) := T_{\text{mem}}(2^\ell)$. Then

$$X(\ell) = T_{\text{mem}}(2^\ell) = T_{\text{mem}}(\mathbf{u}) = T_{\text{mem}}(\sqrt{\mathbf{u}}) + 1$$

Implementation 3: Recursion

$$T_{\text{mem}}(\mathbf{u}) = T_{\text{mem}}(\sqrt{\mathbf{u}}) + 1:$$

Set $\ell := \log u$ and $X(\ell) := T_{\text{mem}}(2^\ell)$. Then

$$\begin{aligned} X(\ell) = T_{\text{mem}}(2^\ell) &= T_{\text{mem}}(\mathbf{u}) = T_{\text{mem}}(\sqrt{\mathbf{u}}) + 1 \\ &= T_{\text{mem}}(2^{\frac{\ell}{2}}) + 1 \end{aligned}$$

Implementation 3: Recursion

$$T_{\text{mem}}(\mathbf{u}) = T_{\text{mem}}(\sqrt{\mathbf{u}}) + 1:$$

Set $\ell := \log u$ and $X(\ell) := T_{\text{mem}}(2^\ell)$. Then

$$\begin{aligned} X(\ell) = T_{\text{mem}}(2^\ell) &= T_{\text{mem}}(\mathbf{u}) = T_{\text{mem}}(\sqrt{\mathbf{u}}) + 1 \\ &= T_{\text{mem}}(2^{\frac{\ell}{2}}) + 1 = X\left(\frac{\ell}{2}\right) + 1 . \end{aligned}$$

Implementation 3: Recursion

$$T_{\text{mem}}(\mathbf{u}) = T_{\text{mem}}(\sqrt{\mathbf{u}}) + 1:$$

Set $\ell := \log u$ and $X(\ell) := T_{\text{mem}}(2^\ell)$. Then

$$\begin{aligned} X(\ell) &= T_{\text{mem}}(2^\ell) = T_{\text{mem}}(u) = T_{\text{mem}}(\sqrt{u}) + 1 \\ &= T_{\text{mem}}(2^{\frac{\ell}{2}}) + 1 = X\left(\frac{\ell}{2}\right) + 1 . \end{aligned}$$

Using Master theorem gives $X(\ell) = \mathcal{O}(\log \ell)$, and hence $T_{\text{mem}}(u) = \mathcal{O}(\log \log u)$.

Implementation 3: Recursion

$$T_{\text{ins}}(u) = 2T_{\text{ins}}(\sqrt{u}) + 1.$$

Implementation 3: Recursion

$$T_{\text{ins}}(\mathbf{u}) = 2T_{\text{ins}}(\sqrt{\mathbf{u}}) + 1.$$

Set $\ell := \log u$ and $X(\ell) := T_{\text{ins}}(2^\ell)$.

Implementation 3: Recursion

$$T_{\text{ins}}(\mathbf{u}) = 2T_{\text{ins}}(\sqrt{\mathbf{u}}) + 1.$$

Set $\ell := \log u$ and $X(\ell) := T_{\text{ins}}(2^\ell)$. Then

Implementation 3: Recursion

$$T_{\text{ins}}(\mathbf{u}) = 2T_{\text{ins}}(\sqrt{\mathbf{u}}) + 1.$$

Set $\ell := \log u$ and $X(\ell) := T_{\text{ins}}(2^\ell)$. Then

$$X(\ell)$$

Implementation 3: Recursion

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Set $\ell := \log u$ and $X(\ell) := T_{\text{ins}}(2^\ell)$. Then

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Set $\ell := \log u$ and $X(\ell) := T_{\text{ins}}(2^\ell)$. Then

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Set $\ell := \log u$ and $X(\ell) := T_{\text{ins}}(2^\ell)$. Then

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Implementation 3: Recursion

$$T_{\text{ins}}(\mathbf{u}) = 2T_{\text{ins}}(\sqrt{\mathbf{u}}) + 1.$$

Set $\ell := \log u$ and $X(\ell) := T_{\text{ins}}(2^\ell)$. Then

$$\begin{aligned} X(\ell) &= T_{\text{ins}}(2^\ell) = T_{\text{ins}}(\mathbf{u}) = 2T_{\text{ins}}(\sqrt{\mathbf{u}}) + 1 \\ &= 2T_{\text{ins}}(2^{\frac{\ell}{2}}) + 1 \end{aligned}$$

Implementation 3: Recursion

$$T_{\text{ins}}(\mathbf{u}) = 2T_{\text{ins}}(\sqrt{\mathbf{u}}) + 1.$$

Set $\ell := \log u$ and $X(\ell) := T_{\text{ins}}(2^\ell)$. Then

$$\begin{aligned} X(\ell) &= T_{\text{ins}}(2^\ell) = T_{\text{ins}}(\mathbf{u}) = 2T_{\text{ins}}(\sqrt{\mathbf{u}}) + 1 \\ &= 2T_{\text{ins}}(2^{\frac{\ell}{2}}) + 1 = 2X\left(\frac{\ell}{2}\right) + 1 . \end{aligned}$$

Implementation 3: Recursion

$$T_{\text{ins}}(u) = 2T_{\text{ins}}(\sqrt{u}) + 1.$$

Set $\ell := \log u$ and $X(\ell) := T_{\text{ins}}(2^\ell)$. Then

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Using Master theorem gives $X(\ell) = \mathcal{O}(\ell)$, and hence $T_{\text{ins}}(u) = \mathcal{O}(\log u)$.

Implementation 3: Recursion

$$T_{\text{ins}}(\mathbf{u}) = 2T_{\text{ins}}(\sqrt{\mathbf{u}}) + 1.$$

Set $\ell := \log u$ and $X(\ell) := T_{\text{ins}}(2^\ell)$. Then

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Using Master theorem gives $X(\ell) = \mathcal{O}(\ell)$, and hence $T_{\text{ins}}(\mathbf{u}) = \mathcal{O}(\log u)$.

The same holds for $T_{\text{max}}(\mathbf{u})$ and $T_{\text{min}}(\mathbf{u})$.

Implementation 3: Recursion

$$T_{\text{del}}(\mathbf{u}) = 2T_{\text{del}}(\sqrt{\mathbf{u}}) + T_{\text{min}}(\sqrt{\mathbf{u}}) + 1 \leq 2T_{\text{del}}(\sqrt{\mathbf{u}}) + c \log(\mathbf{u}).$$

Implementation 3: Recursion

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Set $\ell := \log u$ and $X(\ell) := T_{\text{del}}(2^\ell)$.

Implementation 3: Recursion

$$T_{\text{del}}(\mathbf{u}) = 2T_{\text{del}}(\sqrt{\mathbf{u}}) + T_{\text{min}}(\sqrt{\mathbf{u}}) + 1 \leq 2T_{\text{del}}(\sqrt{\mathbf{u}}) + c \log(\mathbf{u}).$$

Set $\ell := \log u$ and $X(\ell) := T_{\text{del}}(2^\ell)$. Then

Implementation 3: Recursion

$$T_{\text{del}}(\mathbf{u}) = 2T_{\text{del}}(\sqrt{\mathbf{u}}) + T_{\text{min}}(\sqrt{\mathbf{u}}) + 1 \leq 2T_{\text{del}}(\sqrt{\mathbf{u}}) + c \log(\mathbf{u}).$$

Set $\ell := \log u$ and $X(\ell) := T_{\text{del}}(2^\ell)$. Then

$$X(\ell)$$

Implementation 3: Recursion

$$T_{\text{del}}(\mathbf{u}) = 2T_{\text{del}}(\sqrt{\mathbf{u}}) + T_{\text{min}}(\sqrt{\mathbf{u}}) + 1 \leq 2T_{\text{del}}(\sqrt{\mathbf{u}}) + c \log(\mathbf{u}).$$

Set $\ell := \log u$ and $X(\ell) := T_{\text{del}}(2^\ell)$. Then

$$X(\ell) = T_{\text{del}}(2^\ell)$$

Implementation 3: Recursion

$$T_{\text{del}}(\mathbf{u}) = 2T_{\text{del}}(\sqrt{\mathbf{u}}) + T_{\text{min}}(\sqrt{\mathbf{u}}) + 1 \leq 2T_{\text{del}}(\sqrt{\mathbf{u}}) + \mathbf{c} \log(\mathbf{u}).$$

Set $\ell := \log u$ and $X(\ell) := T_{\text{del}}(2^\ell)$. Then

$$X(\ell) = T_{\text{del}}(2^\ell) = T_{\text{del}}(u)$$

Implementation 3: Recursion

$$T_{\text{del}}(\mathbf{u}) = 2T_{\text{del}}(\sqrt{\mathbf{u}}) + T_{\text{min}}(\sqrt{\mathbf{u}}) + 1 \leq 2T_{\text{del}}(\sqrt{\mathbf{u}}) + c \log(\mathbf{u}).$$

Set $\ell := \log u$ and $X(\ell) := T_{\text{del}}(2^\ell)$. Then

$$X(\ell) = T_{\text{del}}(2^\ell) = T_{\text{del}}(u) = 2T_{\text{del}}(\sqrt{u}) + c \log u$$

Implementation 3: Recursion

$$T_{\text{del}}(\mathbf{u}) = 2T_{\text{del}}(\sqrt{\mathbf{u}}) + T_{\text{min}}(\sqrt{\mathbf{u}}) + 1 \leq 2T_{\text{del}}(\sqrt{\mathbf{u}}) + c \log(\mathbf{u}).$$

Set $\ell := \log u$ and $X(\ell) := T_{\text{del}}(2^\ell)$. Then

$$\begin{aligned} X(\ell) &= T_{\text{del}}(2^\ell) = T_{\text{del}}(u) = 2T_{\text{del}}(\sqrt{u}) + c \log u \\ &= 2T_{\text{del}}(2^{\frac{\ell}{2}}) + c\ell \end{aligned}$$

Implementation 3: Recursion

$$T_{\text{del}}(\mathbf{u}) = 2T_{\text{del}}(\sqrt{\mathbf{u}}) + T_{\text{min}}(\sqrt{\mathbf{u}}) + 1 \leq 2T_{\text{del}}(\sqrt{\mathbf{u}}) + c \log(\mathbf{u}).$$

Set $\ell := \log u$ and $X(\ell) := T_{\text{del}}(2^\ell)$. Then

$$\begin{aligned} X(\ell) &= T_{\text{del}}(2^\ell) = T_{\text{del}}(u) = 2T_{\text{del}}(\sqrt{u}) + c \log u \\ &= 2T_{\text{del}}(2^{\frac{\ell}{2}}) + c\ell = 2X\left(\frac{\ell}{2}\right) + c\ell . \end{aligned}$$

Implementation 3: Recursion

$$T_{\text{del}}(\mathbf{u}) = 2T_{\text{del}}(\sqrt{\mathbf{u}}) + T_{\text{min}}(\sqrt{\mathbf{u}}) + 1 \leq 2T_{\text{del}}(\sqrt{\mathbf{u}}) + c \log(\mathbf{u}).$$

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Using Master theorem gives $X(\ell) = \Theta(\ell \log \ell)$, and hence $T_{\text{del}}(u) = \mathcal{O}(\log u \log \log u)$.

Implementation 3: Recursion

$$T_{\text{del}}(\mathbf{u}) = 2T_{\text{del}}(\sqrt{\mathbf{u}}) + T_{\text{min}}(\sqrt{\mathbf{u}}) + 1 \leq 2T_{\text{del}}(\sqrt{\mathbf{u}}) + c \log(\mathbf{u}).$$

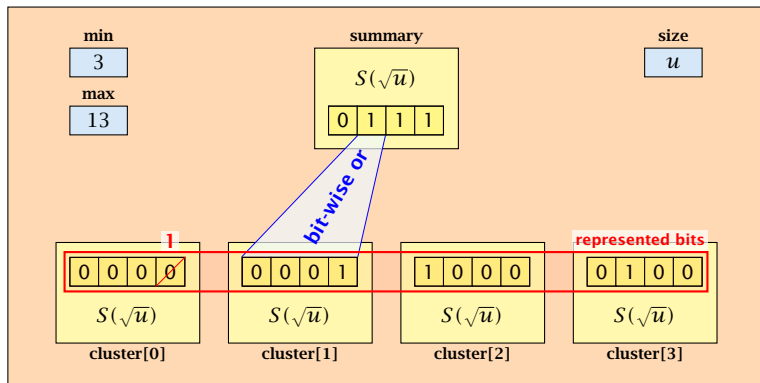
Set $\ell := \log u$ and $X(\ell) := T_{\text{del}}(2^\ell)$. Then

$$\begin{aligned} X(\ell) &= T_{\text{del}}(2^\ell) = T_{\text{del}}(u) = 2T_{\text{del}}(\sqrt{u}) + c \log u \\ &= 2T_{\text{del}}(2^{\frac{\ell}{2}}) + c\ell = 2X(\frac{\ell}{2}) + c\ell . \end{aligned}$$

Using Master theorem gives $X(\ell) = \Theta(\ell \log \ell)$, and hence $T_{\text{del}}(u) = \mathcal{O}(\log u \log \log u)$.

The same holds for $T_{\text{pred}}(u)$ and $T_{\text{succ}}(u)$.

Implementation 4: van Emde Boas Trees



- ▶ The bit referenced by **min** is **not** set within sub-datastructures.
- ▶ The bit referenced by **max** is set within sub-datastructures (if $\text{max} \neq \text{min}$).

Implementation 4: van Emde Boas Trees

Advantages of having max/min pointers:

Implementation 4: van Emde Boas Trees

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- ▶ Recursive calls for **min** and **max** are constant time.

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- ▶ Recursive calls for `min` and `max` are constant time.
- ▶ `min = null` means that the data-structure is empty.

Implementation 4: van Emde Boas Trees

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- ▶ Recursive calls for **min** and **max** are constant time.
- ▶ **min = null** means that the data-structure is empty.
- ▶ **min = max \neq null** means that the data-structure contains exactly one element.

Implementation 4: van Emde Boas Trees

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Implementation 4: van Emde Boas Trees

Advantages of having max/min pointers:

- ▶ Recursive calls for **min** and **max** are constant time.
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- ▶ **min = max \neq null** means that the data-structure contains exactly one element.
- ▶ We can insert into an empty datastructure in constant time by only setting **min = max = x** .
- ▶ We can delete from a data-structure that just contains one element in constant time by setting **min = max = null**.

Implementation 4: van Emde Boas Trees

Algorithm 20 `max()`

1: **return** `max`;

Algorithm 21 `min()`

1: **return** `min`;

- ▶ Constant time.

Implementation 4: van Emde Boas Trees

Algorithm 22 `member(x)`

```
1: if  $x = \min$  then return 1; // TRUE  
2: return cluster[high(x)].member(low(x));
```

- ▶ $T_{\text{mem}}(u) = T_{\text{mem}}(\sqrt{u}) + 1 \Rightarrow T(u) = \mathcal{O}(\log \log u)$.

Implementation 4: van Emde Boas Trees

Algorithm 23 $\text{succ}(x)$

```
1: if  $\text{min} \neq \text{null} \wedge x < \text{min}$  then return  $\text{min}$ ;  
2:  $\text{maxincluster} \leftarrow \text{cluster}[\text{high}(x)].\text{max}()$ ;  
3: if  $\text{maxincluster} \neq \text{null} \wedge \text{low}(x) < \text{maxincluster}$  then  
4:    $\text{offs} \leftarrow \text{cluster}[\text{high}(x)].\text{succ}(\text{low}(x))$ ;  
5:   return  $\text{high}(x) \circ \text{offs}$ ;  
6: else  
7:    $\text{succcluster} \leftarrow \text{summary}.\text{succ}(\text{high}(x))$ ;  
8:   if  $\text{succcluster} = \text{null}$  then return  $\text{null}$ ;  
9:    $\text{offs} \leftarrow \text{cluster}[\text{succcluster}].\text{min}()$ ;  
10:  return  $\text{succcluster} \circ \text{offs}$ ;
```

► $T_{\text{succ}}(u) = T_{\text{succ}}(\sqrt{u}) + 1 \implies T_{\text{succ}}(u) = \mathcal{O}(\log \log u)$.

Implementation 4: van Emde Boas Trees

Algorithm 35 insert(x)

```
1: if min = null then
2:     min =  $x$ ; max =  $x$ ;
3: else
4:     if  $x < \text{min}$  then exchange  $x$  and min;
5:     if  $x > \text{max}$  then max =  $x$ ;
6:     if cluster[high( $x$ )].min = null; then
7:         summary.insert(high( $x$ ));
8:         cluster[high( $x$ )].insert(low( $x$ ));
9:     else
10:        cluster[high( $x$ )].insert(low( $x$ ));
```

► $T_{\text{ins}}(u) = T_{\text{ins}}(\sqrt{u}) + 1 \Rightarrow T_{\text{ins}}(u) = \mathcal{O}(\log \log u)$.

Implementation 4: van Emde Boas Trees

Note that the recursive call in Line 8 takes constant time as the if-condition in Line 6 ensures that we are inserting in an empty sub-tree.

The only non-constant recursive calls are the call in Line 7 and in Line 10. These are mutually exclusive, i.e., only one of these calls will actually occur.

From this we get that $T_{\text{ins}}(u) = T_{\text{ins}}(\sqrt{u}) + 1$.

Implementation 4: van Emde Boas Trees

- ▶ **Assumes that x is contained in the structure.**

Algorithm 36 delete(x)

```
1: if min = max then
2:     min = max = null;
3: else
4:     if  $x$  = min then
5:         firstcluster  $\leftarrow$  summary.min();
6:         offs  $\leftarrow$  cluster[firstcluster].min();
7:          $x \leftarrow$  firstcluster  $\circ$  offs;
8:         min  $\leftarrow$   $x$ ;
9:     cluster[high( $x$ )].delete(low( $x$ ));
```

continued...

Implementation 4: van Emde Boas Trees

- Assumes that x is contained in the structure.

Algorithm 36 delete(x)

```
1: if min = max then
2:     min = max = null;
3: else
4:     if  $x = \text{min}$  then find new minimum
5:          $\text{firstcluster} \leftarrow \text{summary.min}()$ ;
6:          $\text{offs} \leftarrow \text{cluster}[\text{firstcluster}].\text{min}()$ ;
7:          $x \leftarrow \text{firstcluster} \circ \text{offs}$ ;
8:         min  $\leftarrow x$ ;
9:         cluster[high( $x$ )].delete(low( $x$ ));
continued...
```

Implementation 4: van Emde Boas Trees

- ▶ **Assumes that x is contained in the structure.**

Algorithm 36 delete(x)

```
1: if min = max then  
2:     min = max = null;  
3: else  
4:     if  $x$  = min then  
5:         firstcluster  $\leftarrow$  summary.min();  
6:         offs  $\leftarrow$  cluster[firstcluster].min();  
7:          $x \leftarrow$  firstcluster  $\circ$  offs;  
8:         min  $\leftarrow$   $x$ ;  
9:     cluster[high( $x$ )].delete(low( $x$ )); delete
```

continued...

Implementation 4: van Emde Boas Trees

Algorithm 36 delete(x)

...continued

```
10:   if cluster[high( $x$ )].min() = null then
11:       summary.delete(high( $x$ ));
12:   if  $x$  = max then
13:       summax  $\leftarrow$  summary.max();
14:       if summax = null then max  $\leftarrow$  min;
15:       else
16:           offs  $\leftarrow$  cluster[summax].max();
17:           max  $\leftarrow$  summax  $\circ$  offs
18:   else
19:       if  $x$  = max then
20:           offs  $\leftarrow$  cluster[high( $x$ )].max();
21:           max  $\leftarrow$  high( $x$ )  $\circ$  offs;
```


Implementation 4: van Emde Boas Trees

Algorithm 36 delete(x)

...continued

fix maximum

```
10:   if cluster[high( $x$ )].min() = null then
11:       summary.delete(high( $x$ ));
12:       if  $x$  = max then
13:            $summax \leftarrow$  summary.max();
14:           if  $summax$  = null then max  $\leftarrow$  min;
15:           else
16:                $offs \leftarrow$  cluster[ $summax$ ].max();
17:               max  $\leftarrow$   $summax \circ offs$ 
18:       else
19:           if  $x$  = max then
20:                $offs \leftarrow$  cluster[high( $x$ )].max();
21:               max  $\leftarrow$  high( $x$ )  $\circ$   $offs$ ;
```

Implementation 4: van Emde Boas Trees

Note that only one of the possible recursive calls in Line 9 and Line 11 in the deletion-algorithm may take non-constant time.

To see this observe that the call in Line 11 only occurs if the cluster where x was deleted is now empty. But this means that the call in Line 9 deleted the last element in $\text{cluster}[\text{high}(x)]$. Such a call only takes constant time.

Hence, we get a recurrence of the form

$$T_{\text{del}}(u) = T_{\text{del}}(\sqrt{u}) + c .$$

This gives $T_{\text{del}}(u) = \mathcal{O}(\log \log u)$.

7.6 van Emde Boas Trees

Space requirements:

- ▶ The space requirement fulfills the recurrence

$$S(u) = (\sqrt{u} + 1)S(\sqrt{u}) + \mathcal{O}(\sqrt{u}) .$$

- ▶ Note that we cannot solve this recurrence by the Master theorem as the branching factor is not constant.
- ▶ One can show by induction that the space requirement is $S(u) = \mathcal{O}(u)$. Exercise.

- ▶ Let the “real” recurrence relation be

$$S(k^2) = (k + 1)S(k) + c_1 \cdot k; S(4) = c_2$$

- ▶ Replacing $S(k)$ by $R(k) := S(k)/c_2$ gives the recurrence

$$R(k^2) = (k + 1)R(k) + ck; R(4) = 1$$

where $c = c_1/c_2 < 1$.

- ▶ Now, we show $R(k^2) \leq k^2 - 2$ for $k^2 \geq 4$.
 - ▶ Obviously, this holds for $k^2 = 4$.
 - ▶ For $k^2 > 4$ we have

$$\begin{aligned} R(k^2) &= (1 + k)R(k) + ck \\ &\leq (1 + k)(k - 2) + k \leq k^2 - 2 \end{aligned}$$

- ▶ This shows that $R(k)$ and, hence, $S(k)$ grows linearly.

7.7 Hashing

Dictionary:

- ▶ **$S.$ insert(x)**: Insert an element x .
- ▶ **$S.$ delete(x)**: Delete the element pointed to by x .
- ▶ **$S.$ search(k)**: Return a pointer to an element e with $\text{key}[e] = k$ in S if it exists; otherwise return **null**.

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So far we have implemented the search for a key by carefully choosing split-elements.

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Then the memory location of an object x with key k is determined by successively comparing k to split-elements.

Hashing tries to **directly** compute the memory location from the given key. The goal is to have constant search time.

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Definitions:

- ▶ Universe U of keys, e.g., $U \subseteq \mathbb{N}_0$. U very large.

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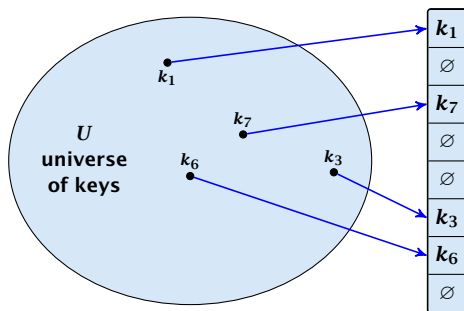
- ▶ Universe U of keys, e.g., $U \subseteq \mathbb{N}_0$. U very large.
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The hash-function h should fulfill:

- ▶ Fast to evaluate.
- ▶ Small storage requirement.
- ▶ Good distribution of elements over the whole table.

Direct Addressing

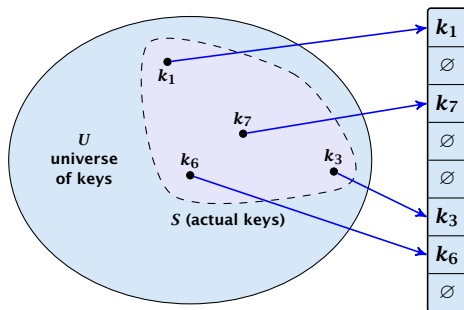
Ideally the hash function maps **all** keys to different memory locations.



This special case is known as **Direct Addressing**. It is usually very unrealistic as the universe of keys typically is quite large, and in particular larger than the available memory.

Perfect Hashing

Suppose that we **know** the set S of actual keys (no insert/no delete). Then we may want to design a **simple** hash-function that maps all these keys to different memory locations.



Such a hash function h is called a **perfect hash function** for set S .

Collisions

If we do not know the keys in advance, the best we can hope for is that the hash function distributes keys evenly across the table.

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Hence, there may be two elements k_1, k_2 from the set S that map to the same memory location (i.e., $h(k_1) = h(k_2)$). This is called a **collision**.

Collisions

Typically, collisions do not appear once the size of the set S of actual keys gets close to n , but already when $|S| \geq \omega(\sqrt{n})$.

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Lemma 20

The probability of having a collision when hashing m elements into a table of size n under uniform hashing is at least

$$1 - e^{-\frac{m(m-1)}{2n}} \approx 1 - e^{-\frac{m^2}{2n}} .$$

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$$1 - e^{-\frac{m(m-1)}{2n}} \approx 1 - e^{-\frac{m^2}{2n}} .$$

Uniform hashing:

Choose a hash function uniformly at random from all functions $f : U \rightarrow [0, \dots, n-1]$.

Collisions

Proof.

Let $A_{m,n}$ denote the event that inserting m keys into a table of size n does **not** generate a collision. Then

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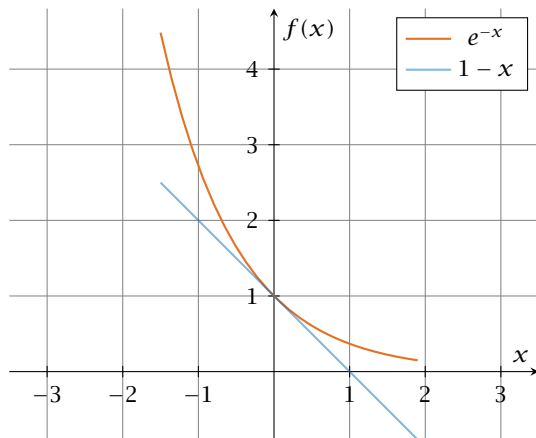
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Here the first equality follows since the ℓ -th element that is hashed has a probability of $\frac{n-\ell+1}{n}$ to not generate a collision under the condition that the previous elements did not induce collisions. □

Collisions



The inequality $1 - x \leq e^{-x}$ is derived by stopping the Taylor-expansion of e^{-x} after the second term.

Resolving Collisions

The methods for dealing with collisions can be classified into the two main types

- ▶ **open addressing**, aka. closed hashing
- ▶ **hashing with chaining**, aka. closed addressing, open hashing.

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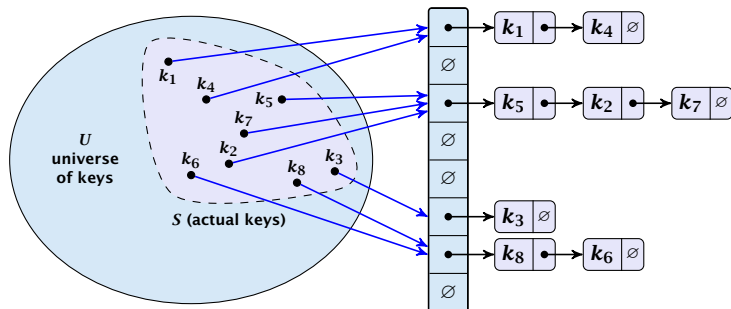
- ▶ **open addressing**, aka. closed hashing
- ▶ **hashing with chaining**, aka. closed addressing, open hashing.

There are applications e.g. computer chess where you do not resolve collisions at all.

Hashing with Chaining

Arrange elements that map to the same position in a linear list.

- ▶ Access: compute $h(x)$ and search list for $\text{key}[x]$.
- ▶ Insert: insert at the front of the list.



Hashing with Chaining

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We assume **uniform hashing** for the following analysis.

Hashing with Chaining

The time required for an unsuccessful search is 1 plus the length of the list that is examined.

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The time required for an unsuccessful search is 1 plus the length of the list that is examined. The average length of a list is $\alpha = \frac{m}{n}$. Hence, if A is the collision resolving strategy “Hashing with Chaining” we have

$$A^- = 1 + \alpha .$$

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For a successful search observe that we do **not** choose a list at random, but we consider a random key k in the hash-table and ask for the search-time for k .

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$$\begin{aligned} \mathbb{E} \left[\frac{1}{m} \sum_{i=1}^m \left(1 + \sum_{j=i+1}^m X_{ij} \right) \right] &= \frac{1}{m} \sum_{i=1}^m \left(1 + \sum_{j=i+1}^m \mathbb{E}[X_{ij}] \right) \\ &= \frac{1}{m} \sum_{i=1}^m \left(1 + \sum_{j=i+1}^m \frac{1}{n} \right) \\ &= 1 + \frac{1}{mn} \sum_{i=1}^m (m - i) \\ &= 1 + \frac{1}{mn} \left(m^2 - \frac{m(m+1)}{2} \right) \\ &= 1 + \frac{m-1}{2n} = 1 + \frac{\alpha}{2} - \frac{\alpha}{2m} . \end{aligned}$$

Hashing with Chaining

$$\begin{aligned} E \left[\frac{1}{m} \sum_{i=1}^m \left(1 + \sum_{j=i+1}^m X_{ij} \right) \right] &= \frac{1}{m} \sum_{i=1}^m \left(1 + \sum_{j=i+1}^m E[X_{ij}] \right) \\ &= \frac{1}{m} \sum_{i=1}^m \left(1 + \sum_{j=i+1}^m \frac{1}{n} \right) \\ &= 1 + \frac{1}{mn} \sum_{i=1}^m (m - i) \\ &= 1 + \frac{1}{mn} \left(m^2 - \frac{m(m+1)}{2} \right) \\ &= 1 + \frac{m-1}{2n} = 1 + \frac{\alpha}{2} - \frac{\alpha}{2m} . \end{aligned}$$

Hence, the expected cost for a successful search is $A^+ \leq 1 + \frac{\alpha}{2}$.

Hashing with Chaining

Disadvantages:

- ▶ pointers increase memory requirements
- ▶ pointers may lead to bad cache efficiency

Advantages:

- ▶ no à priori limit on the number of elements
- ▶ deletion can be implemented efficiently
- ▶ by using balanced trees instead of linked list one can also obtain worst-case guarantees.

Open Addressing

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All objects are stored in the table itself.

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Define a function $h(k, j)$ that determines the table-position to be examined in the j -th step. The values $h(k, 0), \dots, h(k, n - 1)$ must form a permutation of $0, \dots, n - 1$.

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Search(k): Try position $h(k, 0)$; if it is empty your search fails; otw. continue with $h(k, 1), h(k, 2), \dots$

Insert(x): Search until you find an empty slot; insert your element there. If your search reaches $h(k, n - 1)$, and this slot is non-empty then your table is full.

Open Addressing

Choices for $h(k, j)$:

- ▶ **Linear probing:**

$$h(k, i) = h(k) + i \bmod n$$

(sometimes: $h(k, i) = h(k) + ci \bmod n$).

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For quadratic probing and double hashing one has to ensure that the search covers all positions in the table (i.e., for double hashing $h_2(k)$ must be relatively prime to n (**teilerfremd**); for quadratic probing c_1 and c_2 have to be chosen carefully).

Linear Probing

- ▶ Advantage: **Cache-efficiency**. The new probe position is very likely to be in the cache.

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Lemma 21

Let L be the method of linear probing for resolving collisions:

$$L^+ \approx \frac{1}{2} \left(1 + \frac{1}{1 - \alpha} \right)$$

$$L^- \approx \frac{1}{2} \left(1 + \frac{1}{(1 - \alpha)^2} \right)$$

Quadratic Probing

- ▶ Not as cache-efficient as Linear Probing.
- ▶ **Secondary clustering**: caused by the fact that all keys mapped to the same position have the same probe sequence.

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Lemma 22

Let Q be the method of quadratic probing for resolving collisions:

$$Q^+ \approx 1 + \ln\left(\frac{1}{1-\alpha}\right) - \frac{\alpha}{2}$$

$$Q^- \approx \frac{1}{1-\alpha} + \ln\left(\frac{1}{1-\alpha}\right) - \alpha$$

Double Hashing

- ▶ Any probe into the hash-table usually creates a cache-miss.

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Lemma 23

Let D be the method of double hashing for resolving collisions:

$$D^+ \approx \frac{1}{\alpha} \ln \left(\frac{1}{1 - \alpha} \right)$$

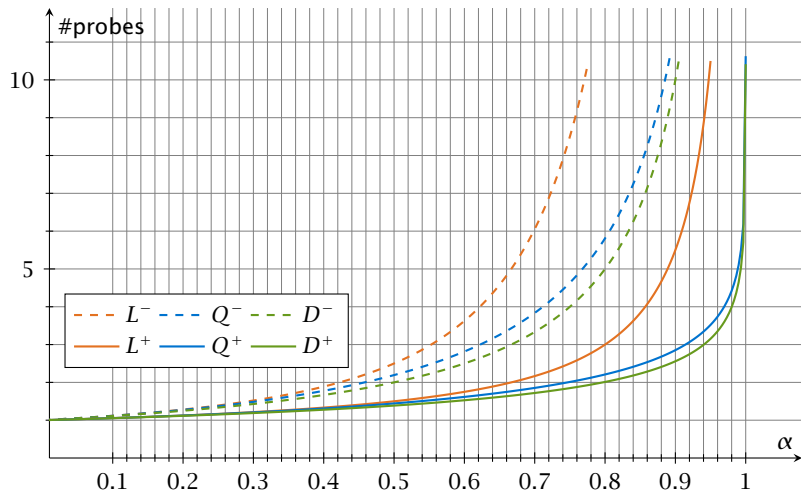
$$D^- \approx \frac{1}{1 - \alpha}$$

Open Addressing

Some values:

α	<i>Linear Probing</i>		<i>Quadratic Probing</i>		<i>Double Hashing</i>	
	L^+	L^-	Q^+	Q^-	D^+	D^-
0.5	1.5	2.5	1.44	2.19	1.39	2
0.9	5.5	50.5	2.85	11.40	2.55	10
0.95	10.5	200.5	3.52	22.05	3.15	20

Open Addressing



Analysis of Idealized Open Address Hashing

We analyze the time for a search in a very idealized Open Addressing scheme.

- ▶ The probe sequence $h(k, 0), h(k, 1), h(k, 2), \dots$ is equally likely to be any permutation of $\langle 0, 1, \dots, n - 1 \rangle$.

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$$\Pr[X \geq i]$$

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$$\Pr[X \geq i] = \frac{m}{n} \cdot \frac{m-1}{n-1} \cdot \frac{m-2}{n-2} \cdot \dots \cdot \frac{m-i+2}{n-i+2}$$

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$E[X]$

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$$E[X] = \sum_{i=1}^{\infty} \Pr[X \geq i]$$

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Analysis of Idealized Open Address Hashing

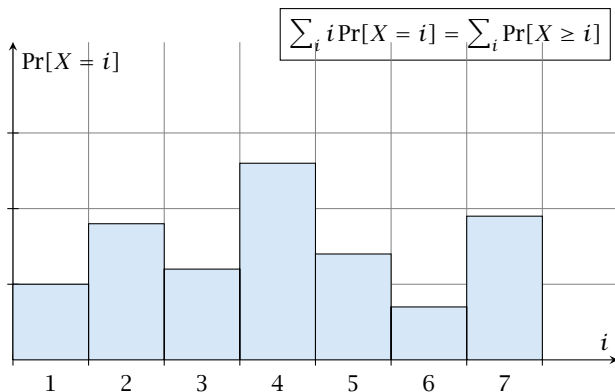
$$E[X] = \sum_{i=1}^{\infty} \Pr[X \geq i] \leq \sum_{i=1}^{\infty} \alpha^{i-1} = \sum_{i=0}^{\infty} \alpha^i = \frac{1}{1-\alpha} .$$

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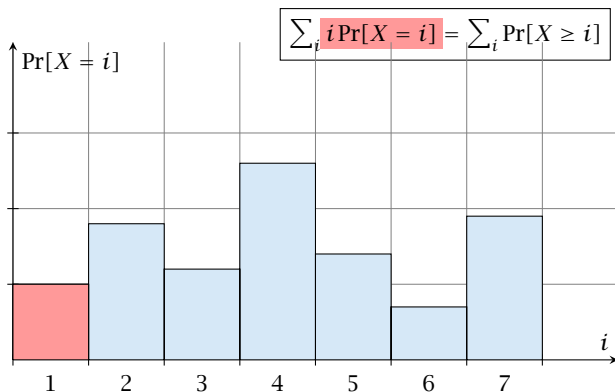
$$\frac{1}{1-\alpha} = 1 + \alpha + \alpha^2 + \alpha^3 + \dots$$

Analysis of Idealized Open Address Hashing



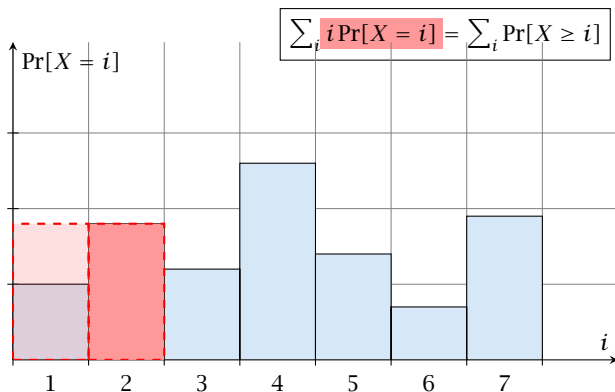
Analysis of Idealized Open Address Hashing

$i = 1$



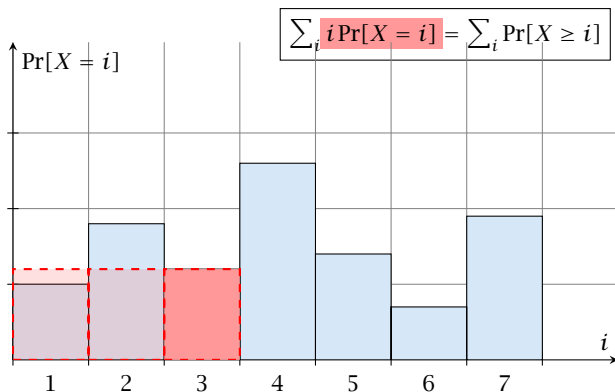
Analysis of Idealized Open Address Hashing

$i = 2$



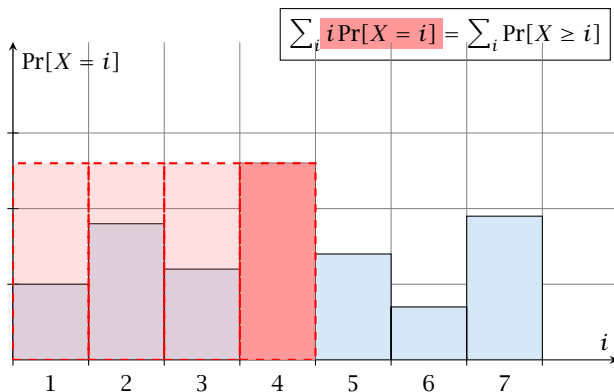
Analysis of Idealized Open Address Hashing

$i = 3$



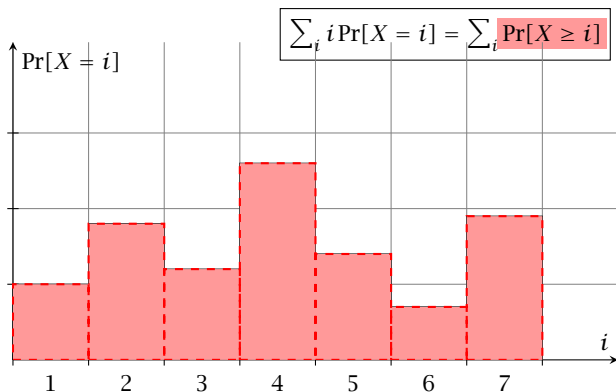
Analysis of Idealized Open Address Hashing

$i = 4$



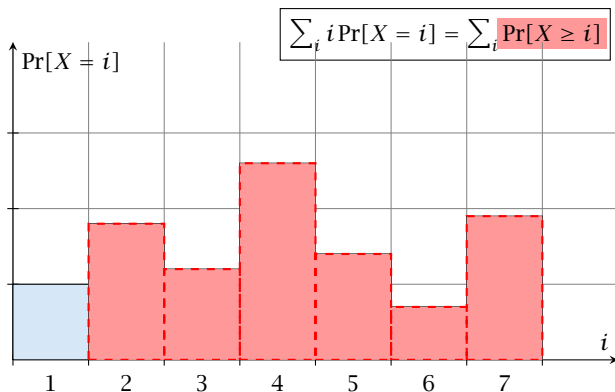
Analysis of Idealized Open Address Hashing

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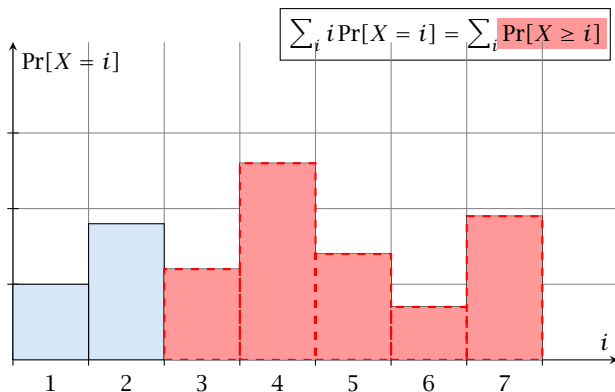
Analysis of Idealized Open Address Hashing

$i = 2$



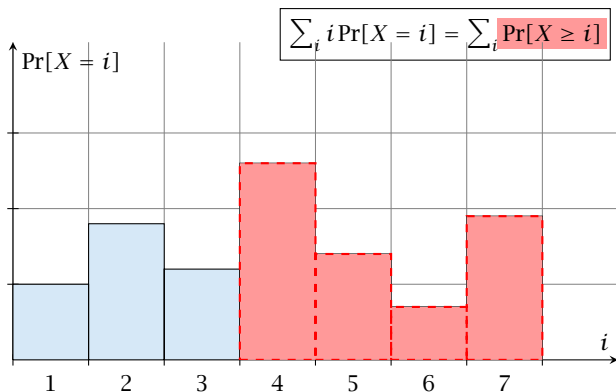
Analysis of Idealized Open Address Hashing

$i = 3$

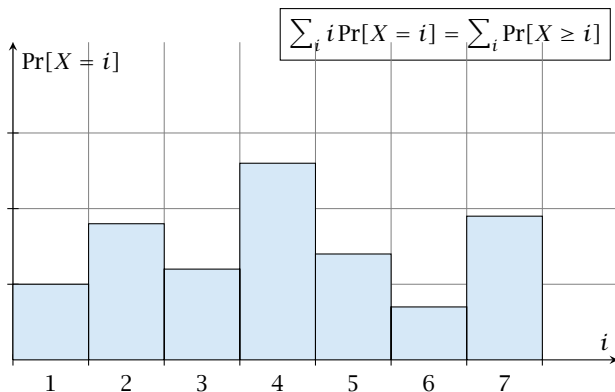


Analysis of Idealized Open Address Hashing

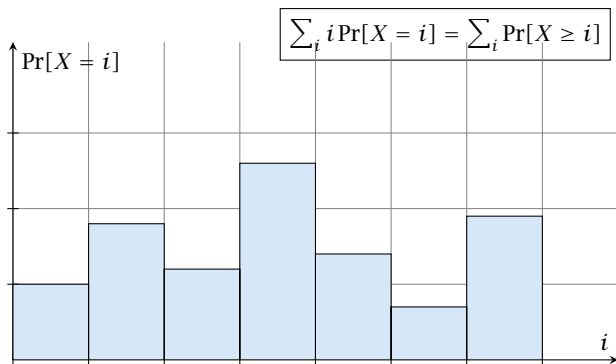
$i = 4$



Analysis of Idealized Open Address Hashing



Analysis of Idealized Open Address Hashing



The j -th rectangle appears in both sums j times. (j times in the first due to multiplication with j ; and j times in the second for summands $i = 1, 2, \dots, j$)

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$$\frac{1}{m} \sum_{i=0}^{m-1} \frac{n}{n-i} = \frac{n}{m} \sum_{i=0}^{m-1} \frac{1}{n-i} = \frac{1}{\alpha} \sum_{k=n-m+1}^n \frac{1}{k}$$

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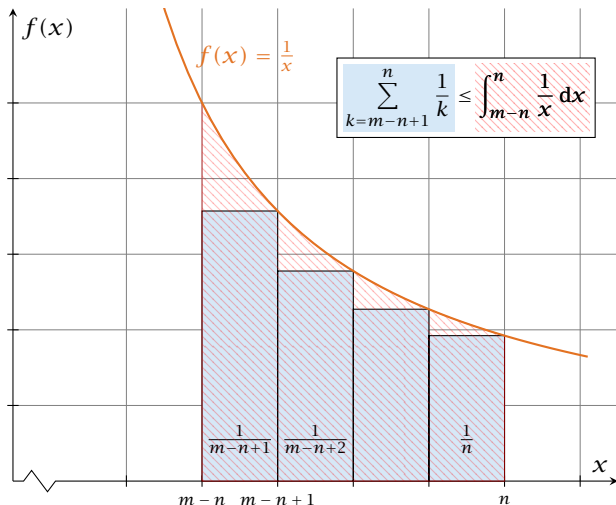
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Deletions in Hashtables

How do we delete in a hash-table?

- ▶ For hashing with chaining this is not a problem. Simply search for the key, and delete the item in the corresponding list.

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- ▶ For open addressing this is difficult.

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- ▶ The table could fill up with **deleted**-markers leading to bad performance.
- ▶ If a table contains many deleted-markers (linear fraction of the keys) one can rehash the whole table and amortize the cost for this rehash against the cost for the deletions.

Deletions for Linear Probing

- ▶ For Linear Probing one can delete elements without using **deletion**-markers.

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- ▶ Upon a deletion elements that are further down in the probe-sequence may be moved to guarantee that they are still found during a search.

Deletions for Linear Probing

Algorithm 37 delete(p)

```
1:  $T[p] \leftarrow \text{null}$ 
2:  $p \leftarrow \text{succ}(p)$ 
3: while  $T[p] \neq \text{null}$  do
4:    $y \leftarrow T[p]$ 
5:    $T[p] \leftarrow \text{null}$ 
6:    $p \leftarrow \text{succ}(p)$ 
7:    $\text{insert}(y)$ 
```

p is the index into the table-cell that contains the object to be deleted.

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Pointers into the hash-table become invalid.

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However, the assumption of uniform hashing that h is chosen randomly from all functions $f : U \rightarrow [0, \dots, n - 1]$ is clearly unrealistic as there are $n^{|U|}$ such functions. Even writing down such a function would take $|U| \log n$ bits.

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Universal hashing tries to define a set \mathcal{H} of functions that is much smaller but still leads to good average case behaviour when selecting a hash-function uniformly at random from \mathcal{H} .

Universal Hashing

Definition 24

A class \mathcal{H} of hash-functions from the universe U into the set $\{0, \dots, n-1\}$ is called **universal** if for all $u_1, u_2 \in U$ with $u_1 \neq u_2$

$$\Pr[h(u_1) = h(u_2)] \leq \frac{1}{n} ,$$

where the probability is w. r. t. the choice of a random hash-function from set \mathcal{H} .

Universal Hashing

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Note that this means that the probability of a collision between two arbitrary elements is at most $\frac{1}{n}$.

Universal Hashing

Definition 25

A class \mathcal{H} of hash-functions from the universe U into the set $\{0, \dots, n-1\}$ is called **2-independent** (pairwise independent) if the following two conditions hold

- ▶ For any key $u \in U$, and $t \in \{0, \dots, n-1\}$ $\Pr[h(u) = t] = \frac{1}{n}$,
i.e., a key is distributed uniformly within the hash-table.
- ▶ For all $u_1, u_2 \in U$ with $u_1 \neq u_2$, and for any two hash-positions t_1, t_2 :

$$\Pr[h(u_1) = t_1 \wedge h(u_2) = t_2] \leq \frac{1}{n^2} .$$

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This requirement clearly implies a universal hash-function.

Universal Hashing

Definition 26

A class \mathcal{H} of hash-functions from the universe U into the set $\{0, \dots, n-1\}$ is called **k -independent** if for any choice of $\ell \leq k$ distinct keys $u_1, \dots, u_\ell \in U$, and for any set of ℓ not necessarily distinct hash-positions t_1, \dots, t_ℓ :

$$\Pr[h(u_1) = t_1 \wedge \dots \wedge h(u_\ell) = t_\ell] \leq \frac{1}{n^\ell} ,$$

where the probability is w. r. t. the choice of a random hash-function from set \mathcal{H} .

Universal Hashing

Definition 27

A class \mathcal{H} of hash-functions from the universe U into the set $\{0, \dots, n-1\}$ is called (μ, k) -independent if for any choice of $\ell \leq k$ distinct keys $u_1, \dots, u_\ell \in U$, and for any set of ℓ not necessarily distinct hash-positions t_1, \dots, t_ℓ :

$$\Pr[h(u_1) = t_1 \wedge \dots \wedge h(u_\ell) = t_\ell] \leq \frac{\mu}{n^\ell},$$

where the probability is w. r. t. the choice of a random hash-function from set \mathcal{H} .

Universal Hashing

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Let $U := \{0, \dots, p - 1\}$ for a prime p . Let $\mathbb{Z}_p := \{0, \dots, p - 1\}$, and let $\mathbb{Z}_p^* := \{1, \dots, p - 1\}$ denote the set of invertible elements in \mathbb{Z}_p .

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Lemma 28

The class

$$\mathcal{H} = \{h_{a,b} \mid a \in \mathbb{Z}_p^*, b \in \mathbb{Z}_p\}$$

is a universal class of hash-functions from U to $\{0, \dots, n-1\}$.

Universal Hashing

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Proof.

Let $x, y \in U$ be two distinct keys. We have to show that the probability of a collision is only $1/n$.

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where we use that \mathbb{Z}_p is a field (**Körper**) and, hence, has no zero divisors (**nullteilerfrei**).

Universal Hashing

- ▶ The hash-function does not generate collisions before the $(\text{mod } n)$ -operation. Furthermore, every choice (a, b) is mapped to a different pair (t_x, t_y) with $t_x := ax + b$ and $t_y := ay + b$.

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$$t_y \equiv ay + b \pmod{p}$$

$$a \equiv (t_x - t_y)(x - y)^{-1} \pmod{p}$$

$$b \equiv t_y - ay \pmod{p}$$

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What happens when we do the $\text{mod } n$ operation?

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From the range $0, \dots, p - 1$ the values $t_x, t_x + n, t_x + 2n, \dots$ map to t_x after the modulo-operation. These are at most $\lceil p/n \rceil$ values.

Universal Hashing

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This happens with probability at most $\frac{1}{n}$.

Universal Hashing

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It is also possible to show that \mathcal{H} is an (almost) pairwise independent class of hash-functions.

$$\Pr_{t_x \neq t_y \in \mathbb{Z}_p^2} \left[\begin{array}{l} t_x \bmod n = h_1 \\ t_y \bmod n = h_2 \end{array} \right]$$

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$$\frac{\left\lfloor \frac{p}{n} \right\rfloor^2}{p(p-1)} \leq \Pr_{t_x \neq t_y \in \mathbb{Z}_p^2} \left[\begin{array}{l} t_x \bmod n = h_1 \\ t_y \bmod n = h_2 \end{array} \right] \leq \frac{\left\lfloor \frac{p}{n} \right\rfloor^2}{p(p-1)}$$

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Note that the middle is the probability that $h(x) = h_1$ and $h(y) = h_2$. The total number of choices for (t_x, t_y) is $p(p-1)$. The number of choices for t_x (t_y) such that $t_x \bmod n = h_1$ ($t_y \bmod n = h_2$) lies between $\lfloor \frac{p}{n} \rfloor$ and $\lceil \frac{p}{n} \rceil$.

Universal Hashing

Definition 29

Let $d \in \mathbb{N}$; $q \geq (d + 1)n$ be a prime; and let $\bar{a} \in \{0, \dots, q - 1\}^{d+1}$. Define for $x \in \{0, \dots, q - 1\}$

$$h_{\bar{a}}(x) := \left(\sum_{i=0}^d a_i x^i \bmod q \right) \bmod n .$$

Let $\mathcal{H}_n^d := \{h_{\bar{a}} \mid \bar{a} \in \{0, \dots, q - 1\}^{d+1}\}$. The class \mathcal{H}_n^d is $(e, d + 1)$ -independent.

Note that in the previous case we had $d = 1$ and chose $a_d \neq 0$.

Universal Hashing

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For the coefficients $\bar{a} \in \{0, \dots, q-1\}^{d+1}$ let $f_{\bar{a}}$ denote the polynomial

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The polynomial is defined by $d+1$ distinct points.

Universal Hashing

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Then

$$h_{\bar{a}} \in A^\ell \Leftrightarrow h_{\bar{a}} = f_{\bar{a}} \bmod n \text{ and}$$

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$$|B_1| \cdot \dots \cdot |B_\ell|$$

possibilities to do this (so that $h_{\bar{a}}(x_i) = t_i$).

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Now, we choose $d - \ell + 1$ other inputs and choose their value arbitrarily. We have $q^{d-\ell+1}$ possibilities to do this.

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Therefore we have

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possibilities to choose \bar{a} such that $h_{\bar{a}} \in A_\ell$.

Universal Hashing

Therefore the probability of choosing $h_{\bar{a}}$ from A_ℓ is only

$$\frac{\left[\frac{q}{n}\right]^\ell \cdot q^{d-\ell+1}}{q^{d+1}}$$

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$$\frac{\lceil \frac{q}{n} \rceil^\ell \cdot q^{d-\ell+1}}{q^{d+1}} \leq \frac{\left(\frac{q+n}{n}\right)^\ell}{q^\ell} \leq \left(\frac{q+n}{q}\right)^\ell \cdot \frac{1}{n^\ell}$$

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Therefore the probability of choosing $h_{\bar{a}}$ from A_ℓ is only

$$\begin{aligned}\frac{\lceil \frac{q}{n} \rceil^\ell \cdot q^{d-\ell+1}}{q^{d+1}} &\leq \frac{(\frac{q+n}{n})^\ell}{q^\ell} \leq \left(\frac{q+n}{q}\right)^\ell \cdot \frac{1}{n^\ell} \\ &\leq \left(1 + \frac{1}{\ell}\right)^\ell \cdot \frac{1}{n^\ell}\end{aligned}$$

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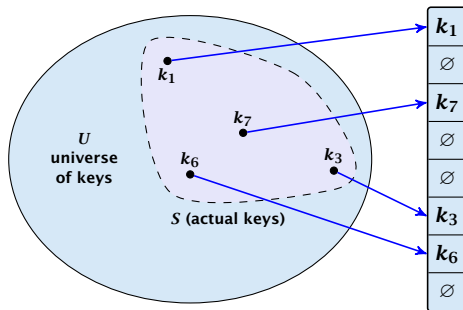
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This shows that the \mathcal{H} is $(e, d+1)$ -universal.

The last step followed from $q \geq (d+1)n$, and $\ell \leq d+1$.

Perfect Hashing

Suppose that we **know** the set S of actual keys (no insert/no delete). Then we may want to design a **simple** hash-function that maps all these keys to different memory locations.



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Let $m = |S|$. We could simply choose the hash-table size very large so that we don't get any collisions.

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$$E[\#\text{Collisions}] = \binom{m}{2} \cdot \frac{1}{n} .$$

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The probability of having **1** or more collisions can be at most $\frac{1}{2}$ as otherwise the expectation would be larger than $\frac{1}{2}$.

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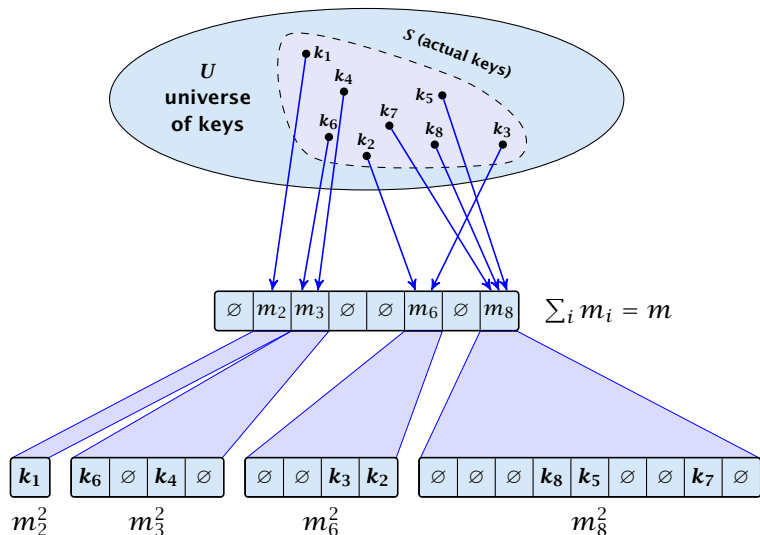
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However, a hash-table size of $n = m^2$ is very very high.

We construct a two-level scheme. We first use a hash-function that maps elements from S to m buckets.

Let m_j denote the number of items that are hashed to the j -th bucket. For each bucket we choose a second hash-function that maps the elements of the bucket into a table of size m_j^2 . The second function can be chosen such that all elements are mapped to different locations.

Perfect Hashing



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The total memory that is required by all hash-tables is $\mathcal{O}(\sum_j m_j^2)$.
Note that m_j is a random variable.

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The first expectation is simply the expected number of collisions, for the first level. Since we use universal hashing we have

Perfect Hashing

The total memory that is required by all hash-tables is $\mathcal{O}(\sum_j m_j^2)$. Note that m_j is a random variable.

$$\begin{aligned} \mathbb{E} \left[\sum_j m_j^2 \right] &= \mathbb{E} \left[2 \sum_j \binom{m_j}{2} + \sum_j m_j \right] \\ &= 2 \mathbb{E} \left[\sum_j \binom{m_j}{2} \right] + \mathbb{E} \left[\sum_j m_j \right] \end{aligned}$$

The first expectation is simply the expected number of collisions, for the first level. Since we use universal hashing we have

$$= 2 \binom{m}{2} \frac{1}{m} + m = 2m - 1 .$$

Perfect Hashing

We need only $\mathcal{O}(m)$ time to construct a hash-function h with $\sum_j m_j^2 = \mathcal{O}(4m)$, because with probability at least $1/2$ a random function from a universal family will have this property.

Then we construct a hash-table h_j for every bucket. This takes expected time $\mathcal{O}(m_j)$ for every bucket. A random function h_j is collision-free with probability at least $1/2$. We need $\mathcal{O}(m_j)$ to test this.

We only need that the hash-functions are chosen from a universal family!!!

Cuckoo Hashing

Cuckoo Hashing

Goal:

Try to generate a hash-table with constant worst-case search time in a dynamic scenario.

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Cuckoo Hashing

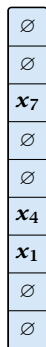
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- ▶ An object x is either stored at location $T_1[h_1(x)]$ or $T_2[h_2(x)]$.
- ▶ A search clearly takes constant time if the above constraint is met.

Cuckoo Hashing

Insert:



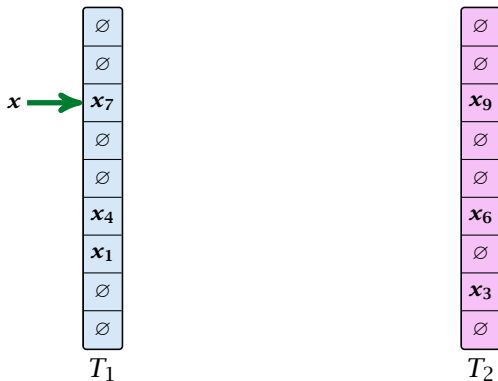
T_1



T_2

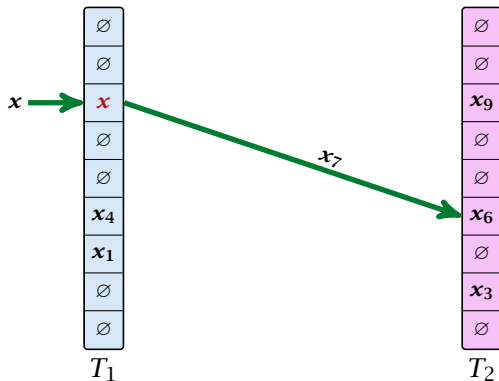
Cuckoo Hashing

Insert:



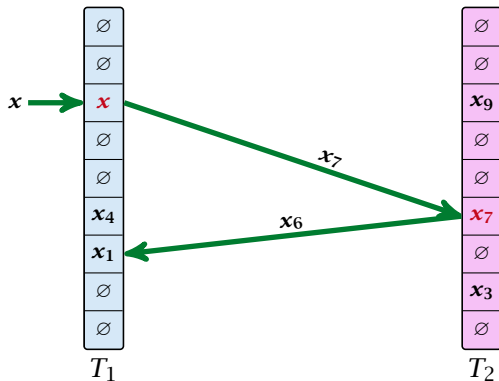
Cuckoo Hashing

Insert:



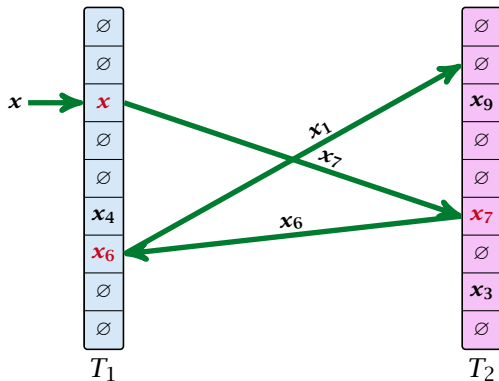
Cuckoo Hashing

Insert:



Cuckoo Hashing

Insert:



Algorithm 38 Cuckoo-Insert(x)

```
1: if  $T_1[h_1(x)] = x \vee T_2[h_2(x)] = x$  then return  
2: steps  $\leftarrow 1$   
3: while steps  $\leq$  maxsteps do  
4:     exchange  $x$  and  $T_1[h_1(x)]$   
5:     if  $x = \text{null}$  then return  
6:     exchange  $x$  and  $T_2[h_2(x)]$   
7:     if  $x = \text{null}$  then return  
8:     steps  $\leftarrow$  steps + 1  
9: rehash() // change hash-functions; rehash everything  
10: Cuckoo-Insert( $x$ )
```

Cuckoo Hashing

- ▶ We call one iteration through the while-loop a **step** of the algorithm.

Cuckoo Hashing

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Cuckoo Hashing

- ▶ We call one iteration through the while-loop a **step** of the algorithm.
- ▶ We call a sequence of iterations through the while-loop without the termination condition becoming true a **phase** of the algorithm.
- ▶ We say a phase is **successful** if it is not terminated by the **maxstep**-condition, but the while loop is left because $x = \text{null}$.

Cuckoo Hashing

What is the expected time for an insert-operation?

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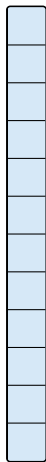
We first analyze the probability that we end-up in an infinite loop (that is then terminated after **maxsteps** steps).

What is the expected time for an insert-operation?

We first analyze the probability that we end-up in an infinite loop (that is then terminated after **maxsteps** steps).

Formally what is the probability to enter an infinite loop that touches s different keys?

Cuckoo Hashing: Insert

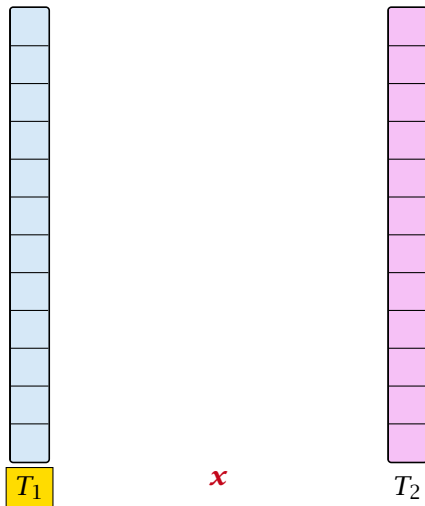


T_1

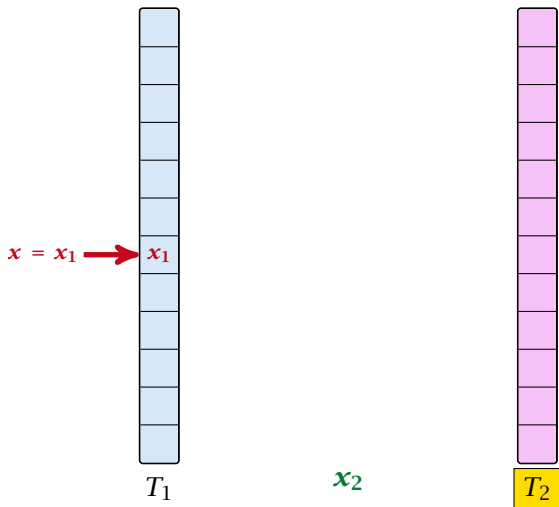


T_2

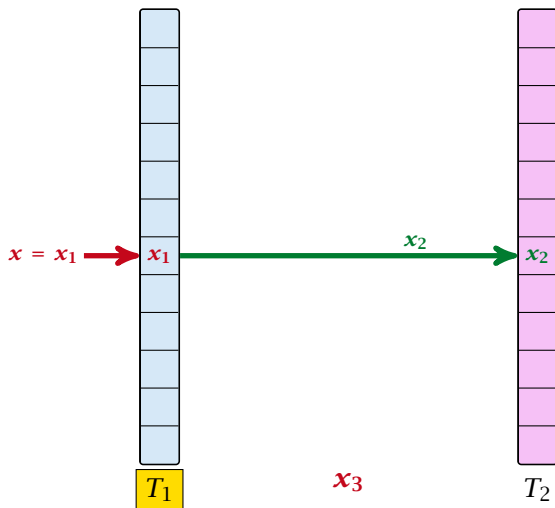
Cuckoo Hashing: Insert



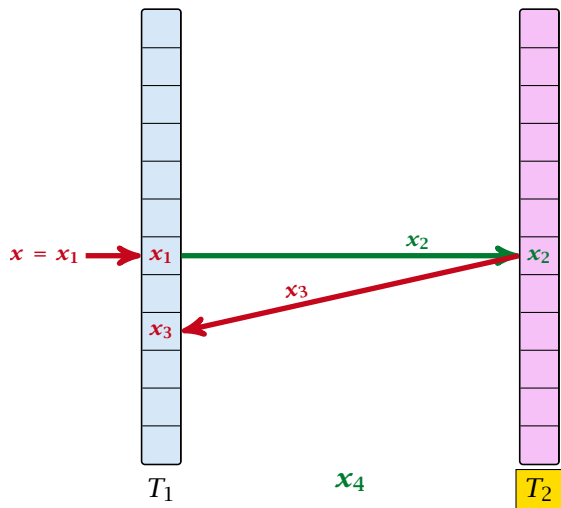
Cuckoo Hashing: Insert



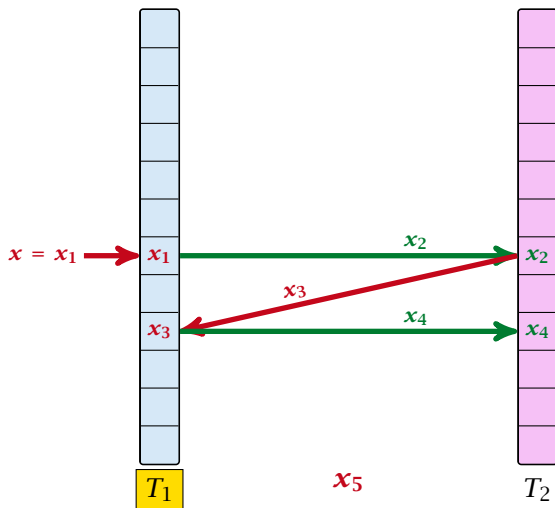
Cuckoo Hashing: Insert



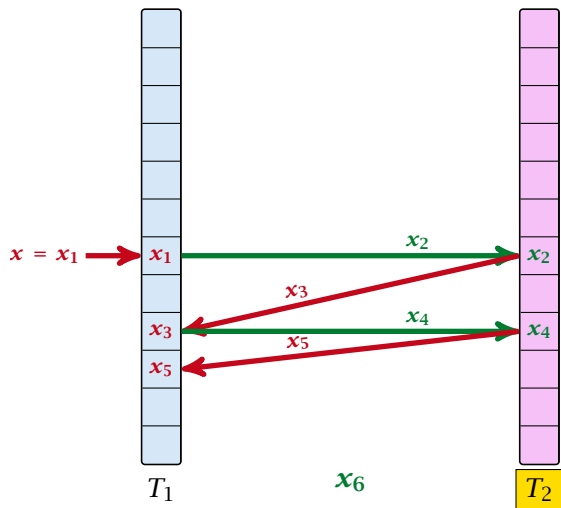
Cuckoo Hashing: Insert



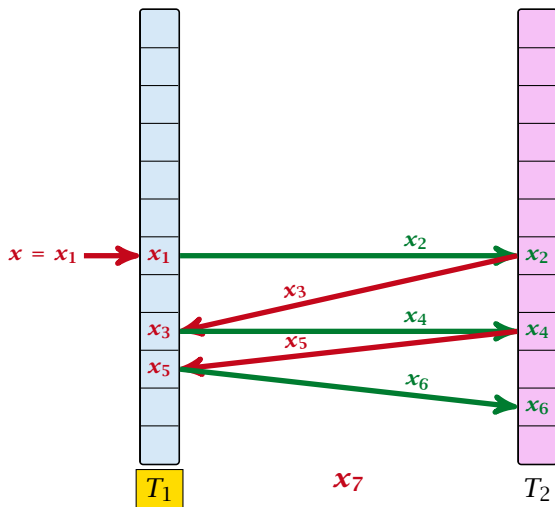
Cuckoo Hashing: Insert



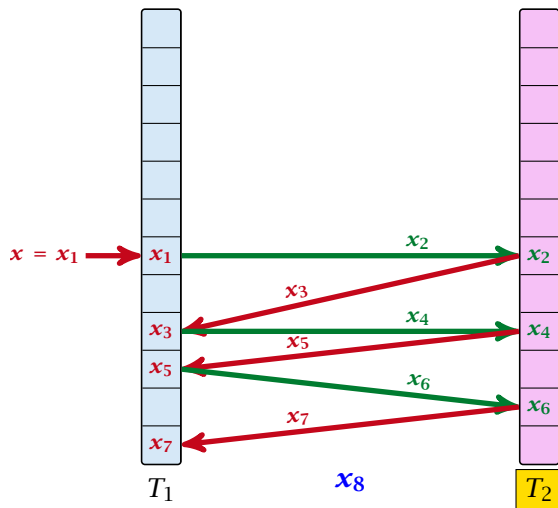
Cuckoo Hashing: Insert



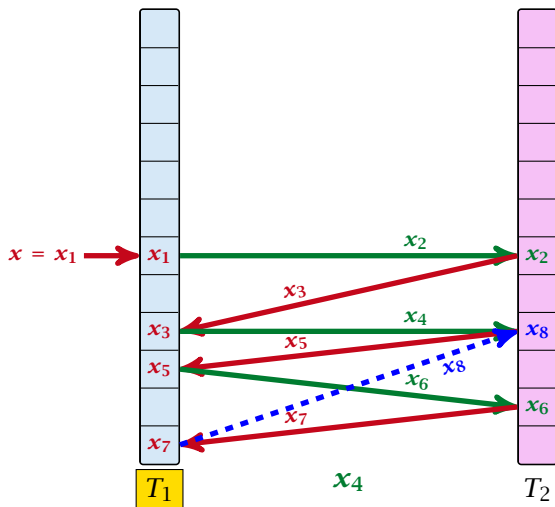
Cuckoo Hashing: Insert



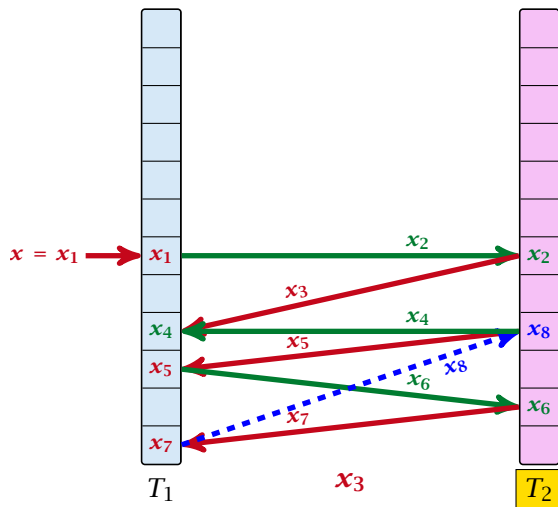
Cuckoo Hashing: Insert



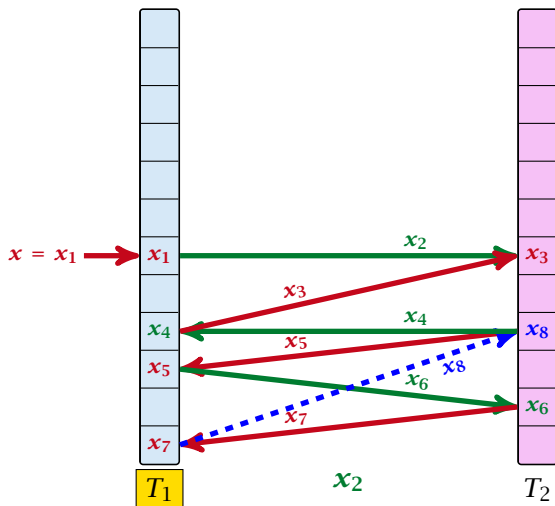
Cuckoo Hashing: Insert



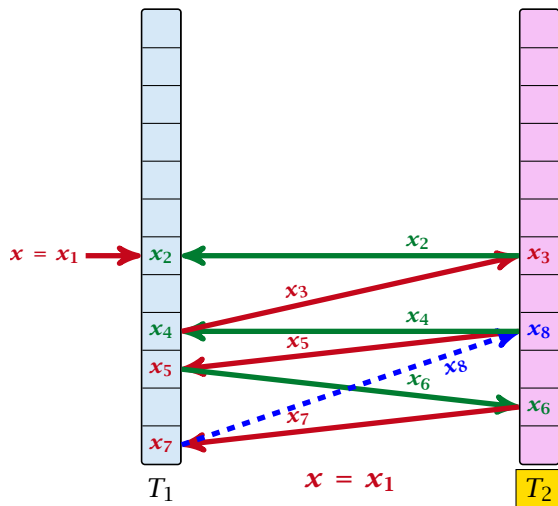
Cuckoo Hashing: Insert



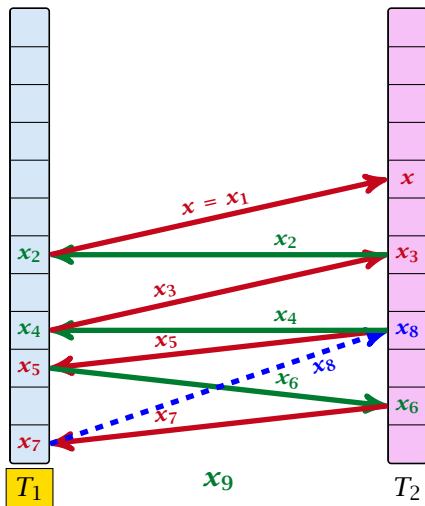
Cuckoo Hashing: Insert



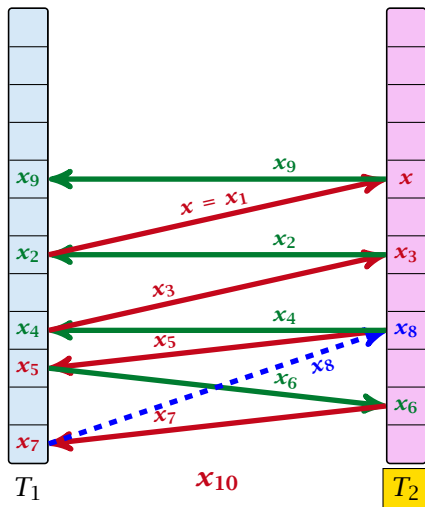
Cuckoo Hashing: Insert



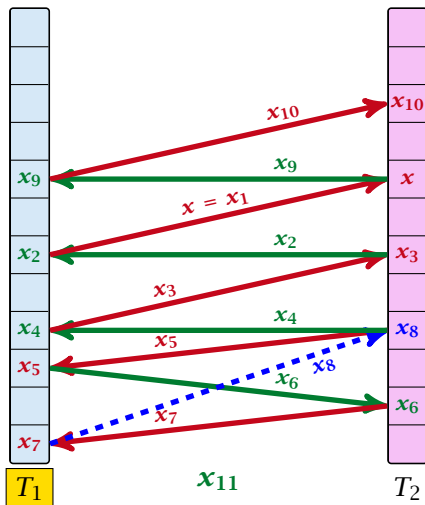
Cuckoo Hashing: Insert



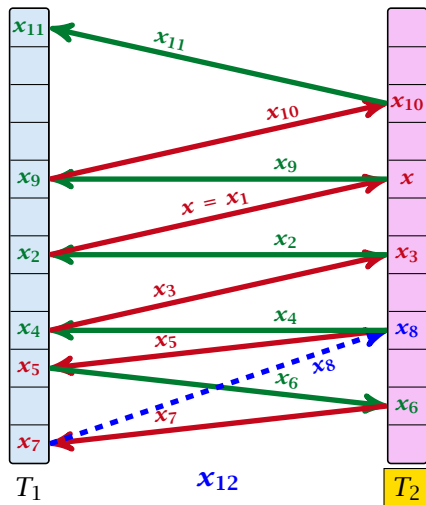
Cuckoo Hashing: Insert



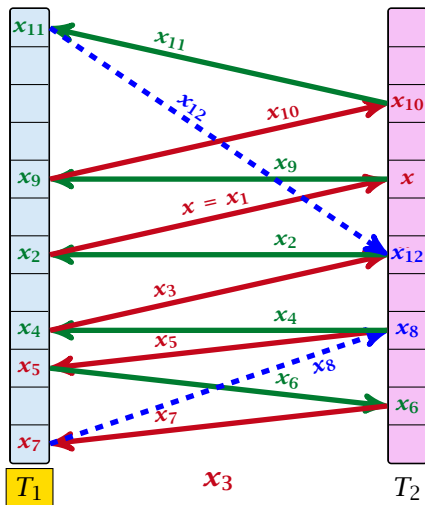
Cuckoo Hashing: Insert



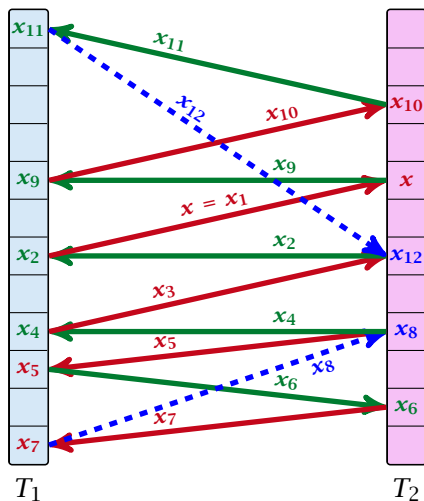
Cuckoo Hashing: Insert



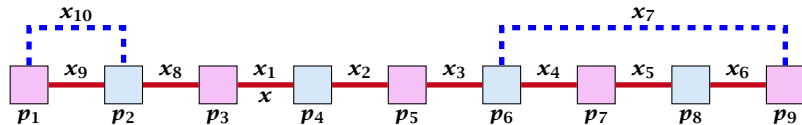
Cuckoo Hashing: Insert



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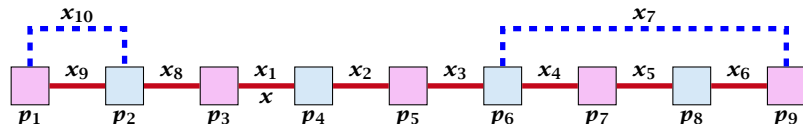


Cuckoo Hashing



A cycle-structure of size s is defined by

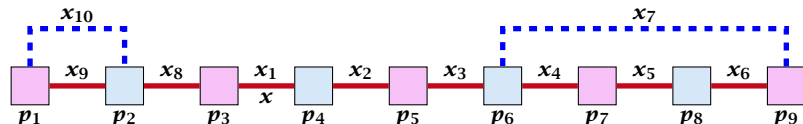
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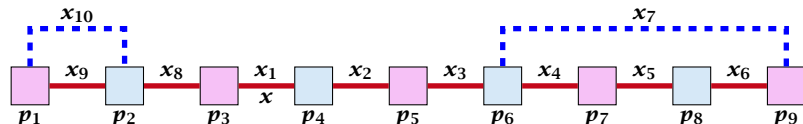
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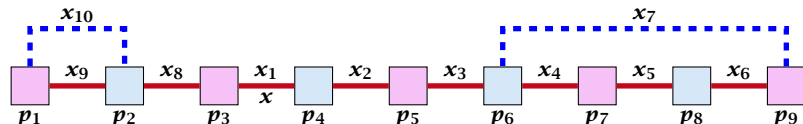
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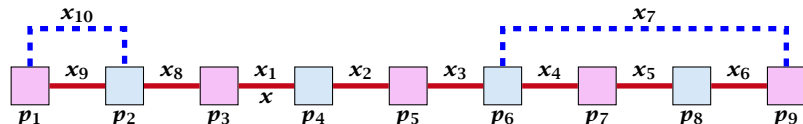
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- ▶ The leftmost cell is “linked forward” to some cell on the right.
- ▶ The rightmost cell is “linked backward” to a cell on the left.
- ▶ One link represents key x ; this is where the counting starts.

Cuckoo Hashing

A cycle-structure is **active** if for every key x_ℓ (linking a cell p_i from T_1 and a cell p_j from T_2) we have

$$h_1(x_\ell) = p_i \quad \text{and} \quad h_2(x_\ell) = p_j$$

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Observation:

If during a phase the insert-procedure runs into a cycle there must exist an active cycle structure of size $s \geq 3$.

Cuckoo Hashing

What is the probability that all keys in a cycle-structure of size s correctly map into their T_1 -cell?

Cuckoo Hashing

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These events are independent.

Cuckoo Hashing

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What is the probability that **there exists** an active cycle structure of size s ?

Cuckoo Hashing

The number of cycle-structures of size s is at most

$$s^3 \cdot n^{s-1} \cdot m^{s-1} .$$

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- ▶ There are m^{s-1} possibilities to choose the keys apart from x .
- ▶ There are n^{s-1} possibilities to choose the cells.

Cuckoo Hashing

The probability that there exists an active cycle-structure is therefore at most

$$\sum_{s=3}^{\infty} s^3 \cdot n^{s-1} \cdot m^{s-1} \cdot \frac{\mu^2}{n^{2s}}$$

Cuckoo Hashing

The probability that there exists an active cycle-structure is therefore at most

$$\sum_{s=3}^{\infty} s^3 \cdot n^{s-1} \cdot m^{s-1} \cdot \frac{\mu^2}{n^{2s}} = \frac{\mu^2}{nm} \sum_{s=3}^{\infty} s^3 \left(\frac{m}{n}\right)^s$$

Cuckoo Hashing

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Here we used the fact that $(1 + \epsilon)m \leq n$.

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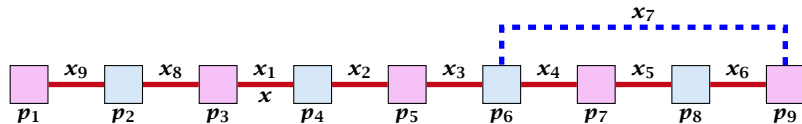
Hence,

$$\Pr[\text{cycle}] = \mathcal{O}\left(\frac{1}{m^2}\right).$$

Cuckoo Hashing

Now, we analyze the probability that a phase is not successful without running into a closed cycle.

Cuckoo Hashing



Sequence of visited keys:

$x = x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_3, x_2, x_1 = x, x_8, x_9, \dots$

Cuckoo Hashing

Consider the sequence of not necessarily distinct keys starting with x in the order that they are visited during the phase.

Cuckoo Hashing

Consider the sequence of not necessarily distinct keys starting with x in the order that they are visited during the phase.

Lemma 30

*If the sequence is of length p then there exists a sub-sequence of at least $\frac{p+2}{3}$ keys starting with x of *distinct* keys.*

Cuckoo Hashing

Proof.

Let i be the number of keys (including x) that we see before the first repeated key. Let j denote the total number of distinct keys.

The sequence is of the form:

$$x = x_1 \rightarrow x_2 \rightarrow \dots \rightarrow x_i \rightarrow x_r \rightarrow x_{r-1} \rightarrow \dots \rightarrow x_1 \rightarrow x_{i+1} \rightarrow \dots \rightarrow x_j$$

As $r \leq i - 1$ the length p of the sequence is

$$p = i + r + (j - i) \leq i + j - 1 .$$

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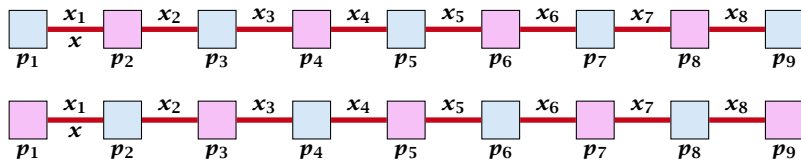
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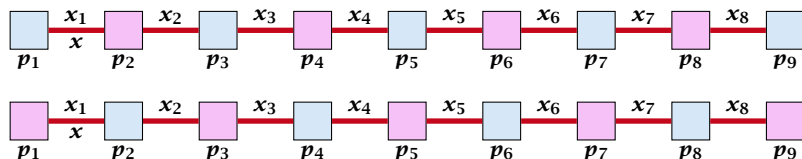
Either sub-sequence $x_1 \rightarrow x_2 \rightarrow \dots \rightarrow x_i$ or sub-sequence $x_1 \rightarrow x_{i+1} \rightarrow \dots \rightarrow x_j$ has at least $\frac{p+2}{3}$ elements. □

Cuckoo Hashing



A path-structure of size s is defined by

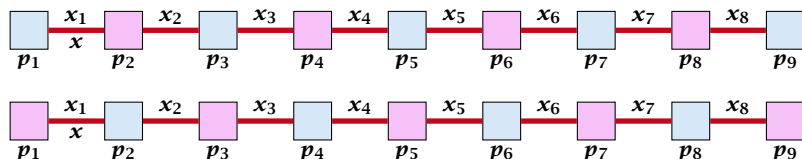
Cuckoo Hashing



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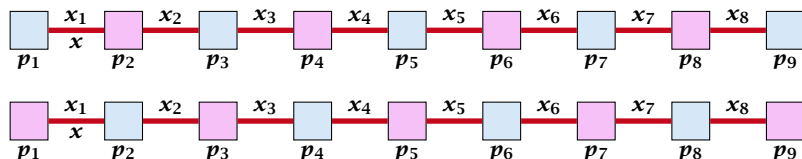
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A path-structure is **active** if for every key x_ℓ (linking a cell p_i from T_1 and a cell p_j from T_2) we have

$$h_1(x_\ell) = p_i \quad \text{and} \quad h_2(x_\ell) = p_j$$

Observation:

If a phase takes at least t steps without running into a cycle there must exist an active path-structure of size $(2t + 2)/3$.

Cuckoo Hashing

The probability that a given path-structure of size s is active is at most $\frac{\mu^2}{n^{2s}}$.

Cuckoo Hashing

The probability that a given path-structure of size s is active is at most $\frac{\mu^2}{n^{2s}}$.

The probability that there exists an active path-structure of size s is at most

$$2 \cdot n^{s+1} \cdot m^{s-1} \cdot \frac{\mu^2}{n^{2s}}$$

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The probability that a given path-structure of size s is active is at most $\frac{\mu^2}{n^{2s}}$.

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This means the expected cost for a successful phase is constant (even after accounting for the cost of the incomplete step that finishes the phase).

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Therefore the expected cost for re-hashes is $\mathcal{O}(m) \cdot \mathcal{O}(p) = \mathcal{O}(1)$.

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Therefore, it is sufficient to have $(\mu, \Theta(\log m))$ -independent hash-functions.

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- ▶ Note that right after a change in table-size we have $m = \alpha n/2$. In order for a table-expand to occur at least $\alpha n/2$ insertions are required. Similar, for a table-shrink at least $\alpha n/4$ deletions must occur.

Cuckoo Hashing

How do we make sure that $n \geq (1 + \epsilon)m$?

- ▶ Let $\alpha := 1/(1 + \epsilon)$.
- ▶ Keep track of the number of elements in the table. When $m \geq \alpha n$ we double n and do a complete re-hash (**table-expand**).
- ▶ Whenever m drops below $\alpha n/4$ we divide n by 2 and do a rehash (**table-shrink**).
- ▶ Note that right after a change in table-size we have $m = \alpha n/2$. In order for a table-expand to occur at least $\alpha n/2$ insertions are required. Similar, for a table-shrink at least $\alpha n/4$ deletions must occur.
- ▶ Therefore we can amortize the rehash cost after a change in table-size against the cost for insertions and deletions.

Cuckoo Hashing

Lemma 31

Cuckoo Hashing has an expected constant insert-time and a worst-case constant search-time.

Cuckoo Hashing

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Cuckoo Hashing has an expected constant insert-time and a worst-case constant search-time.

Note that the above lemma only holds if the fill-factor (number of keys/total number of hash-table slots) is at most $\frac{1}{2(1+\epsilon)}$.

The $1/(2(1+\epsilon))$ fill-factor comes from the fact that the total hash-table is of size $2n$ (because we have two tables of size n); moreover $m \leq (1+\epsilon)n$.

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- ▶ **boolean S . is-empty()**: Returns **true** if the data-structure is empty and false otherwise.

Sometimes we also have

- ▶ **S . merge(S')**: $S := S \cup S'$; $S' := \emptyset$.

8 Priority Queues

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- ▶ **handle S . insert(x)**: Adds element x to the data-structure, and returns a **handle** to the object for future reference.

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8 Priority Queues

An **addressable Priority Queue** also supports:

- ▶ **handle S . insert(x)**: Adds element x to the data-structure, and returns a **handle** to the object for future reference.
- ▶ **S . delete(h)**: Deletes element specified through handle h .
- ▶ **S . decrease-key(h, k)**: Decreases the key of the element specified by handle h to k . Assumes that the key is at least k before the operation.

Dijkstra's Shortest Path Algorithm

Algorithm 39 Shortest-Path($G = (V, E, d), s \in V$)

```
1: Input: weighted graph  $G = (V, E, d)$ ; start vertex  $s$ ;  
2: Output: key-field of every node contains distance from  $s$ ;  
3:  $S.build()$ ; // build empty priority queue  
4: for all  $v \in V \setminus \{s\}$  do  
5:      $v.key \leftarrow \infty$ ;  
6:      $h_v \leftarrow S.insert(v)$ ;  
7:  $s.key \leftarrow 0$ ;  $S.insert(s)$ ;  
8: while  $S.is-empty() = false$  do  
9:      $v \leftarrow S.delete-min()$ ;  
10:    for all  $x \in V$  s.t.  $(v, x) \in E$  do  
11:        if  $x.key > v.key + d(v, x)$  then  
12:             $S.decrease-key(h_x, v.key + d(v, x))$ ;  
13:             $x.key \leftarrow v.key + d(v, x)$ ;
```

Prim's Minimum Spanning Tree Algorithm

Algorithm 40 Prim-MST($G = (V, E, d), s \in V$)

```
1: Input: weighted graph  $G = (V, E, d)$ ; start vertex  $s$ ;  
2: Output: pred-fields encode MST;  
3:  $S.build()$ ; // build empty priority queue  
4: for all  $v \in V \setminus \{s\}$  do  
5:      $v.key \leftarrow \infty$ ;  
6:      $h_v \leftarrow S.insert(v)$ ;  
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14:             $x.pred \leftarrow v$ ;
```

Analysis of Dijkstra and Prim

Both algorithms require:

- ▶ 1 build() operation
- ▶ $|V|$ insert() operations
- ▶ $|V|$ delete-min() operations
- ▶ $|V|$ is-empty() operations
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Analysis of Dijkstra and Prim

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- ▶ $|E|$ decrease-key() operations

How good a running time can we obtain?

8 Priority Queues

<i>Operation</i>	<i>Binary Heap</i>	<i>BST</i>	<i>Binomial Heap</i>	<i>Fibonacci Heap*</i>
build	n	$n \log n$	$n \log n$	n
minimum	1	$\log n$	$\log n$	1
is-empty	1	1	1	1
insert	$\log n$	$\log n$	$\log n$	1
delete	$\log n^{**}$	$\log n$	$\log n$	$\log n$
delete-min	$\log n$	$\log n$	$\log n$	$\log n$
decrease-key	$\log n$	$\log n$	$\log n$	1
merge	n	$n \log n$	$\log n$	1

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Note that most applications use **build()** only to create an empty heap which then costs time 1.

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The standard version of binary heaps is not addressable, and hence does not support a delete operation.

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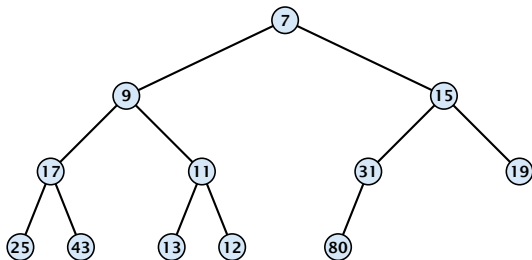
Fibonacci heaps only give an **amortized** guarantee.

8 Priority Queues

Using Binary Heaps, Prim and Dijkstra run in time $\mathcal{O}((|V| + |E|) \log |V|)$.

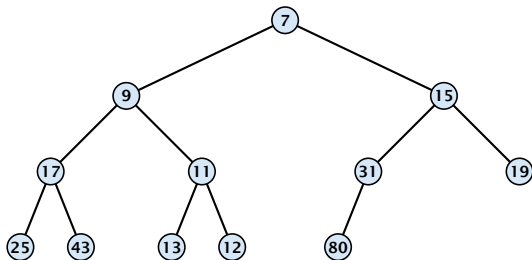
Using Fibonacci Heaps, Prim and Dijkstra run in time $\mathcal{O}(|V| \log |V| + |E|)$.

8.1 Binary Heaps



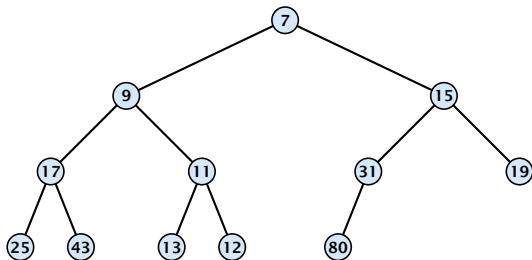
8.1 Binary Heaps

- ▶ Nearly complete binary tree; only the last level is not full, and this one is filled from left to right.



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- ▶ Nearly complete binary tree; only the last level is not full, and this one is filled from left to right.
- ▶ **Heap property:** A node's key is not larger than the key of one of its children.



Binary Heaps

Operations:

Binary Heaps

Operations:

- ▶ **minimum()**: return the root-element. Time $\mathcal{O}(1)$.

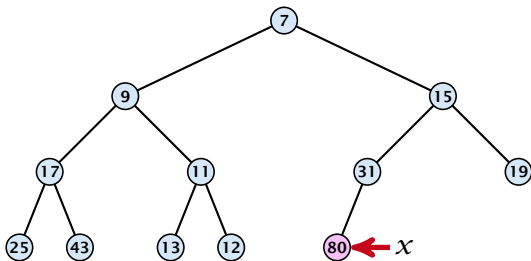
Binary Heaps

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- ▶ **minimum()**: return the root-element. Time $\mathcal{O}(1)$.
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8.1 Binary Heaps

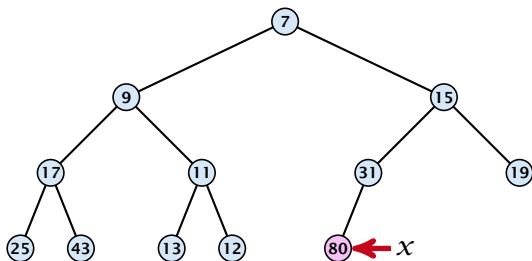
Maintain a pointer to the **last element** x .



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Maintain a pointer to the **last element** x .

- ▶ We can compute the predecessor of x (last element when x is deleted) in time $\mathcal{O}(\log n)$.



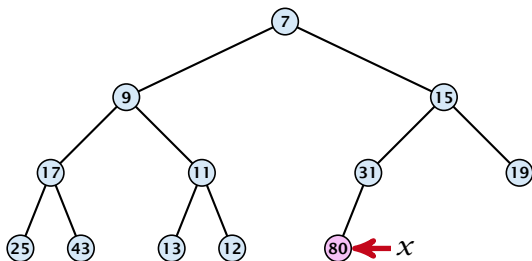
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go up until the last edge used was a right edge.

go left; go right until you reach a leaf



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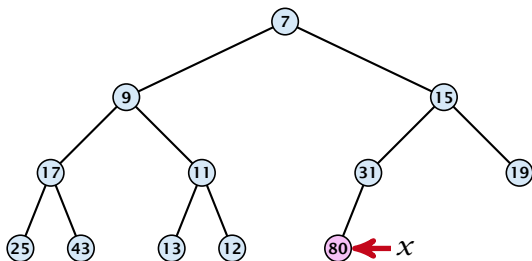
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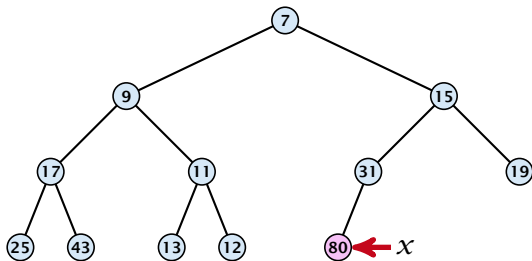
go left; go right until you reach a leaf

if you hit the root on the way up, go to the rightmost element



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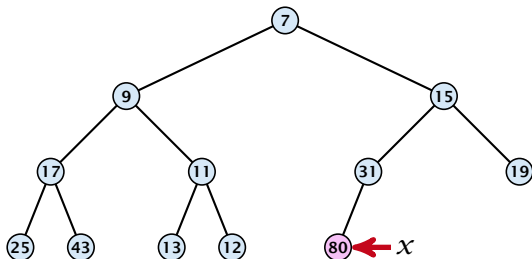
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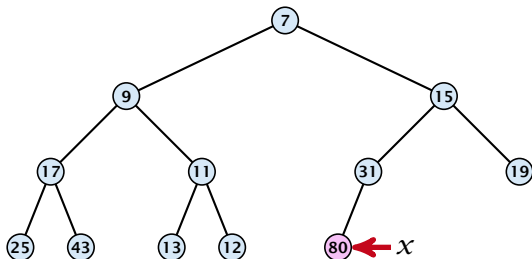
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go right; go left until you reach a **null**-pointer.



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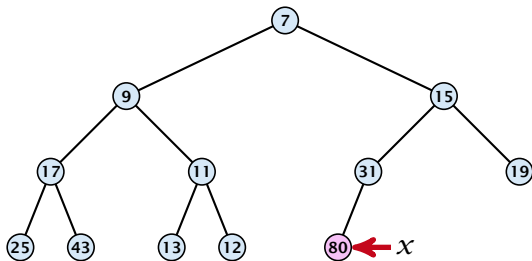
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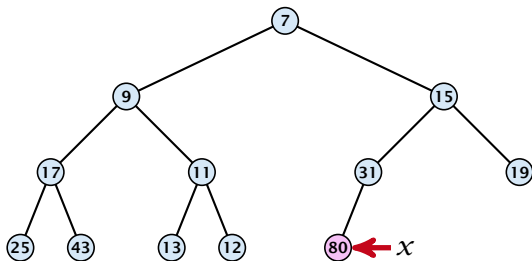
if you hit the root on the way up, go to the leftmost element;

insert a new element as a left child;



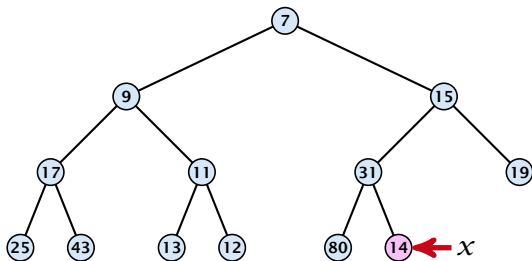
Insert

1. Insert element at successor of x .



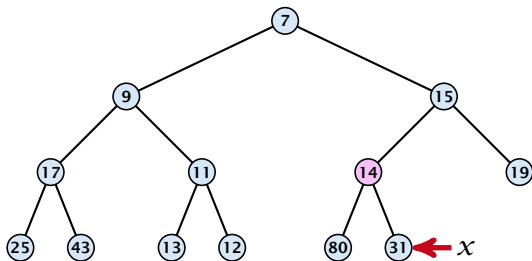
Insert

1. Insert element at successor of x .
2. Exchange with parent until heap property is fulfilled.



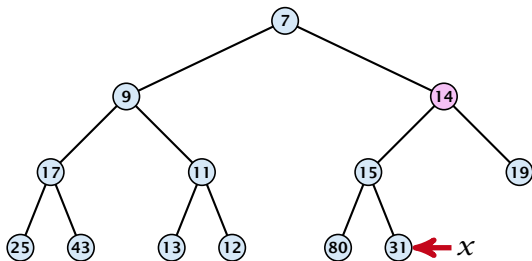
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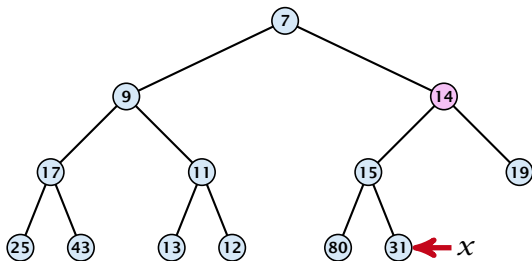
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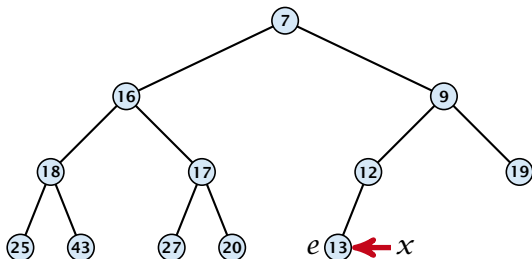
1. Insert element at successor of x .
2. Exchange with parent until heap property is fulfilled.



Note that an exchange can either be done by moving the data or by changing pointers. The latter method leads to an addressable priority queue.

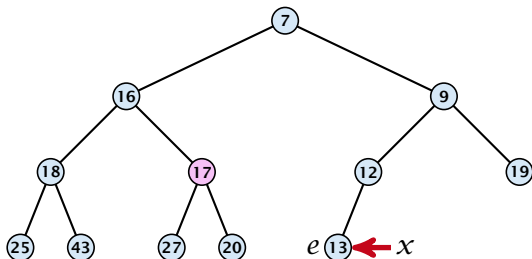
Delete

1. Exchange the element to be deleted with the element e pointed to by x .



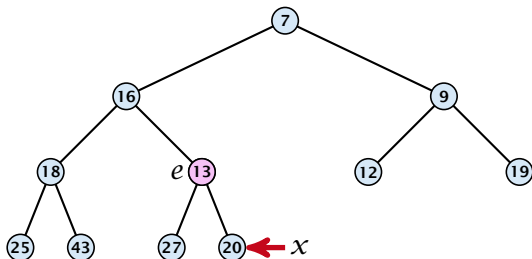
Delete

1. Exchange the element to be deleted with the element e pointed to by x .
2. Restore the heap-property for the element e .



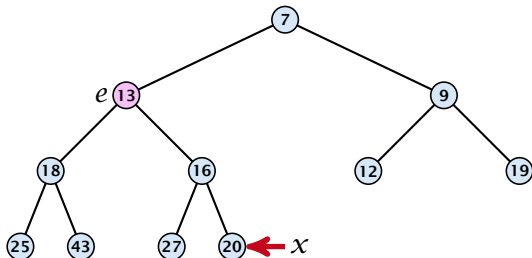
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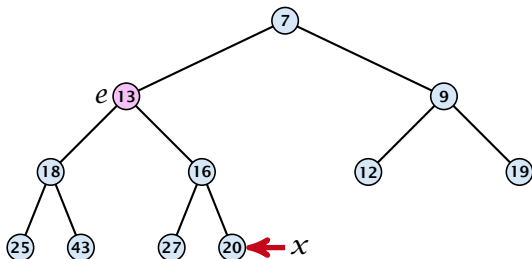
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Delete

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2. Restore the heap-property for the element e .



At its new position e may either travel up or down in the tree (but not both directions).

Binary Heaps

Operations:

- ▶ **minimum()**: return the root-element. Time $\mathcal{O}(1)$.
- ▶ **is-empty()**: check whether root-pointer is **null**. Time $\mathcal{O}(1)$.
- ▶ **insert(k)**: insert at successor of x and bubble up. Time $\mathcal{O}(\log n)$.
- ▶ **delete(h)**: swap with x and bubble up or sift-down. Time $\mathcal{O}(\log n)$.

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- ▶ **delete(h)**: Swap with x and bubble up or sift-down. Time $\mathcal{O}(\log n)$.
- ▶ **build(x_1, \dots, x_n)**: Insert elements arbitrarily; then do sift-down operations starting with the lowest layer in the tree. Time $\mathcal{O}(n)$.

Binary Heaps

Binary Heaps

The standard implementation of binary heaps is via arrays. Let $A[0, \dots, n - 1]$ be an array

- ▶ The parent of i -th element is at position $\lfloor \frac{i-1}{2} \rfloor$.
- ▶ The left child of i -th element is at position $2i + 1$.
- ▶ The right child of i -th element is at position $2i + 2$.

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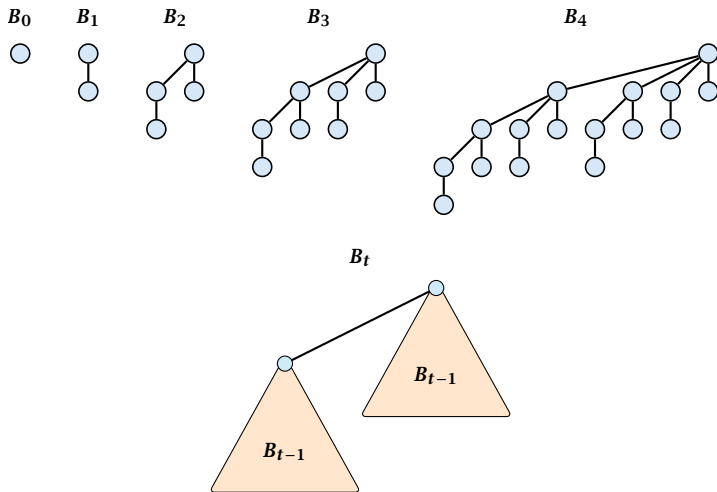
Finding the successor of x is much easier than in the description on the previous slide. Simply increase or decrease x .

The resulting binary heap is not addressable. The elements don't maintain their positions and therefore there are no stable handles.

8.2 Binomial Heaps

<i>Operation</i>	<i>Binary Heap</i>	<i>BST</i>	<i>Binomial Heap</i>	<i>Fibonacci Heap*</i>
build	n	$n \log n$	$n \log n$	n
minimum	1	$\log n$	$\log n$	1
is-empty	1	1	1	1
insert	$\log n$	$\log n$	$\log n$	1
delete	$\log n^{**}$	$\log n$	$\log n$	$\log n$
delete-min	$\log n$	$\log n$	$\log n$	$\log n$
decrease-key	$\log n$	$\log n$	$\log n$	1
merge	n	$n \log n$	$\log n$	1

Binomial Trees



Binomial Trees

Properties of Binomial Trees

- ▶ B_k has 2^k nodes.

Binomial Trees

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Binomial Trees

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- ▶ B_k has 2^k nodes.
- ▶ B_k has height k .
- ▶ The root of B_k has degree k .

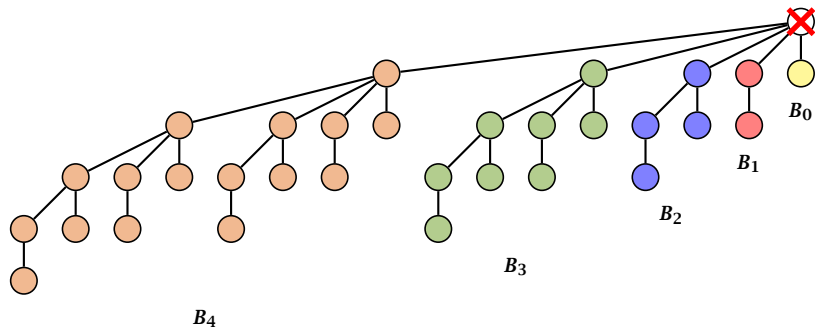
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- ▶ B_k has 2^k nodes.
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- ▶ B_k has $\binom{k}{\ell}$ nodes on level ℓ .

Properties of Binomial Trees

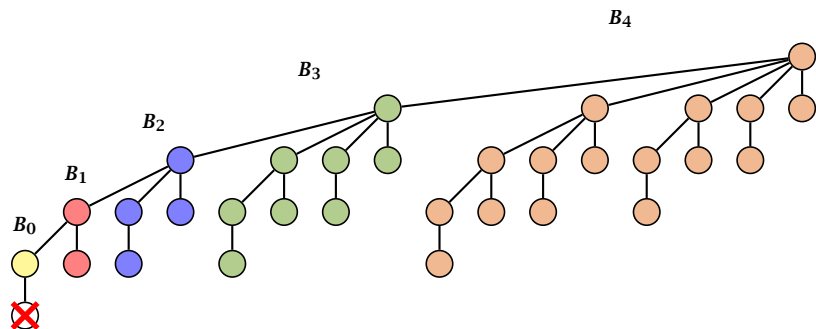
- ▶ B_k has 2^k nodes.
- ▶ B_k has height k .
- ▶ The root of B_k has degree k .
- ▶ B_k has $\binom{k}{\ell}$ nodes on level ℓ .
- ▶ Deleting the root of B_k gives trees B_0, B_1, \dots, B_{k-1} .

Binomial Trees



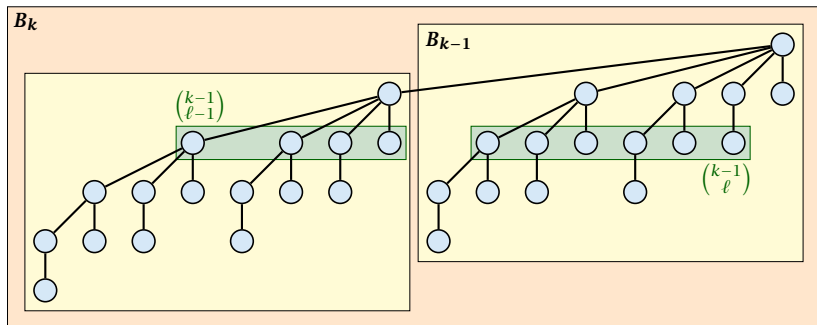
Deleting the root of B_5 leaves sub-trees B_4 , B_3 , B_2 , B_1 , and B_0 .

Binomial Trees



Deleting the leaf furthest from the root (in B_5) leaves a path that connects the roots of sub-trees B_4 , B_3 , B_2 , B_1 , and B_0 .

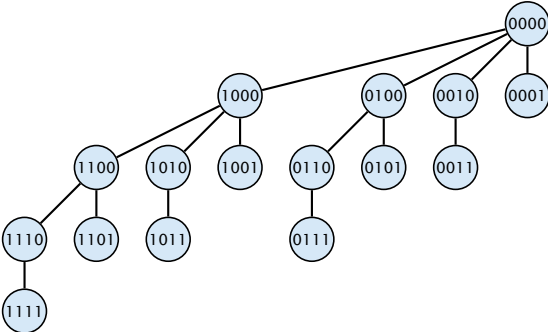
Binomial Trees



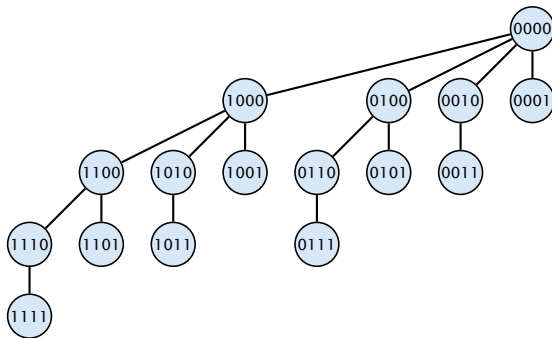
The number of nodes on level ℓ in tree B_k is therefore

$$\binom{k-1}{\ell-1} + \binom{k-1}{\ell} = \binom{k}{\ell}$$

Binomial Trees

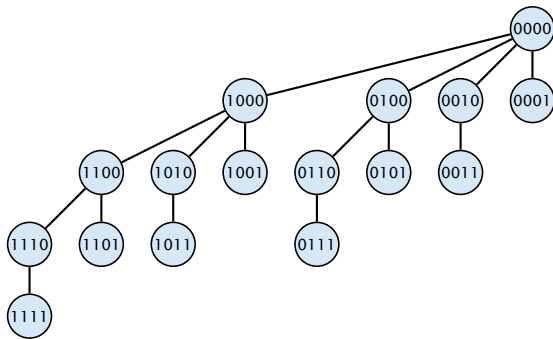


Binomial Trees



The binomial tree B_k is a sub-graph of the hypercube H_k .

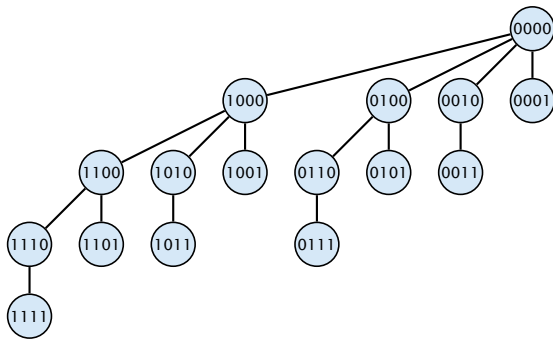
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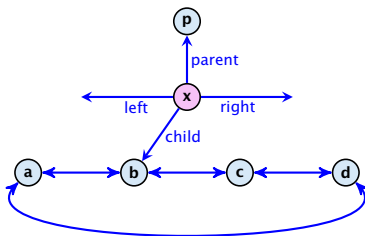
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The ℓ -th level contains nodes that have ℓ 1's in their label.

8.2 Binomial Heaps

How do we implement trees with non-constant degree?

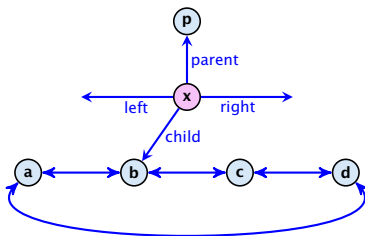
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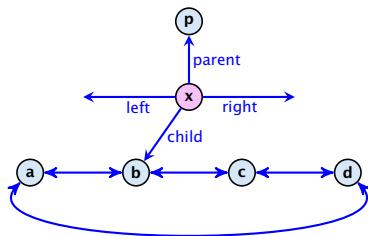
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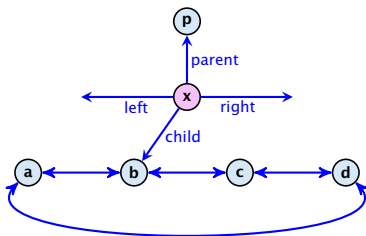
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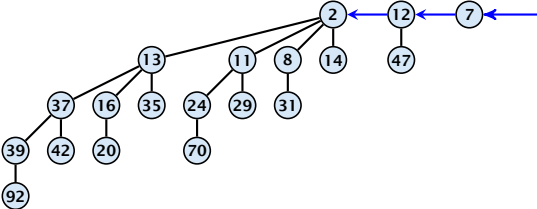
- ▶ The children of a node are arranged in a **circular linked list**.
- ▶ A child-pointer points to an arbitrary node within the list.
- ▶ A parent-pointer points to the parent node.
- ▶ Pointers $x.\text{left}$ and $x.\text{right}$ point to the left and right sibling of x (if x does not have siblings then $x.\text{left} = x.\text{right} = x$).



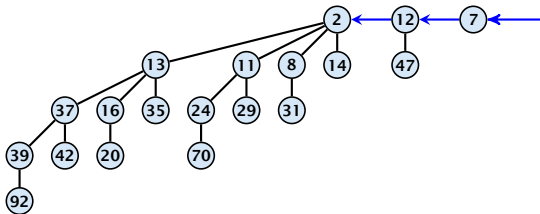
8.2 Binomial Heaps

- ▶ Given a pointer to a node x we can splice out the sub-tree rooted at x in constant time.
- ▶ We can add a child-tree T to a node x in constant time if we are given a pointer to x and a pointer to the root of T .

Binomial Heap

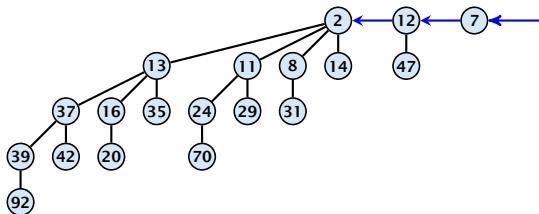


Binomial Heap



In a binomial heap the keys are arranged in a collection of binomial trees.

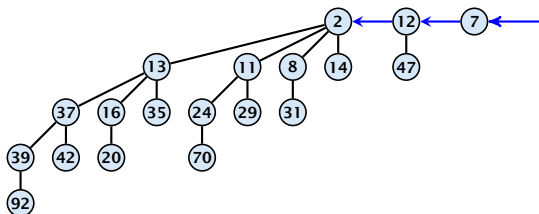
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There is at most one tree for every dimension/order. For example the above heap contains trees B_0 , B_1 , and B_4 .

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Let $B_{k_1}, B_{k_2}, B_{k_3}, k_i < k_{i+1}$ denote the binomial trees in the collection and recall that every tree may be contained at most once.

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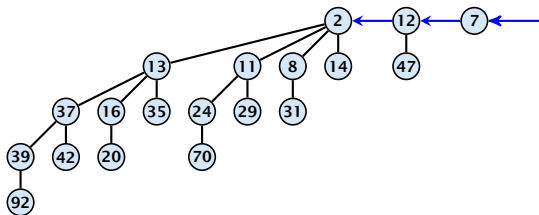
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Then $n = \sum_i 2^{k_i}$ must hold. But since the k_i are all distinct this means that the k_i define the non-zero bit-positions in the binary representation of n .

Binomial Heap

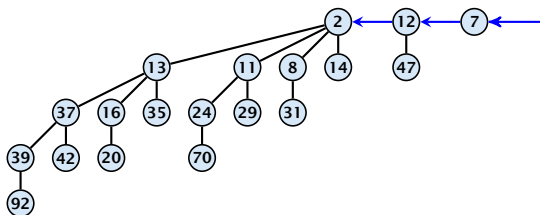
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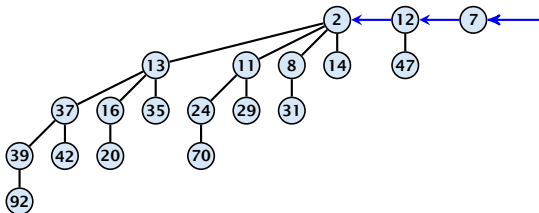
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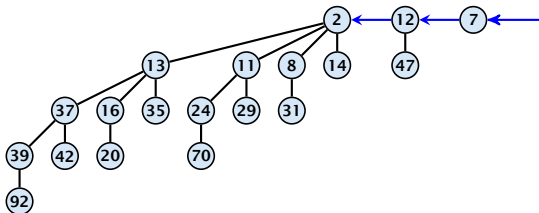
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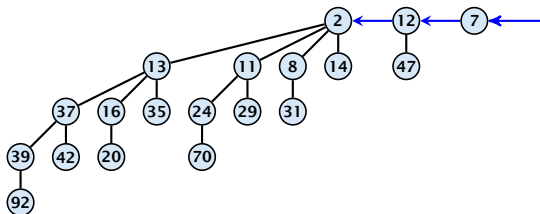
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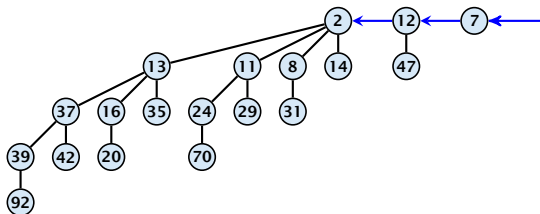
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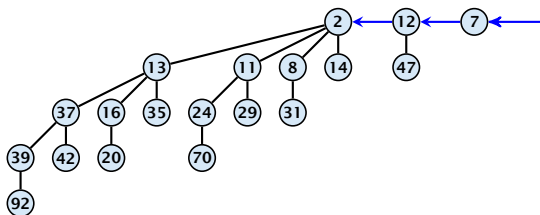
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- ▶ Hence, at most $\lfloor \log n \rfloor + 1$ trees.
- ▶ The minimum must be contained in one of the roots.
- ▶ The height of the largest tree is at most $\lfloor \log n \rfloor$.
- ▶ The trees are stored in a single-linked list; ordered by dimension/size.



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Otherwise, we cannot do this because the merged heap is not allowed to contain two trees of the same order.

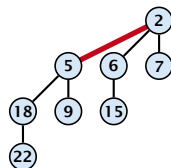
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Merging two trees of the same size: Add the tree with larger root-value as a child to the other tree.



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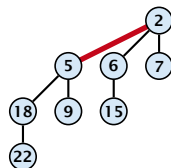
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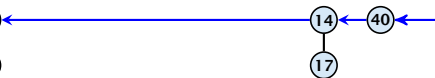
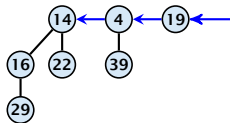
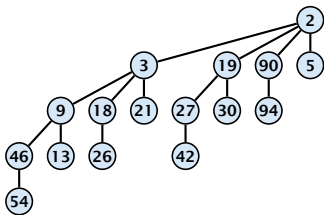
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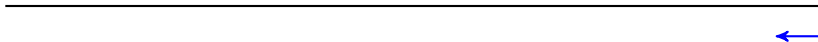
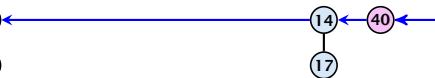
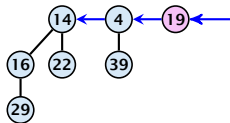
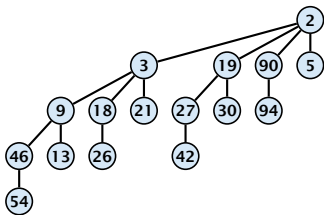
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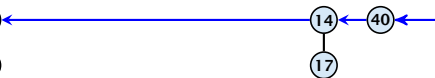
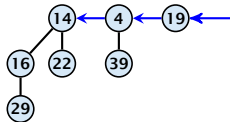
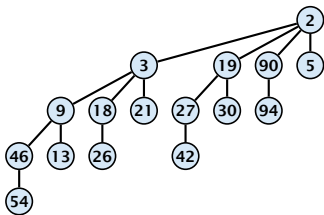
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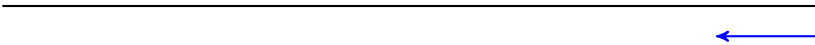
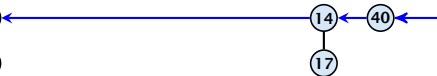
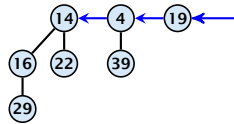
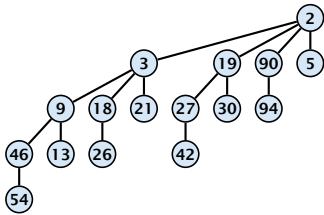
For more trees the technique is analogous to binary addition.

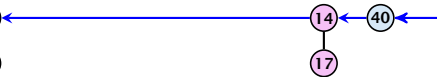
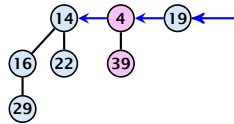
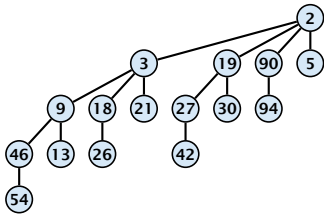


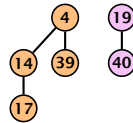
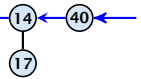
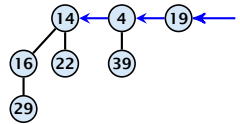
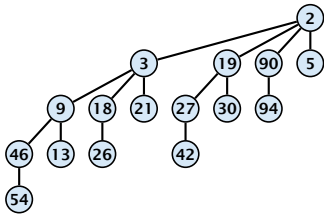


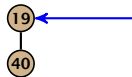
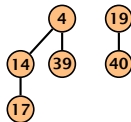
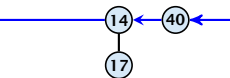
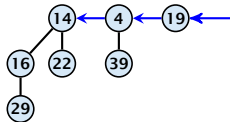
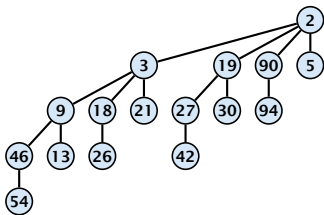


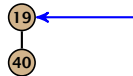
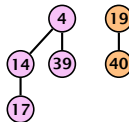
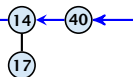
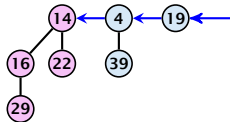
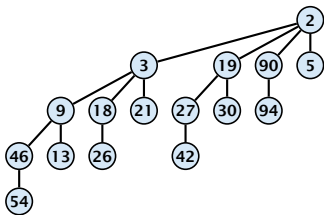


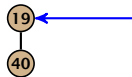
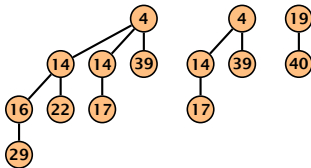
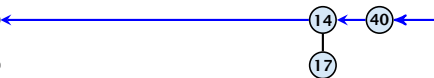
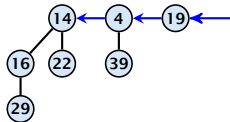
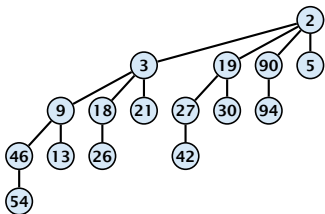


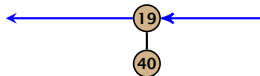
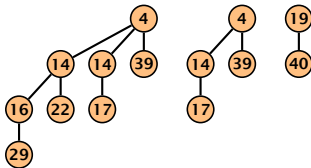
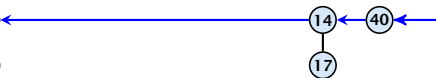
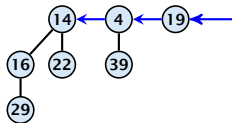
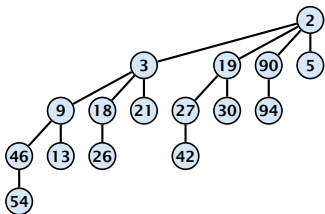


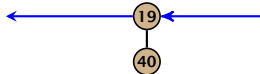
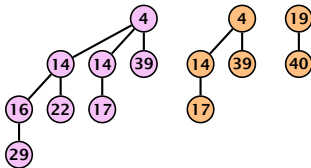
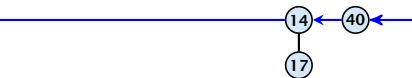
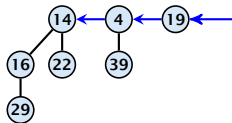
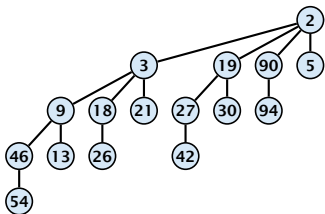




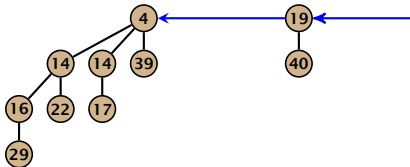
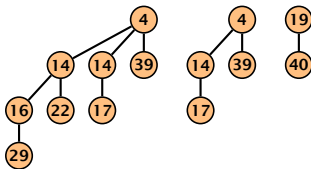
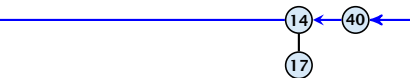
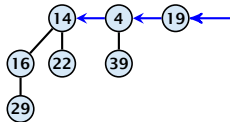
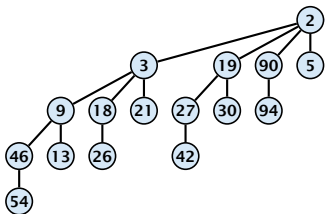




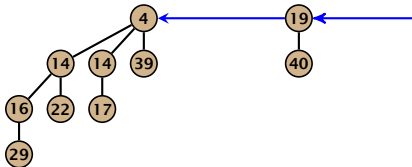
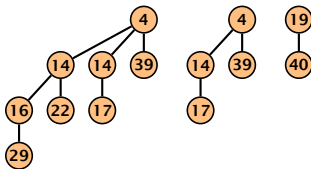
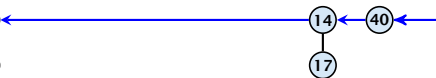
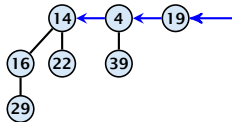
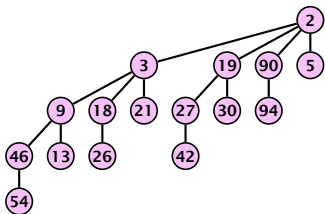




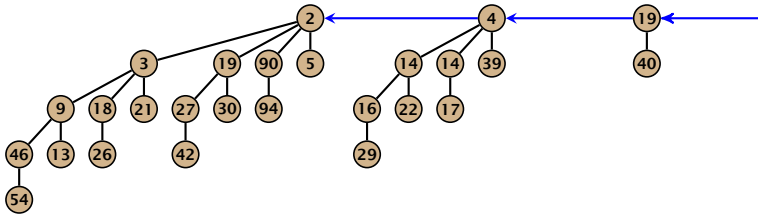
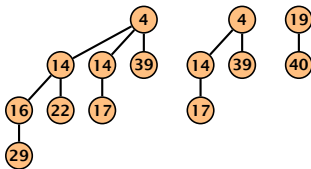
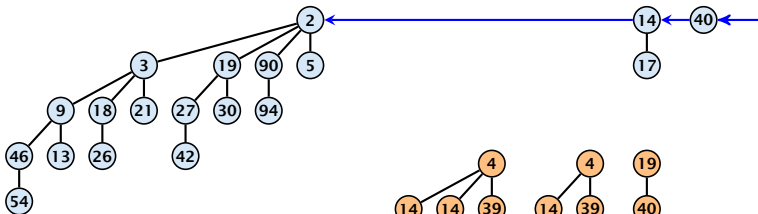
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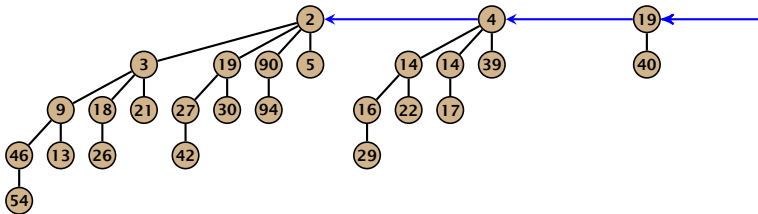
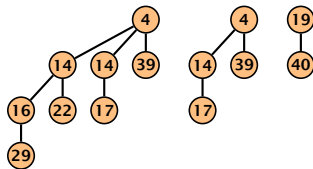
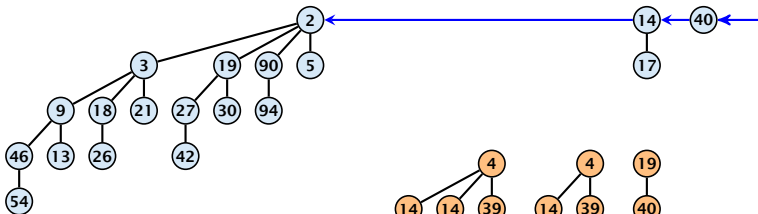


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8.2 Binomial Heaps

S_1 . merge(S_2):

- ▶ Analogous to binary addition.

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8.2 Binomial Heaps

S. minimum():

- ▶ Find the minimum key-value among all roots.
- ▶ Time: $\mathcal{O}(\log n)$.

8.2 Binomial Heaps

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- ▶ Bubble the element up in the tree until the heap property is fulfilled.
- ▶ Time: $\mathcal{O}(\log n)$ since the trees have height $\mathcal{O}(\log n)$.

8.2 Binomial Heaps

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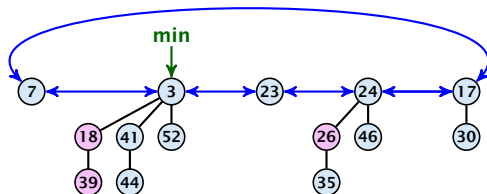
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- ▶ Execute S . decrease-key($h, -\infty$).
- ▶ Execute S . delete-min().
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8.3 Fibonacci Heaps

Collection of trees that fulfill the heap property.

Structure is much more relaxed than binomial heaps.



8.3 Fibonacci Heaps

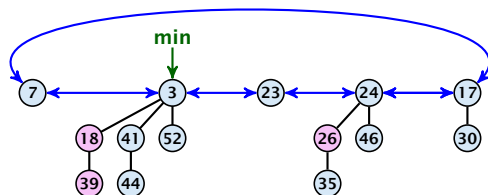
Additional implementation details:

- ▶ Every node x stores its degree in a field $x.degree$. Note that this can be updated in constant time when adding a child to x .
- ▶ Every node stores a boolean value $x.marked$ that specifies whether x is **marked** or not.

8.3 Fibonacci Heaps

The potential function:

- ▶ $t(S)$ denotes the number of trees in the heap.
- ▶ $m(S)$ denotes the number of marked nodes.
- ▶ We use the potential function $\Phi(S) = t(S) + 2m(S)$.



The potential is $\Phi(S) = 5 + 2 \cdot 3 = 11$.

8.3 Fibonacci Heaps

We assume that one unit of potential can pay for a constant amount of work, where the constant is chosen “big enough” (to take care of the constants that occur).

To make this more explicit we use c to denote the amount of work that a unit of potential can pay for.

8.3 Fibonacci Heaps

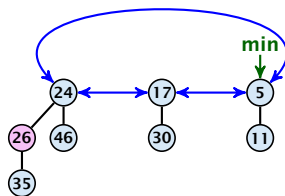
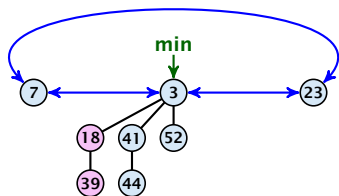
S. minimum()

- ▶ Access through the min-pointer.
- ▶ Actual cost $\mathcal{O}(1)$.
- ▶ No change in potential.
- ▶ Amortized cost $\mathcal{O}(1)$.

8.3 Fibonacci Heaps

S . merge(S')

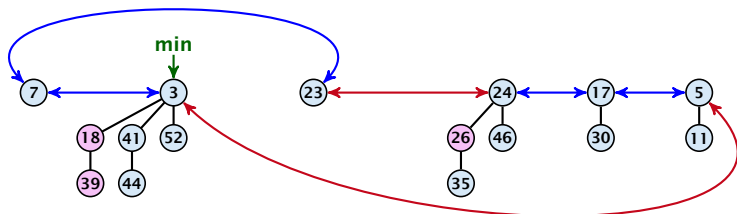
- ▶ Merge the root lists.
- ▶ Adjust the min-pointer



8.3 Fibonacci Heaps

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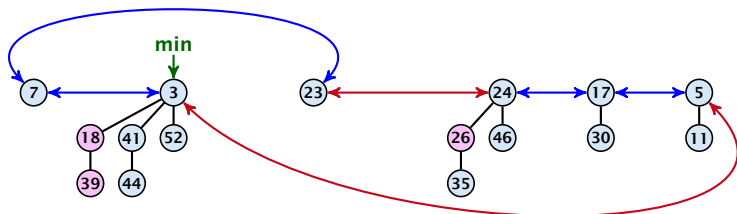
Running time:

- ▶ Actual cost $\mathcal{O}(1)$.

8.3 Fibonacci Heaps

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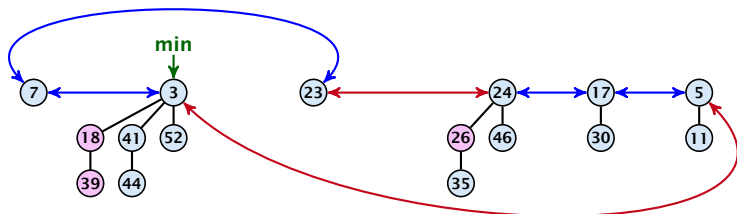
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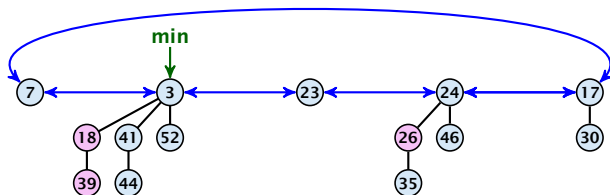
Running time:

- ▶ Actual cost $\mathcal{O}(1)$.
- ▶ No change in potential.
- ▶ Hence, amortized cost is $\mathcal{O}(1)$.

8.3 Fibonacci Heaps

S. insert(x)

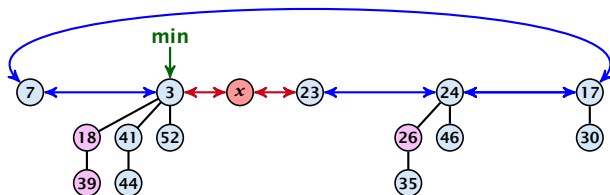
- ▶ Create a new tree containing x .
- ▶ Insert x into the root-list.
- ▶ Update min-pointer, if necessary.



8.3 Fibonacci Heaps

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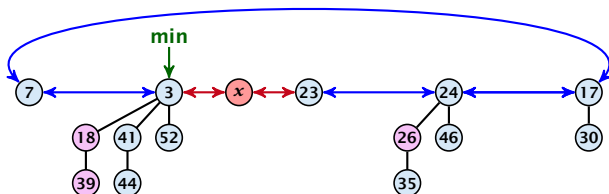
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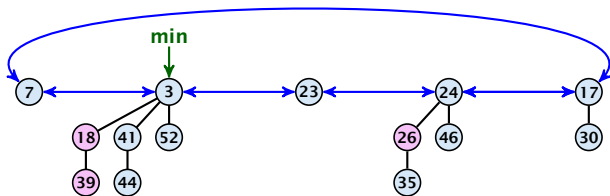


Running time:

- ▶ Actual cost $\mathcal{O}(1)$.
- ▶ Change in potential is $+1$.
- ▶ Amortized cost is $c + \mathcal{O}(1) = \mathcal{O}(1)$.

8.3 Fibonacci Heaps

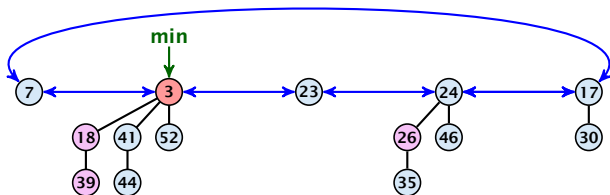
S. delete-min(x)



8.3 Fibonacci Heaps

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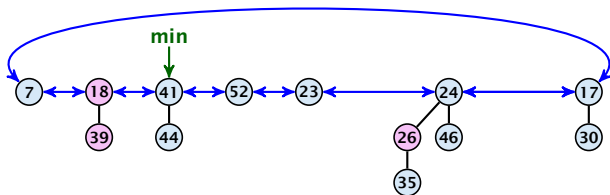
- ▶ Delete minimum; add child-trees to heap;
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8.3 Fibonacci Heaps

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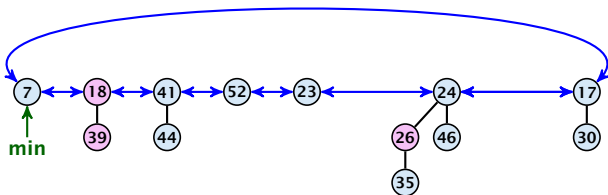
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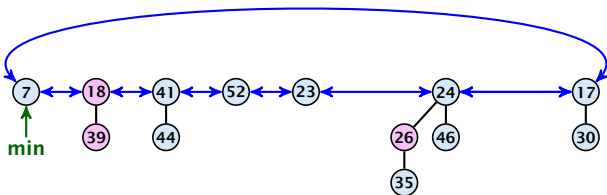
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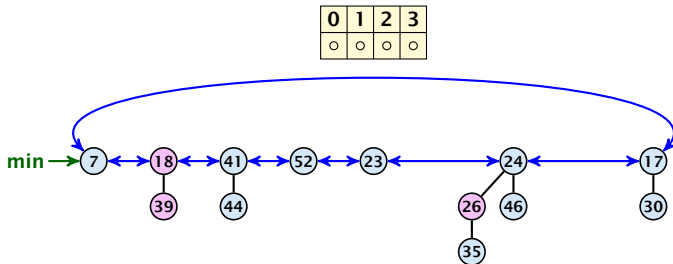
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- ▶ Consolidate root-list so that no roots have the same degree. Time $t \cdot \mathcal{O}(1)$ (see next slide).

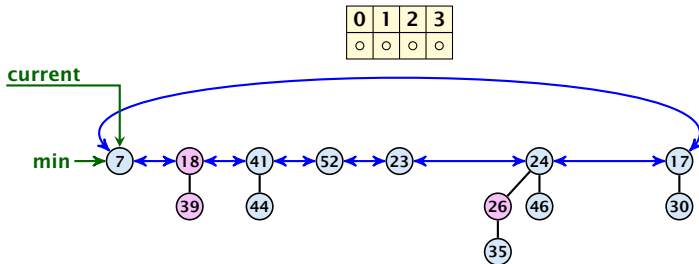
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Consolidate:



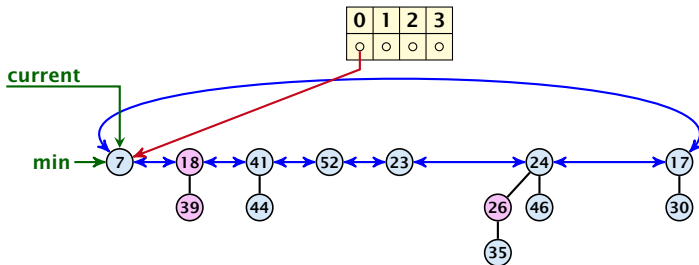
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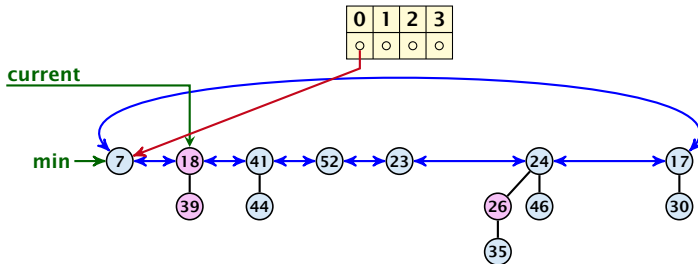
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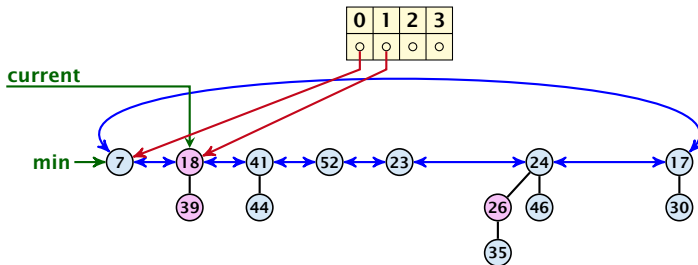
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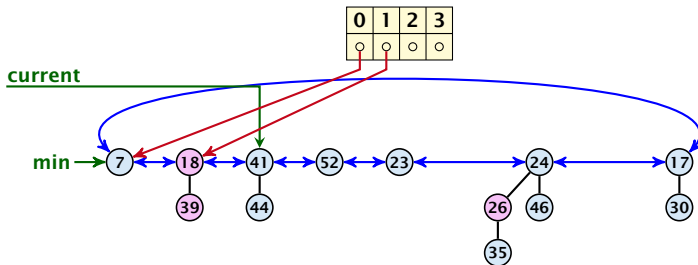
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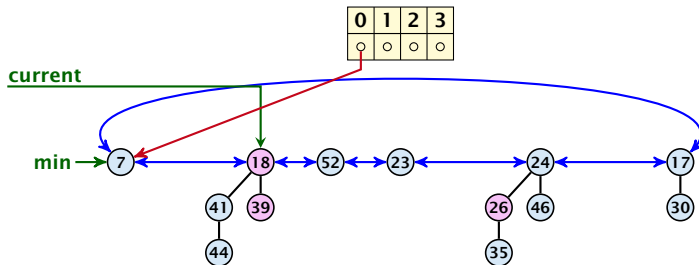
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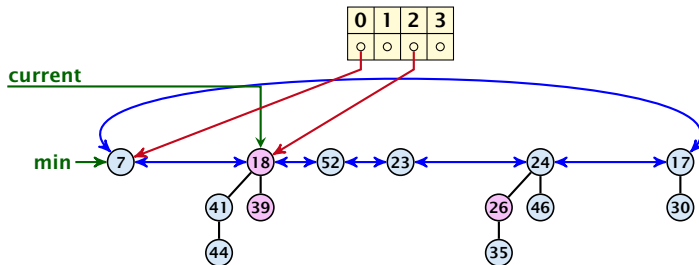
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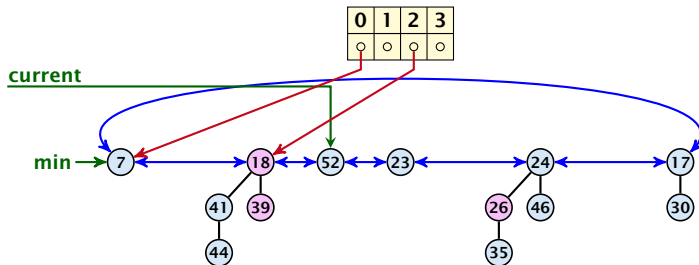
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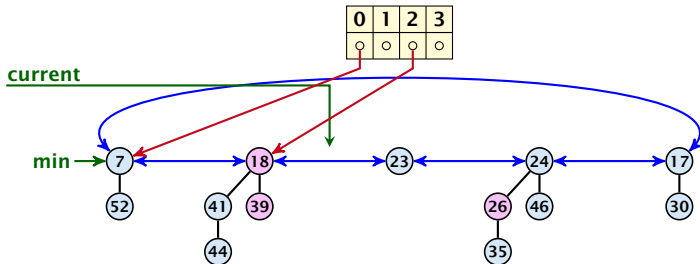
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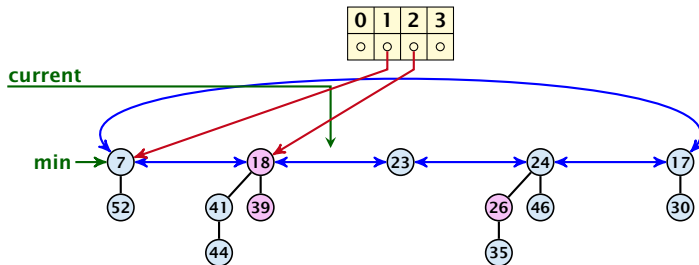
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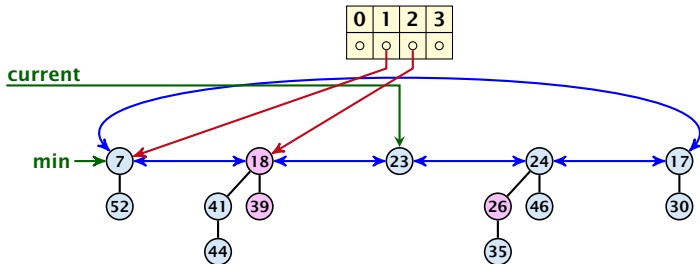
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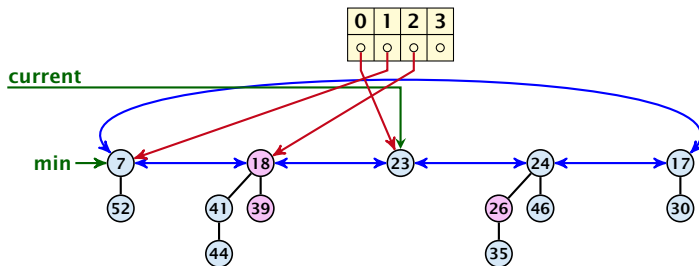
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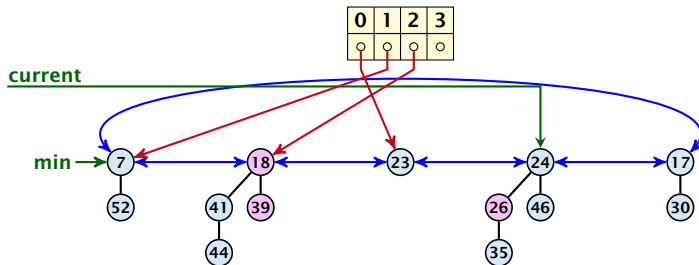
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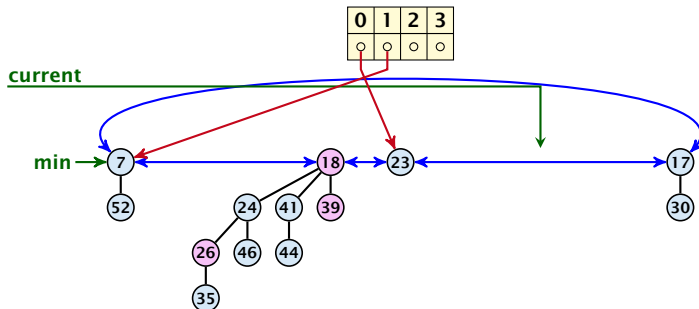
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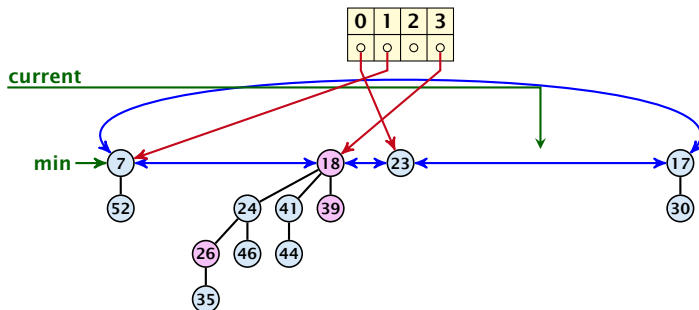
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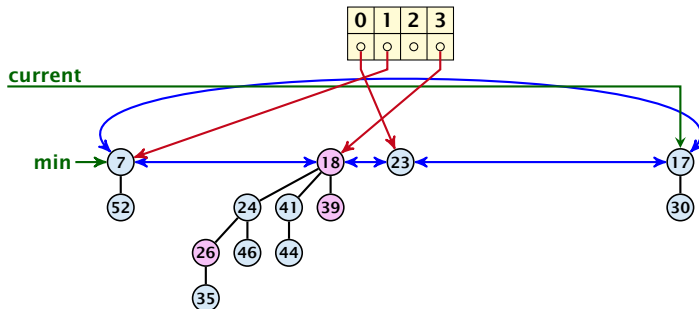
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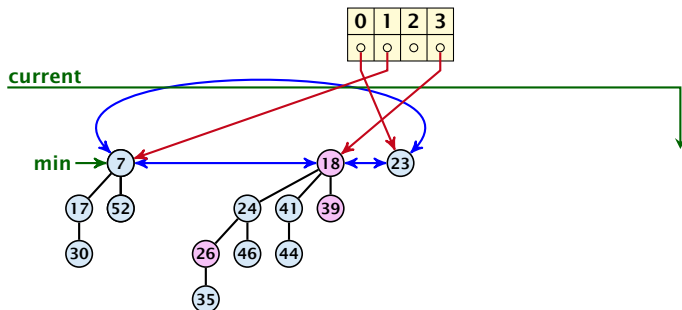
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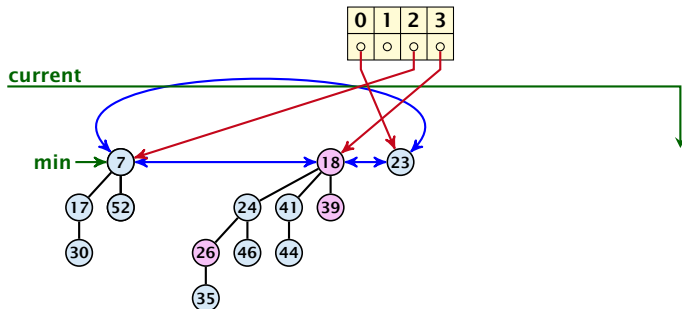
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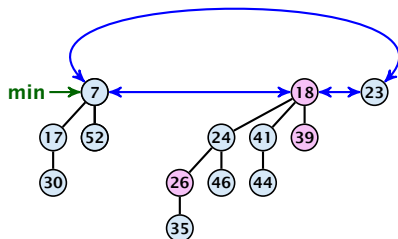
8.3 Fibonacci Heaps

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8.3 Fibonacci Heaps

Actual cost for delete-min()

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8.3 Fibonacci Heaps

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Hence, there exists c_1 s.t. actual cost is at most $c_1 \cdot (D_n + t)$.

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$$\begin{aligned}c_1 \cdot (D_n + t) - c \cdot (t - D_n - 1) \\ \leq (c_1 + c)D_n + (c_1 - c)t + c\end{aligned}$$

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$$\begin{aligned}c_1 \cdot (D_n + t) - c \cdot (t - D_n - 1) \\ \leq (c_1 + c)D_n + (c_1 - c)t + c \leq 2c(D_n + 1)\end{aligned}$$

8.3 Fibonacci Heaps

Actual cost for delete-min()

- ▶ At most $D_n + t$ elements in root-list before consolidate.
- ▶ Actual cost for a delete-min is at most $\mathcal{O}(1) \cdot (D_n + t)$.
Hence, there exists c_1 s.t. actual cost is at most $c_1 \cdot (D_n + t)$.

Amortized cost for delete-min()

- ▶ $t' \leq D_n + 1$ as degrees are different after consolidating.
- ▶ Therefore $\Delta\Phi \leq D_n + 1 - t$;
- ▶ We can pay $c \cdot (t - D_n - 1)$ from the potential decrease.
- ▶ The amortized cost is

$$\begin{aligned}c_1 \cdot (D_n + t) - c \cdot (t - D_n - 1) \\ \leq (c_1 + c)D_n + (c_1 - c)t + c \leq 2c(D_n + 1) \leq \mathcal{O}(D_n)\end{aligned}$$

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for $c \geq c_1$.

8.3 Fibonacci Heaps

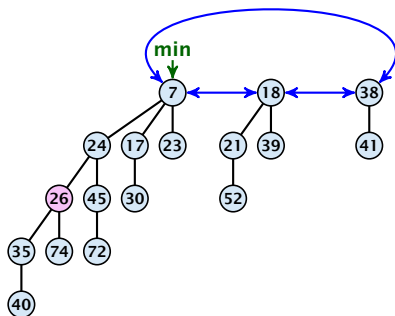
If the input trees of the consolidation procedure are binomial trees (for example only singleton vertices) then the output will be a set of distinct binomial trees, and, hence, the Fibonacci heap will be (more or less) a Binomial heap right after the consolidation.

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If we do not have delete or decrease-key operations then
 $D_n \leq \log n$.

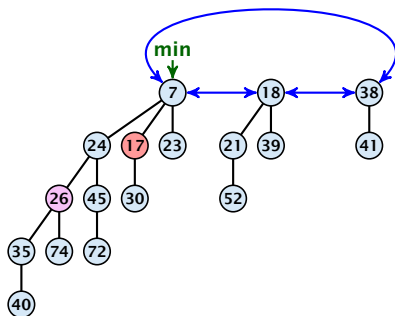
Fibonacci Heaps: decrease-key(handle h, v)



Case 1: decrease-key does not violate heap-property

- ▶ Just decrease the key-value of element referenced by h . Nothing else to do.

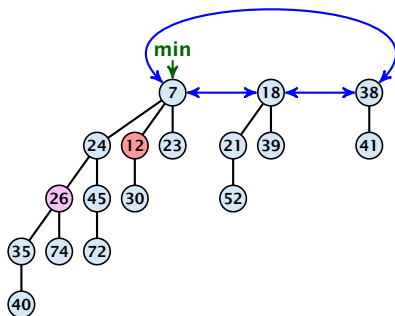
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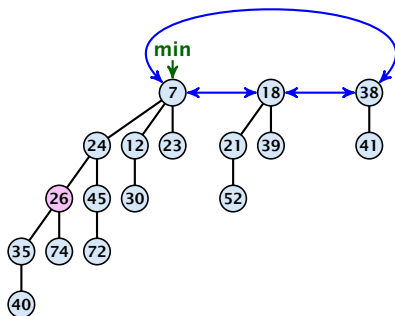
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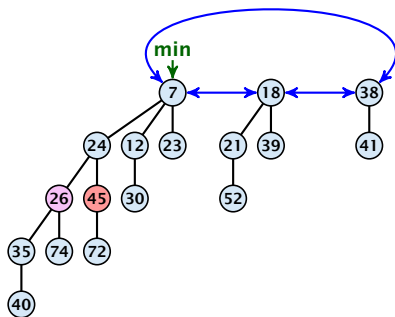
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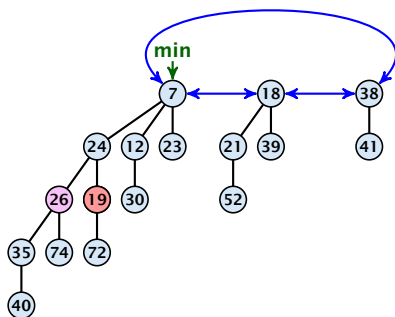
Fibonacci Heaps: decrease-key(handle h, v)



Case 2: heap-property is violated, but parent is not marked

- ▶ Decrease key-value of element x reference by h .
- ▶ If the heap-property is violated, cut the parent edge of x , and make x into a root.
- ▶ Adjust min-pointers, if necessary.
- ▶ Mark the (previous) parent of x (unless it's a root).

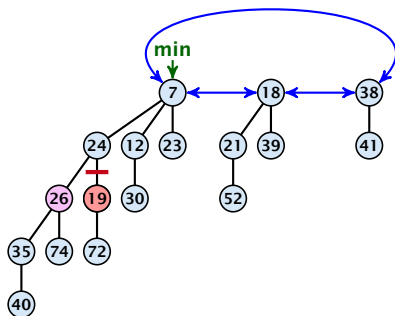
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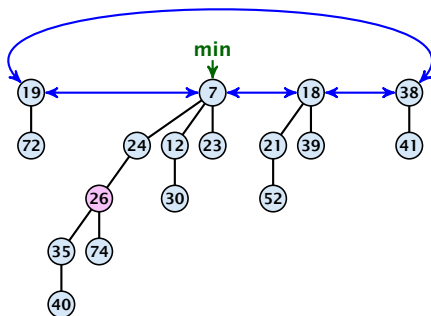
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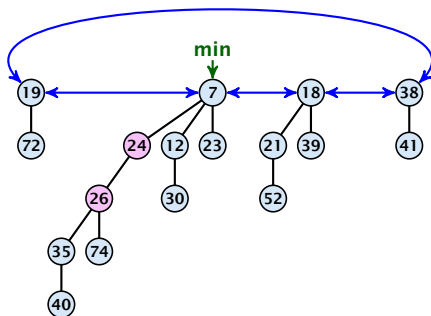
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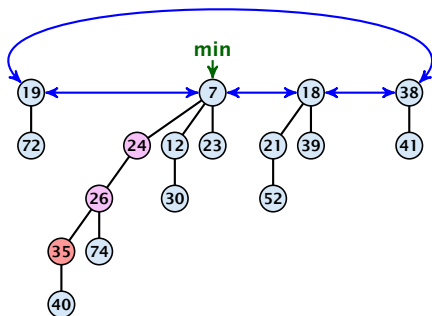
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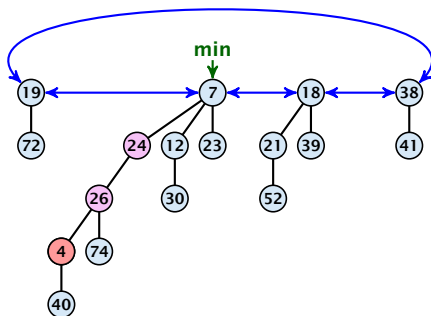
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- ▶ Decrease key-value of element x reference by h .
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- ▶ Continue cutting the parent until you arrive at an unmarked node.

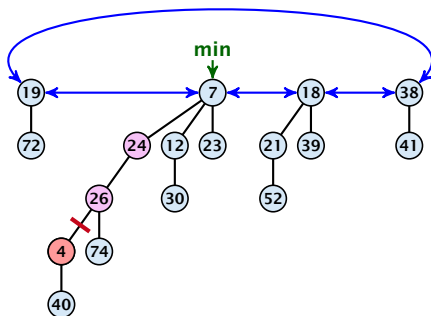
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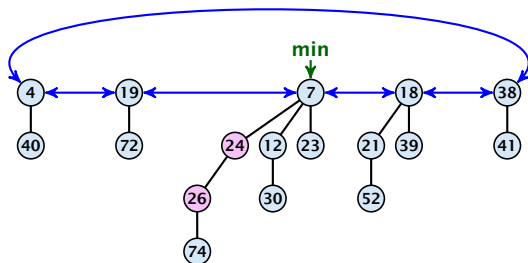
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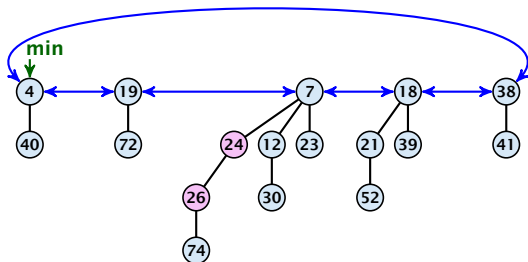
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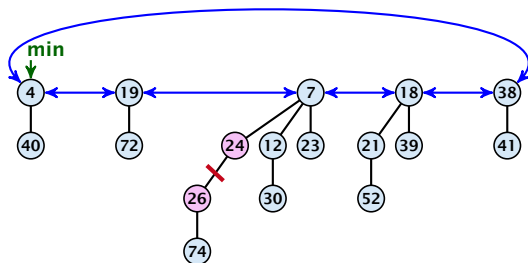
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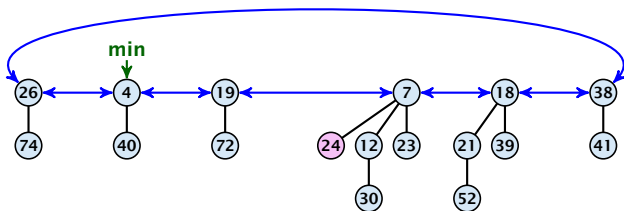
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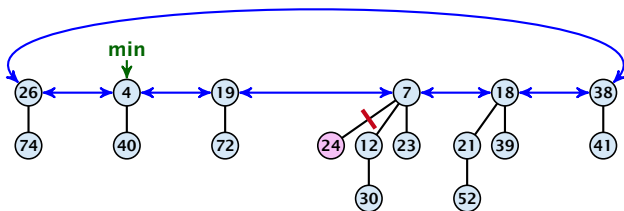
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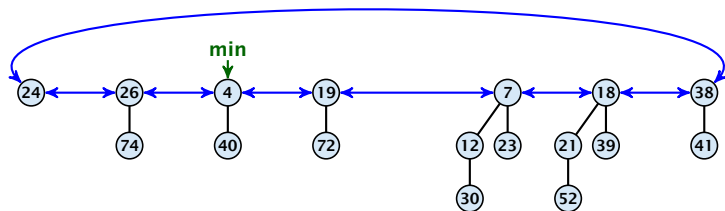
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- ▶ Cut the parent edge of x , and make x into a root.
- ▶ Adjust min-pointers, if necessary.
- ▶ Execute the following:

```
 $p \leftarrow \text{parent}[x];$   
while ( $p$  is marked)  
     $pp \leftarrow \text{parent}[p];$   
    cut of  $p$ ; make it into a root; unmark it;  
     $p \leftarrow pp;$   
if  $p$  is unmarked and not a root mark it;
```

Fibonacci Heaps: decrease-key(handle h, v)

Actual cost:

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$$c_2(\ell + 1) + c(4 - \ell)$$

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$$c_2(\ell + 1) + c(4 - \ell) \leq (c_2 - c)\ell + 4c + c_2 = \mathcal{O}(1),$$
if $c \geq c_2$.

Delete node

H. delete(x):

- ▶ decrease value of x to $-\infty$.
- ▶ delete-min.

Amortized cost: $\mathcal{O}(D_n)$

- ▶ $\mathcal{O}(1)$ for decrease-key.
- ▶ $\mathcal{O}(D_n)$ for delete-min.

8.3 Fibonacci Heaps

Lemma 32

Let x be a node with degree k and let y_1, \dots, y_k denote the children of x in the order that they were linked to x . Then

$$\text{degree}(y_i) \geq \begin{cases} 0 & \text{if } i = 1 \\ i - 2 & \text{if } i > 1 \end{cases}$$

8.3 Fibonacci Heaps

Proof

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- ▶ Since, then y_i has lost at most one child.
- ▶ Therefore, $\text{degree}(y_i) \geq i - 2$.

8.3 Fibonacci Heaps

- ▶ Let s_k be the minimum possible size of a sub-tree rooted at a node of degree k that can occur in a Fibonacci heap.

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$$s_k = 2 + \sum_{i=2}^k \text{size}(y_i)$$

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$$\begin{aligned} s_k &= 2 + \sum_{i=2}^k \text{size}(y_i) \\ &\geq 2 + \sum_{i=2}^k s_{i-2} \\ &= 2 + \sum_{i=0}^{k-2} s_i \end{aligned}$$

8.3 Fibonacci Heaps

$\phi = \frac{1}{2}(1 + \sqrt{5})$ denotes the *golden ratio*.
Note that $\phi^2 = 1 + \phi$.

Definition 33

Consider the following non-standard Fibonacci type sequence:

$$F_k = \begin{cases} 1 & \text{if } k = 0 \\ 2 & \text{if } k = 1 \\ F_{k-1} + F_{k-2} & \text{if } k \geq 2 \end{cases}$$

Facts:

1. $F_k \geq \phi^k$.
2. For $k \geq 2$: $F_k = 2 + \sum_{i=0}^{k-2} F_i$.

The above facts can be easily proved by induction. From this it follows that $s_k \geq F_k \geq \phi^k$, which gives that the maximum degree in a Fibonacci heap is logarithmic.

$$k=0: \quad 1 = F_0 \geq \Phi^0 = 1$$

$$k=1: \quad 2 = F_1 \geq \Phi^1 \approx 1.61$$

$$k-2, k-1 \rightarrow k: \quad F_k = F_{k-1} + F_{k-2} \geq \Phi^{k-1} + \Phi^{k-2} = \Phi^{k-2} \underbrace{(\Phi + 1)}_{\Phi^2} = \Phi^k$$

$$k=2: \quad 3 = F_2 = 2 + 1 = 2 + F_0$$

$$k-1 \rightarrow k: \quad F_k = F_{k-1} + F_{k-2} = 2 + \sum_{i=0}^{k-3} F_i + F_{k-2} = 2 + \sum_{i=0}^{k-2} F_i$$

9 Union Find

Union Find Data Structure \mathcal{P} : Maintains a partition of **disjoint** sets over elements.

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- ▶ **\mathcal{P} . find(x):** Given a handle for an element x ; find the set that contains x . Returns a representative/identifier for this set.
- ▶ **\mathcal{P} . union(x, y):** Given two elements x , and y that are currently in sets S_x and S_y , respectively, the function replaces S_x and S_y by $S_x \cup S_y$ and returns an identifier for the new set.

9 Union Find

Applications:

- ▶ Keep track of the connected components of a dynamic graph that changes due to insertion of nodes and edges.

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- ▶ Keep track of the connected components of a dynamic graph that changes due to insertion of nodes and edges.
- ▶ Kruskals Minimum Spanning Tree Algorithm

9 Union Find

Algorithm 1 Kruskal-MST($G = (V, E), w$)

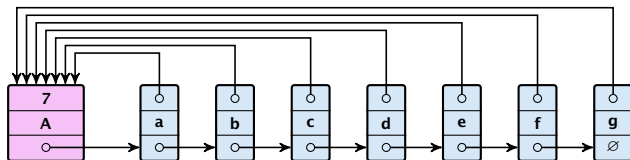
```
1:  $A \leftarrow \emptyset$ ;  
2: for all  $v \in V$  do  
3:    $v.\text{set} \leftarrow \mathcal{P}.\text{makeset}(v.\text{label})$   
4: sort edges in non-decreasing order of weight  $w$   
5: for all  $(u, v) \in E$  in non-decreasing order do  
6:   if  $\mathcal{P}.\text{find}(u.\text{set}) \neq \mathcal{P}.\text{find}(v.\text{set})$  then  
7:      $A \leftarrow A \cup \{(u, v)\}$   
8:      $\mathcal{P}.\text{union}(u.\text{set}, v.\text{set})$ 
```

List Implementation

- ▶ The elements of a set are stored in a list; each node has a backward pointer to the head.

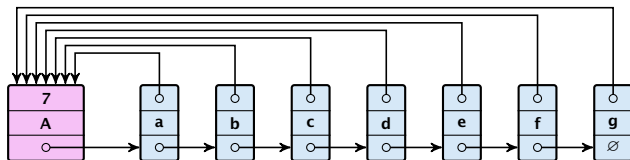
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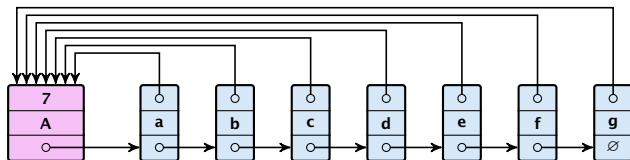
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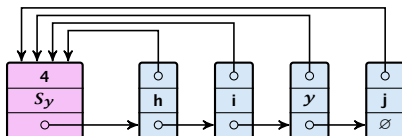
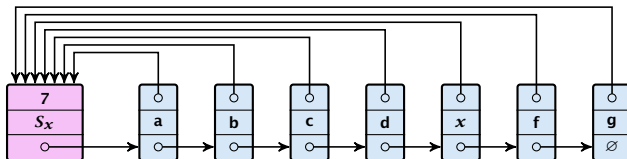
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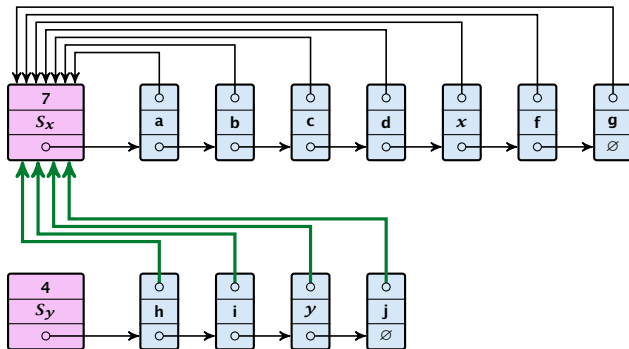
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- ▶ Time: $\min\{|S_x|, |S_y|\}$.

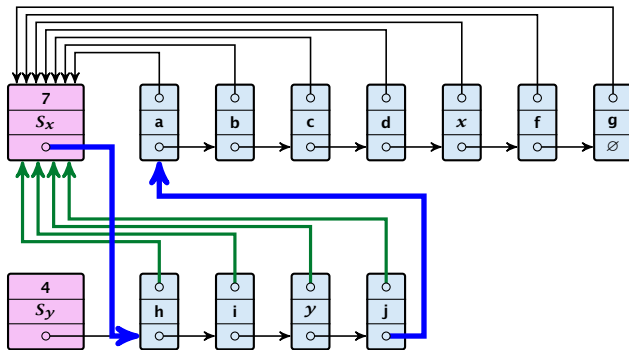
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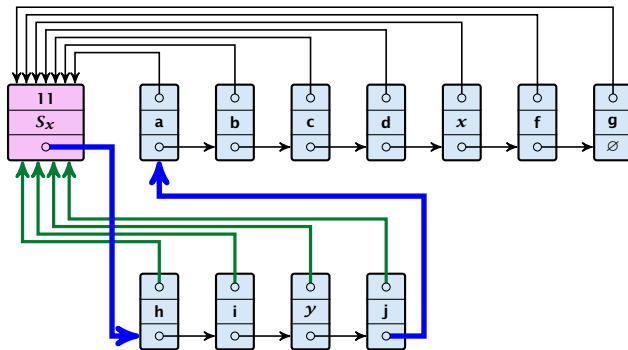
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Running times:

- ▶ $\text{find}(x)$: constant
- ▶ $\text{makeset}(x)$: constant
- ▶ $\text{union}(x, y)$: $\mathcal{O}(n)$, where n denotes the number of elements contained in the set system.

List Implementation

Lemma 34

The list implementation for the ADT union find fulfills the following amortized time bounds:

- ▶ $\text{find}(x): \mathcal{O}(1)$.
- ▶ $\text{makeset}(x): \mathcal{O}(\log n)$.
- ▶ $\text{union}(x, y): \mathcal{O}(1)$.

The Accounting Method for Amortized Time Bounds

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- ▶ If we can find a charging scheme that guarantees that balances always stay positive the amortized time bounds are proven.

List Implementation

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- ▶ Later operations charge the account but the balance never drops below zero.

List Implementation

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- ▶ Assume wlog. that S_x is the smaller set; let c denote the hidden constant, i.e., the actual cost is at most $c \cdot |S_x|$.
- ▶ Charge c to every element in set S_x .

List Implementation

Lemma 35

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Proof.

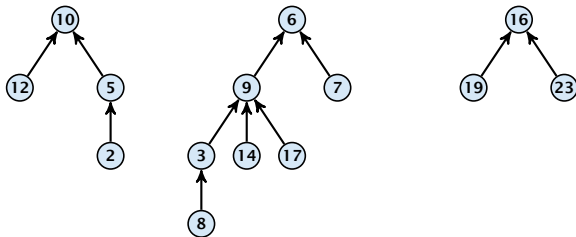
Whenever an element x is charged the number of elements in x 's set doubles. This can happen at most $\lceil \log n \rceil$ times. \square

Implementation via Trees

- ▶ Maintain nodes of a set in a tree.
- ▶ The root of the tree is the label of the set.
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- ▶ Example:



Set system $\{2, 5, 10, 12\}$, $\{3, 6, 7, 8, 9, 14, 17\}$, $\{16, 19, 23\}$.

Implementation via Trees

makeset(x)

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Implementation via Trees

makeiset(x)

- ▶ Create a singleton tree. Return pointer to the root.
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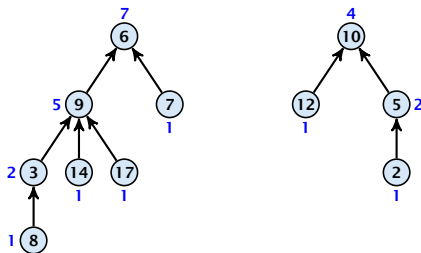
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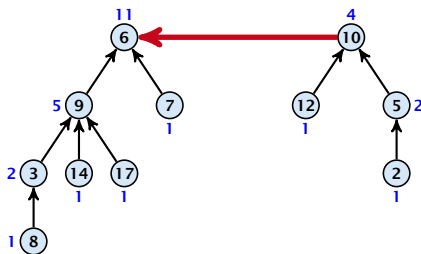


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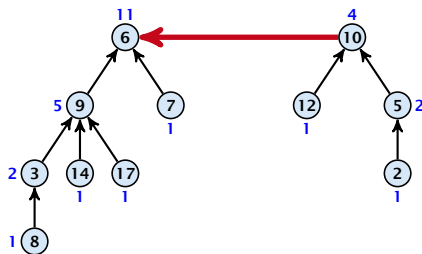


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- ▶ Time: constant for $\text{link}(a, b)$ plus two find-operations.

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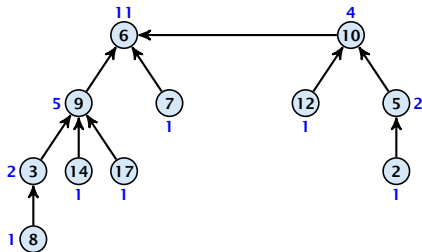
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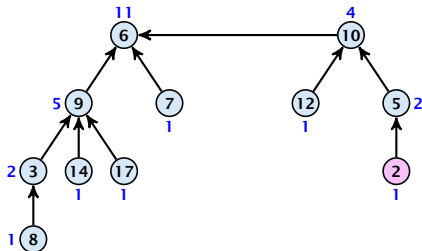
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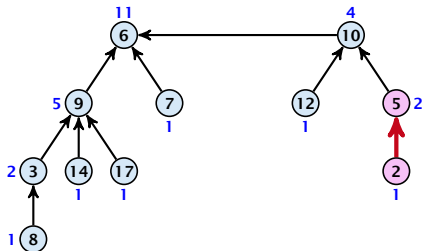
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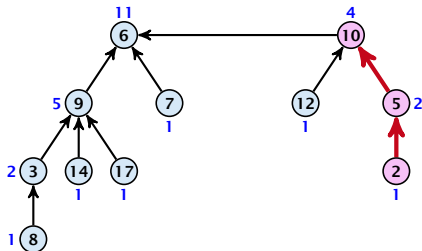
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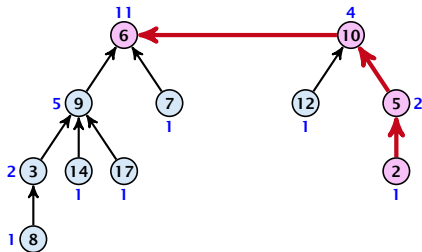
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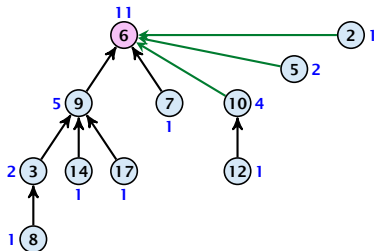
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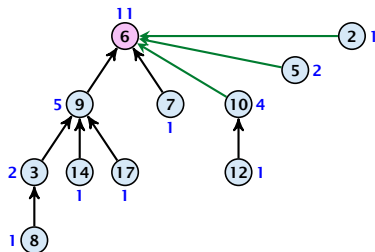
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- ▶ Note that the size-fields now only give an upper bound on the size of a sub-tree.

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However, for a worst-case analysis there is no improvement on the running time. It can still happen that a find-operation takes time $\mathcal{O}(\log n)$.

Amortized Analysis

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- ▶ $\text{size}(v)$:= the number of nodes that were in the sub-tree rooted at v when v became the child of another node (or the number of nodes if v is the root).

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Lemma 37

The rank of a parent must be strictly larger than the rank of a child.

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There are at most $n/2^s$ nodes of rank s .

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- ▶ This holds because the rank-sequence of the roots of the different trees that contain v during the running time of the algorithm is a strictly increasing sequence.
- ▶ Hence, every node sees at most one rank s node, but every rank s node is seen by at least 2^s different nodes. □

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Theorem 39

Union find with path compression fulfills the following amortized running times:

- ▶ $\text{makeset}(x) : \mathcal{O}(\log^*(n))$
- ▶ $\text{find}(x) : \mathcal{O}(\log^*(n))$
- ▶ $\text{union}(x, y) : \mathcal{O}(\log^*(n))$

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- ▶ The maximum non-empty rank group is $\log^*(\lfloor \log n \rfloor) \leq \log^*(n) - 1$ (which holds for $n \geq 2$).
- ▶ Hence, the total number of rank-groups is at most $\log^* n$.

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- ▶ Otherwise we charge the cost to the find-account.

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- ▶ The total charge made to a node in rank-group g is at most $\text{tow}(g) - \text{tow}(g - 1) - 1 \leq \text{tow}(g)$.

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- ▶ The total charge is at most

$$\sum_g n(g) \cdot \text{tow}(g) ,$$

where $n(g)$ is the number of nodes in group g .

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This means if we inflate the cost of **makeset** to $\log^* n$ and add this to the node account of v then the balances of all node accounts will sum up to a positive value (this is sufficient to obtain an amortized bound).

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The analysis is not tight. In fact it has been shown that the amortized time for the union-find data structure with path compression is $\mathcal{O}(\alpha(m, n))$, where $\alpha(m, n)$ is the inverse Ackermann function which grows a lot lot slower than $\log^* n$. (Here, we consider the average running time of m operations on at most n elements).

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There is also a lower bound of $\Omega(\alpha(m, n))$.

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$$A(x, y) = \begin{cases} y + 1 & \text{if } x = 0 \\ A(x - 1, 1) & \text{if } y = 0 \\ A(x - 1, A(x, y - 1)) & \text{otw.} \end{cases}$$

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- ▶ $A(0, y) = y + 1$
- ▶ $A(1, y) = y + 2$
- ▶ $A(2, y) = 2y + 3$
- ▶ $A(3, y) = 2^{y+3} - 3$
- ▶ $A(4, y) = \underbrace{2^{2^{2^2}}}_{y+3 \text{ times}} - 3$