7 Augmenting Path Algorithms

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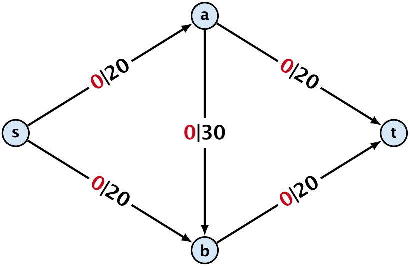
- start with $f(e) = 0$ everywhere
- find an s-t path with $f(e) < c(e)$ on every edge
- augment flow along the path
- repeat as long as possible



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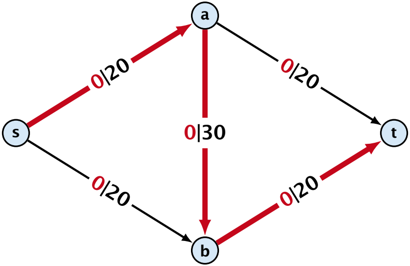

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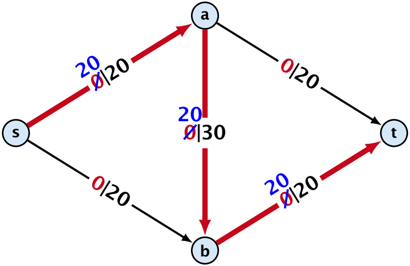

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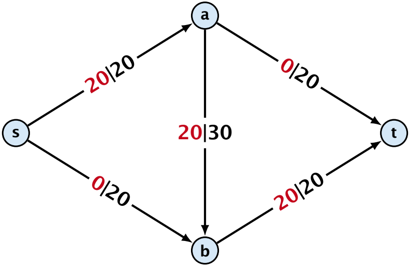

flow value: 20

The Residual Graph

From the graph $G = (V, E, c)$ and the current flow f we construct an auxiliary graph $G_f = (V, E_f, c_f)$ (the residual graph):

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- G_f has edge e'_1 with capacity $\max\{0, c(e_1) - f(e_1) + f(e_2)\}$ and e'_2 with with capacity $\max\{0, c(e_2) - f(e_2) + f(e_1)\}$.

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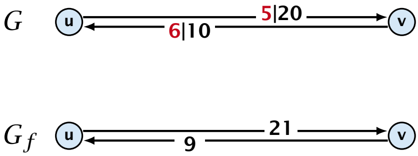
- G_f has edge e_1' with capacity $\max\{0, c(e_1) - f(e_1) + f(e_2)\}$ and e_2' with with capacity $\max\{0, c(e_2) - f(e_2) + f(e_1)\}$.



Augmenting Path Algorithm

Definition 37

An augmenting path with respect to flow f, is a path from s to t in the auxiliary graph G_f that contains only edges with non-zero capacity.

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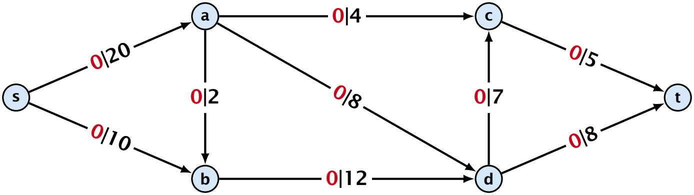
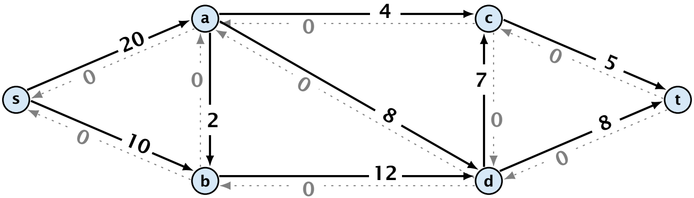
Algorithm 1 FordFulkerson$(G = (V, E, c))$

```
Initialize f(e) ← 0 for all edges.
while ∃ augmenting path p in G_f do
    augment as much flow along p as possible.
```

Augmenting Paths



flow value: 0



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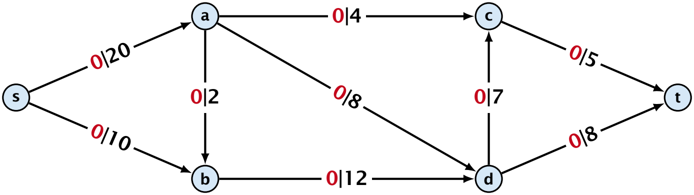

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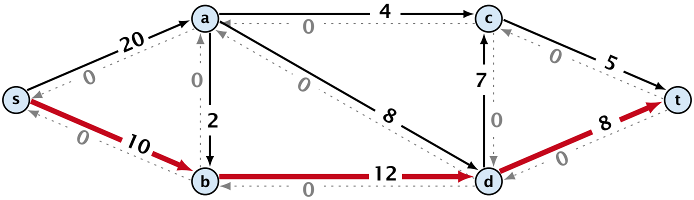

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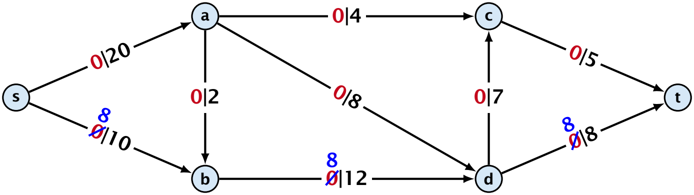

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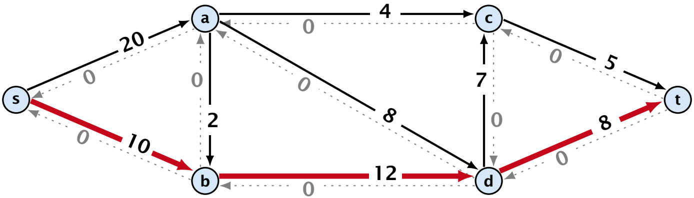

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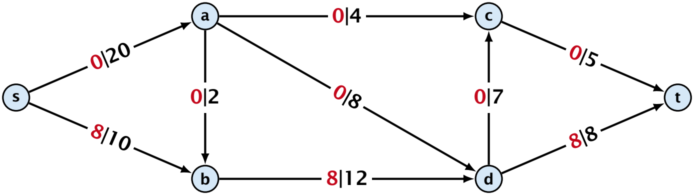
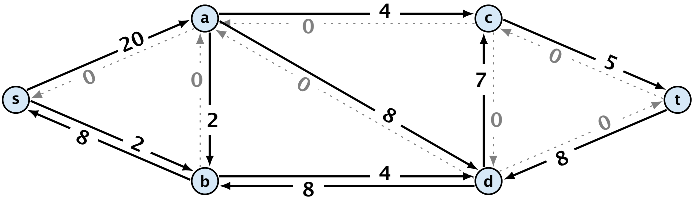


flow value: 8



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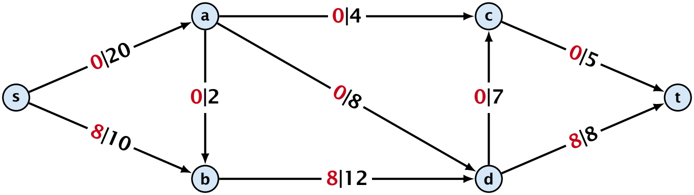

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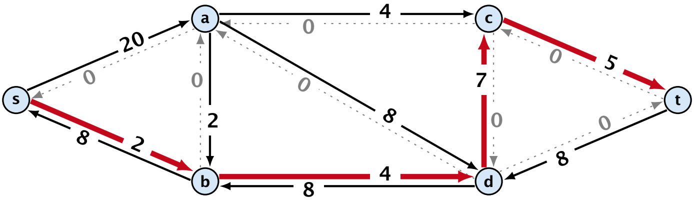

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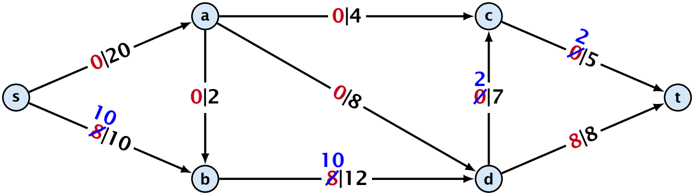

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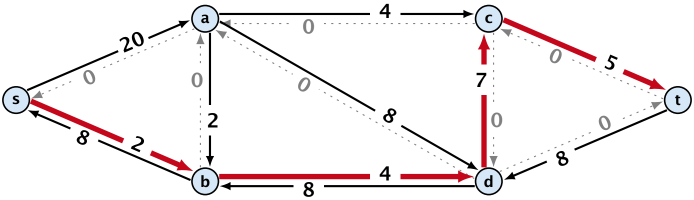

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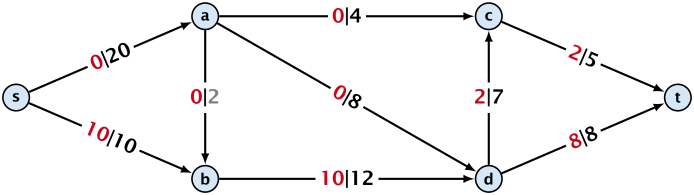
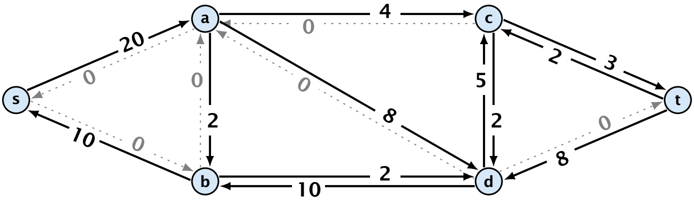


flow value: 10



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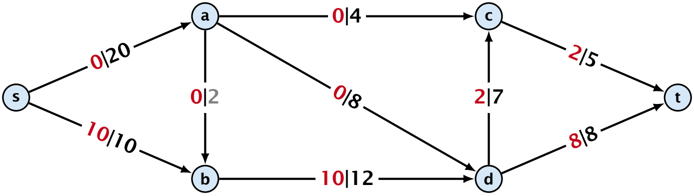

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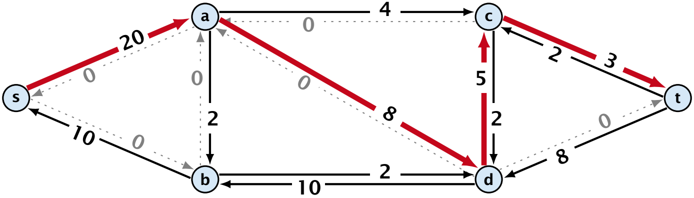

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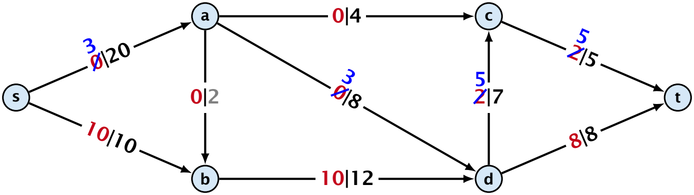

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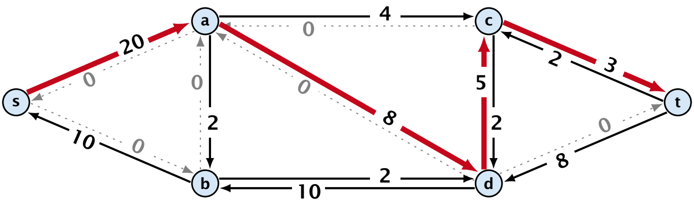

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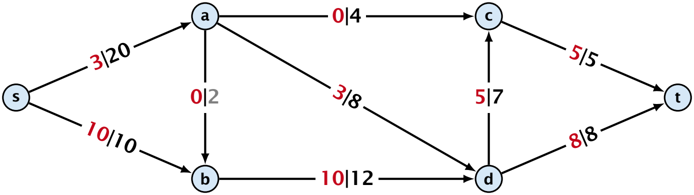
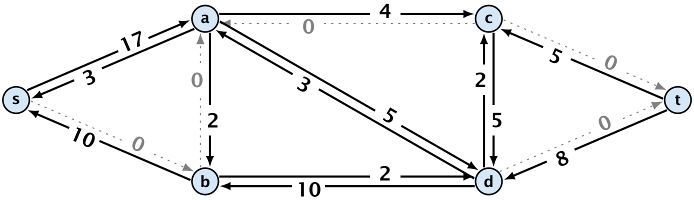


flow value: 13



Augmenting Path Algorithm

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Theorem 38

*A flow f is a maximum flow **iff** there are no augmenting paths.*

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The value of a maximum flow is equal to the value of a minimum cut.

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Proof.

Let f be a flow. The following are equivalent:

1. There exists a cut A such that $\mathrm{val}(f) = \mathrm{cap}(A, V \setminus A)$.

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Proof.
Let f be a flow. The following are equivalent:

1. There exists a cut A such that $\mathrm{val}(f) = \mathrm{cap}(A, V \setminus A)$.
2. Flow f is a maximum flow.
3. There is no augmenting path w.r.t. f.

□

Augmenting Path Algorithm

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This we already showed.

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If there were an augmenting path, we could improve the flow. Contradiction.

3. $\Longrightarrow$ 1.

- Let f be a flow with no augmenting paths.
- Let A be the set of vertices reachable from s in the residual graph along non-zero capacity edges.
- Since there is no augmenting path we have $s \in A$ and $t \notin A$.

Augmenting Path Algorithm

$\text{val}(f)$

Augmenting Path Algorithm

$$\mathrm{val}(f) = \sum_{e \in \mathrm{out}(A)} f(e) - \sum_{e \in \mathrm{into}(A)} f(e)$$

Augmenting Path Algorithm

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This finishes the proof.

Here the first equality uses the flow value lemma, and the second exploits the fact that the flow along incoming edges must be 0 as the residual graph does not have edges leaving A.

Analysis

Assumption:
All capacities are integers between 1 and C.

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Invariant:
Every flow value $f(e)$ and every residual capacity $c_f(e)$ remains integral troughout the algorithm.

Lemma 40

The algorithm terminates in at most $\mathrm{val}(f^*) \leq nC$ *iterations, where* f^* *denotes the maximum flow. Each iteration can be implemented in time* $\mathcal{O}(m)$*. This gives a total running time of* $\mathcal{O}(nmC)$*.*

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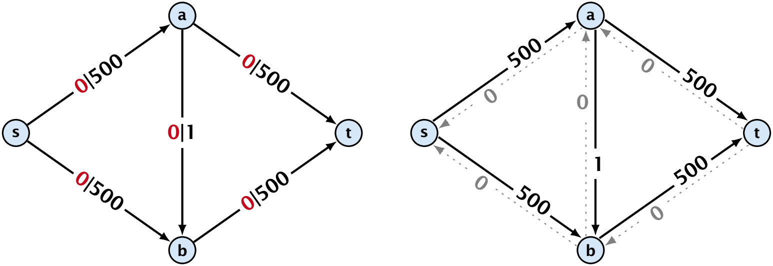
Theorem 41

If all capacities are integers, then there exists a maximum flow for which every flow value $f(e)$ *is integral.*

A Bad Input

Problem: The running time may not be polynomial



flow value: 0

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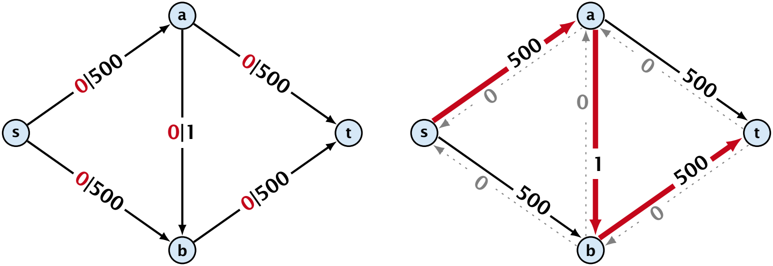

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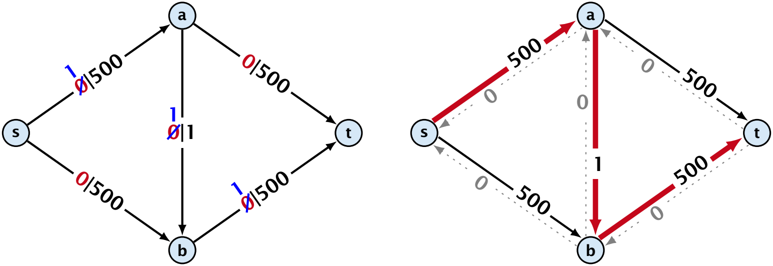

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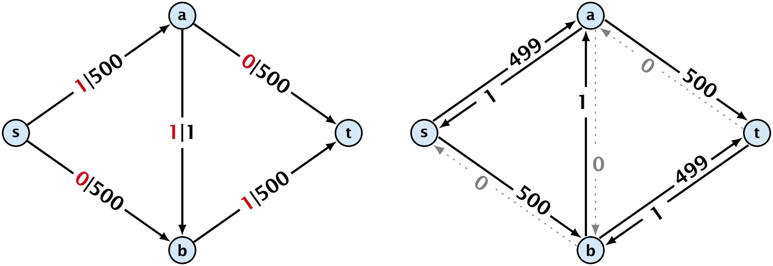

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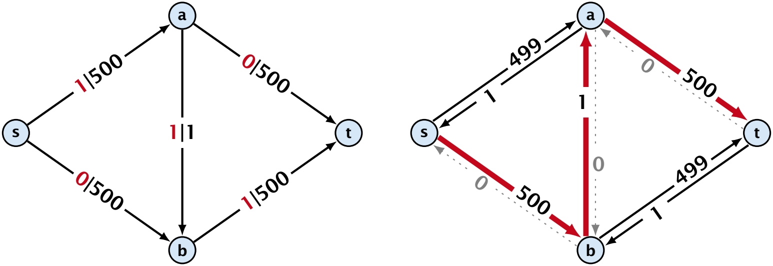

flow value: 1

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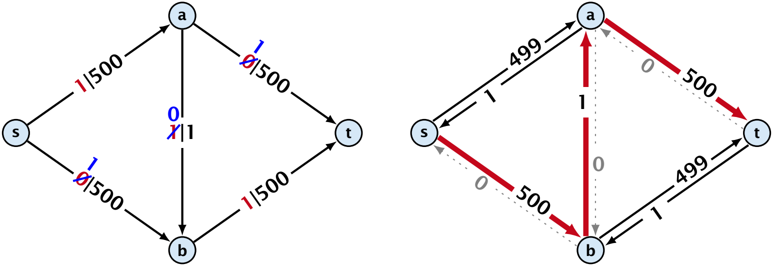

flow value: 1

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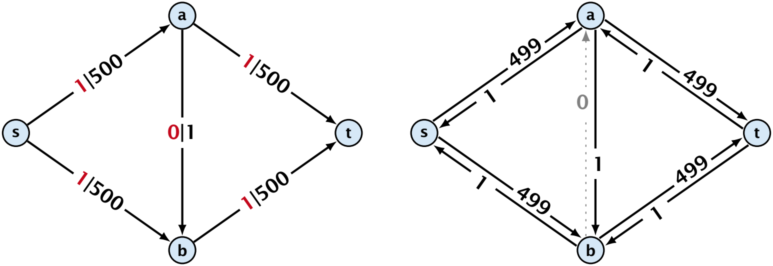
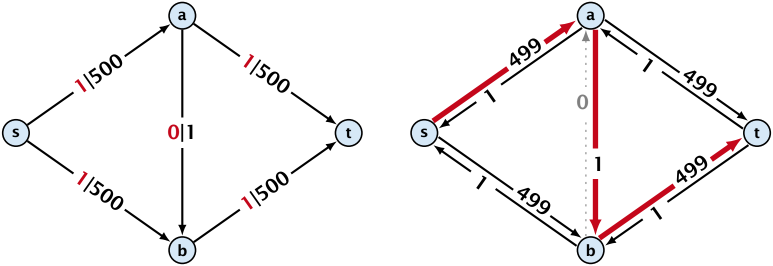


flow value: 2

A Bad Input

Problem: The running time may not be polynomial



flow value: 2

A Bad Input

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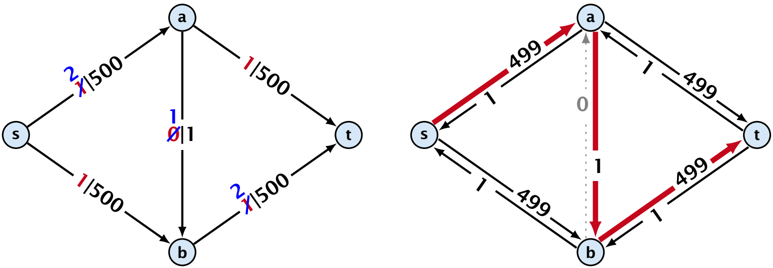

flow value: 2

A Bad Input

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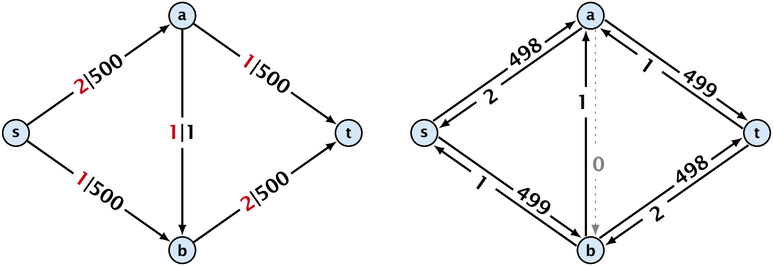
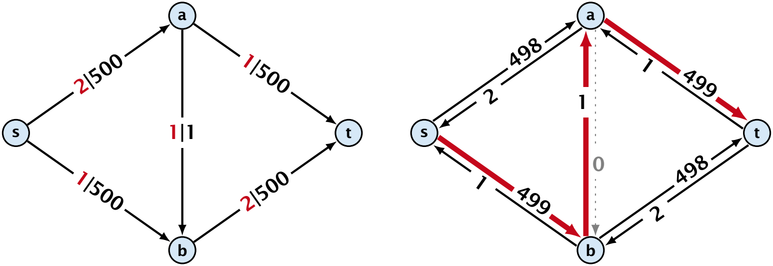


flow value: 3

A Bad Input

Problem: The running time may not be polynomial



flow value: 3

A Bad Input

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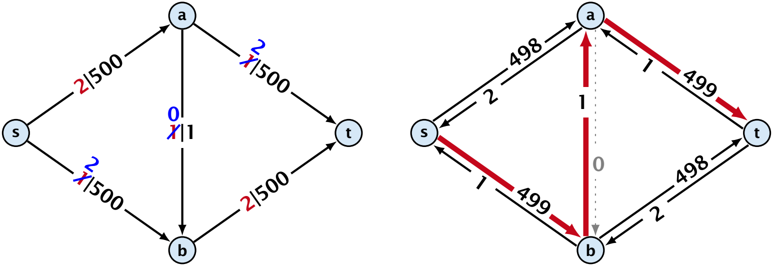

flow value: 3

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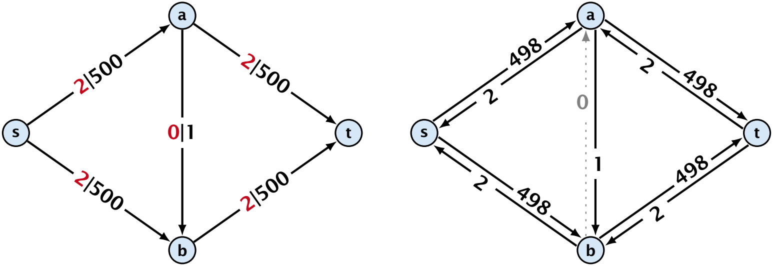
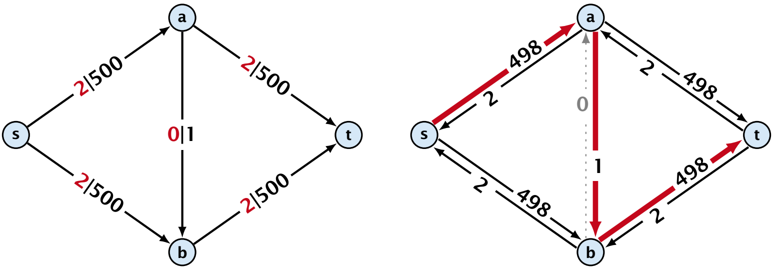


flow value: 4

A Bad Input

Problem: The running time may not be polynomial



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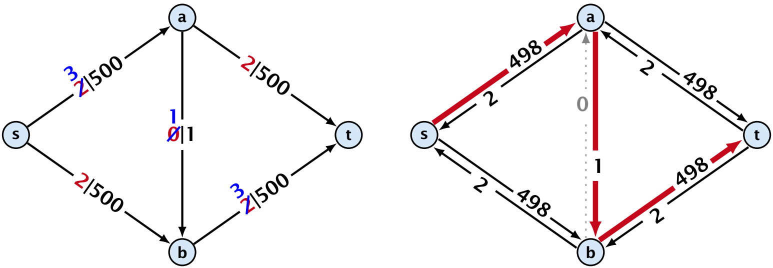

flow value: 4

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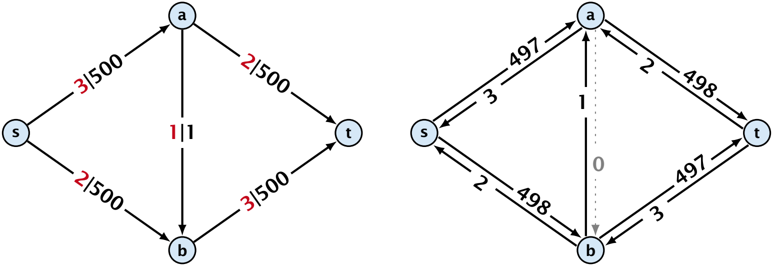
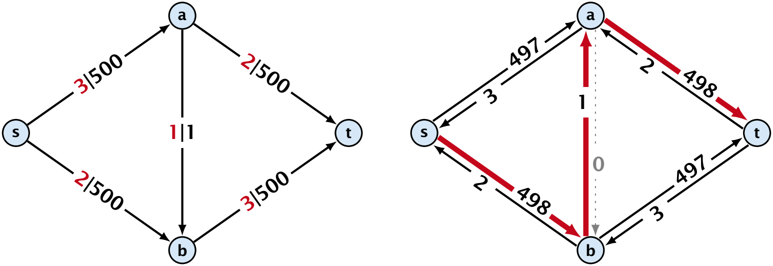


flow value: 5

A Bad Input

Problem: The running time may not be polynomial



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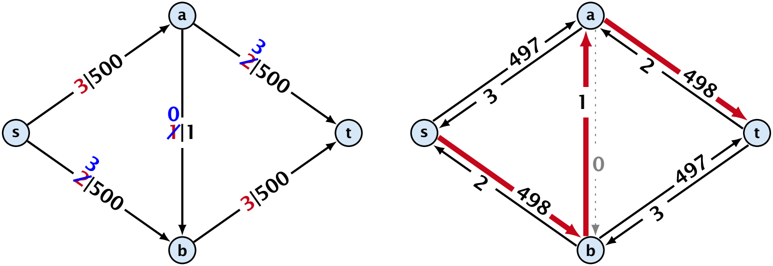

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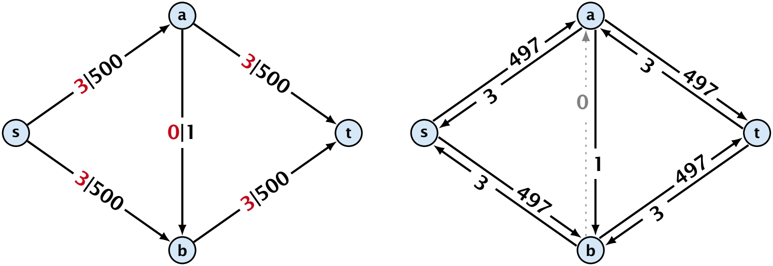
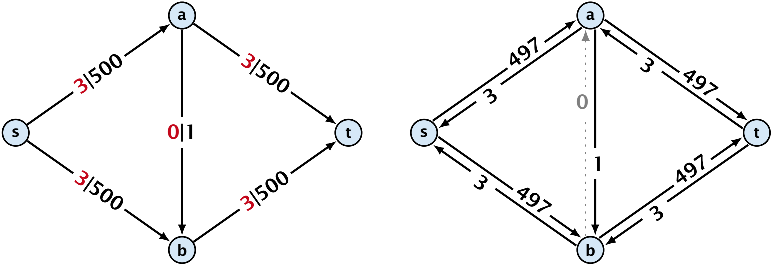


flow value: 6

A Bad Input

Problem: The running time may not be polynomial



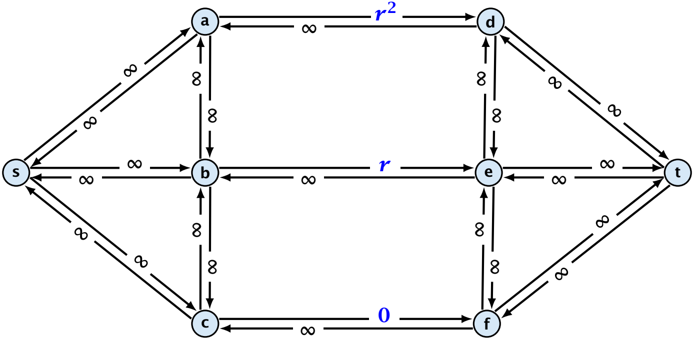
flow value: 6

Question:

Can we tweak the algorithm so that the running time is polynomial in the input length?

A Pathological Input

Let $r = \frac{1}{2}(\sqrt{5} - 1)$. Then $r^{n+2} = r^n - r^{n+1}$.



flow value: 0

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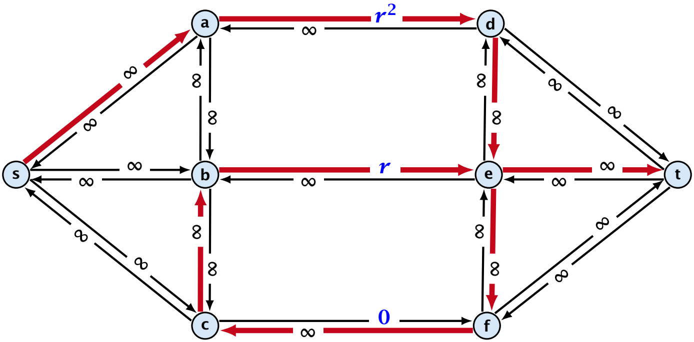

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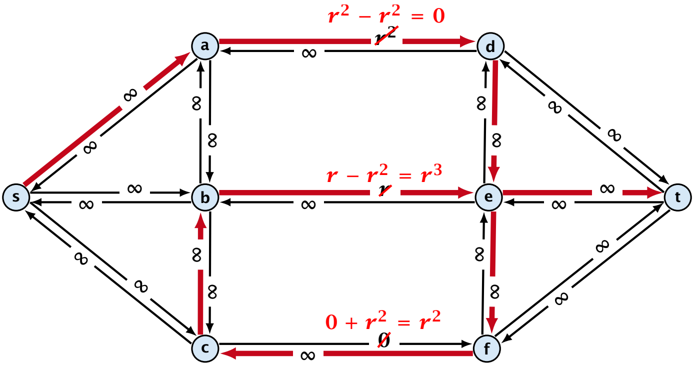

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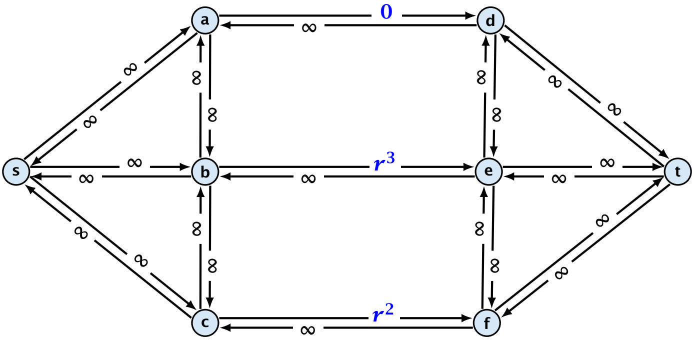

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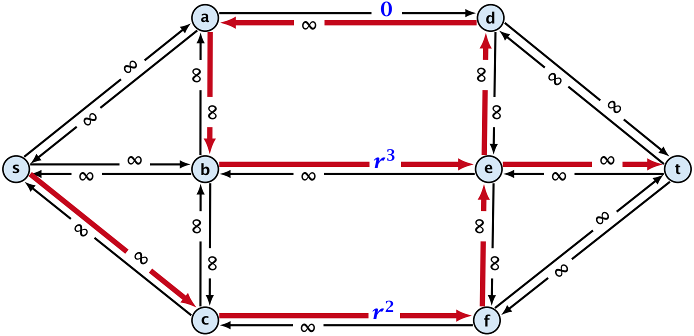

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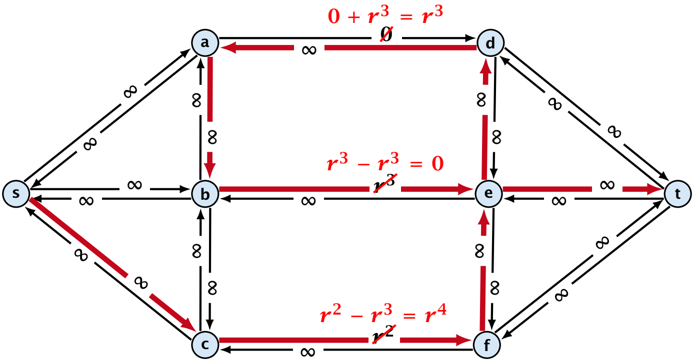

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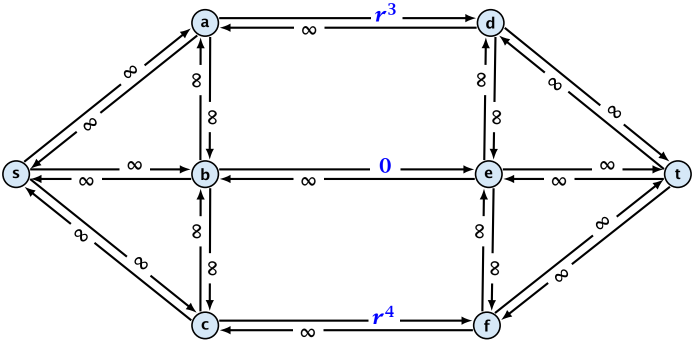

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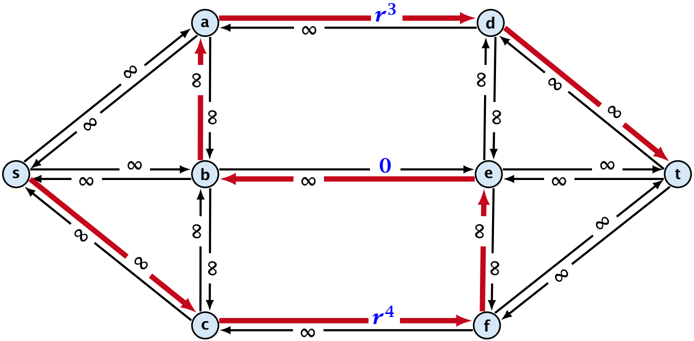

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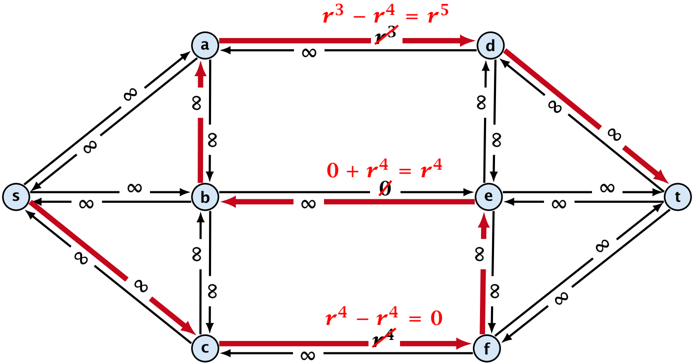

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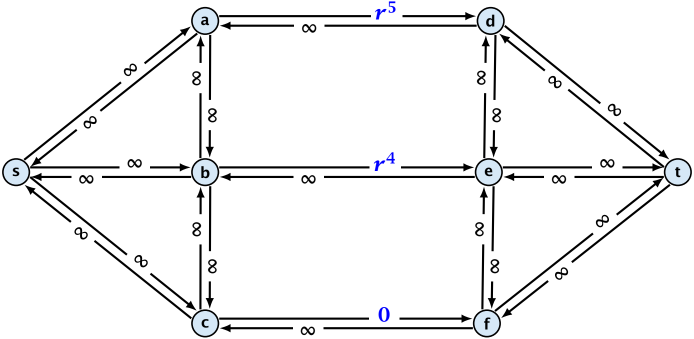

flow value: $r^2 + r^3 + r^4$

Running time may be infinite!!!

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Overview: Shortest Augmenting Paths

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Lemma 42

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After at most $\mathcal{O}(m)$ augmentations, the length of the shortest augmenting path strictly increases.

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- $\mathcal{O}(m)$ augmentations for paths of exactly $k < n$ edges.

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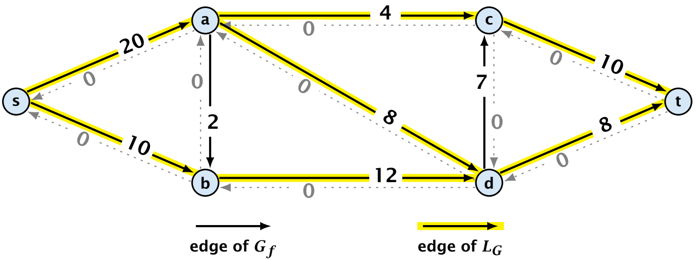

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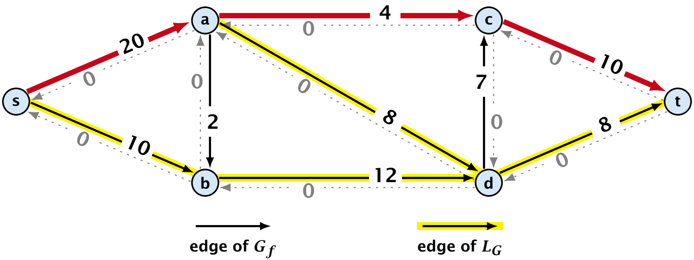

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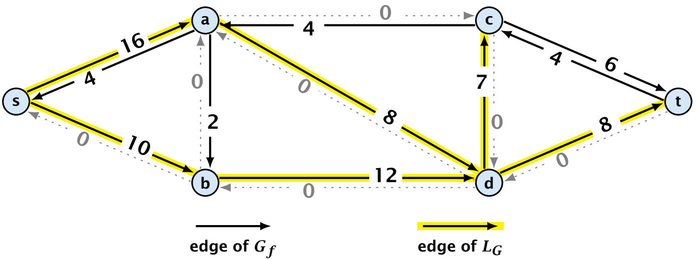

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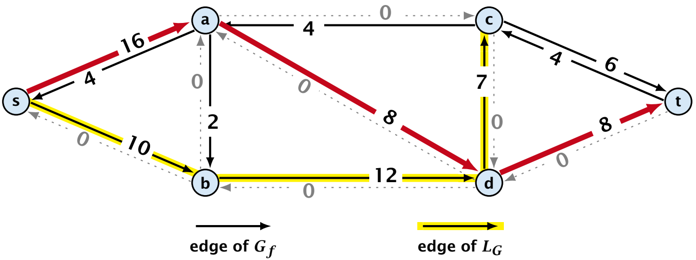

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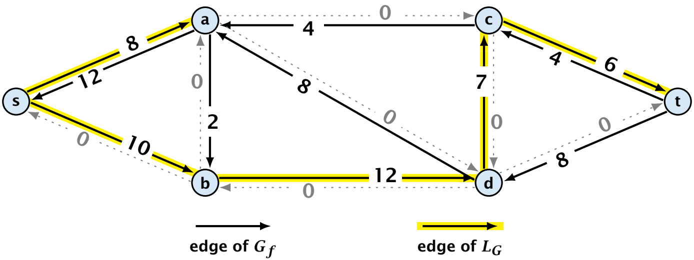

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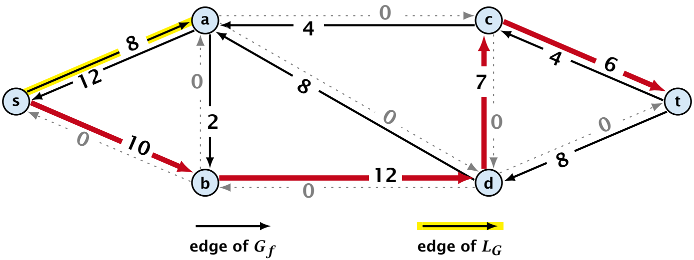

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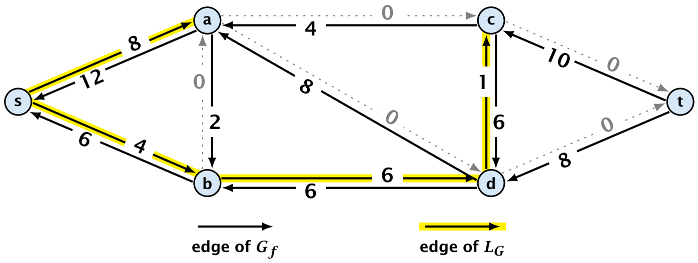

In the following we assume that the residual graph G_f does not contain zero capacity edges.

This means, we construct it in the usual sense and then delete edges of zero capacity.

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The length of the shortest augmenting path never decreases.

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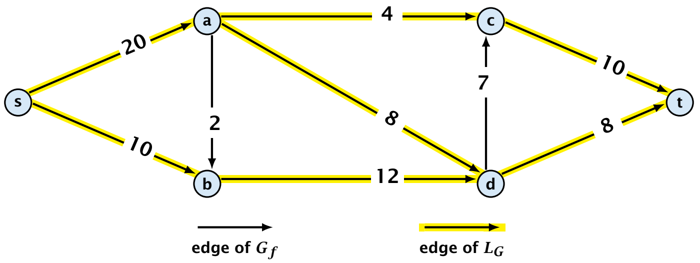

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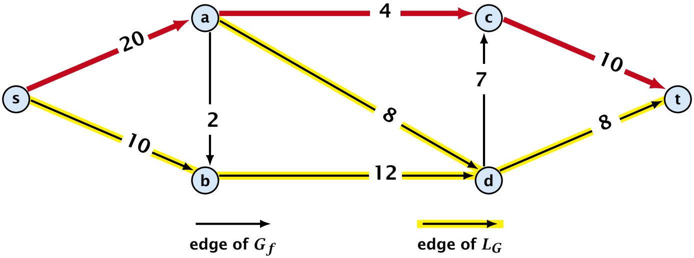

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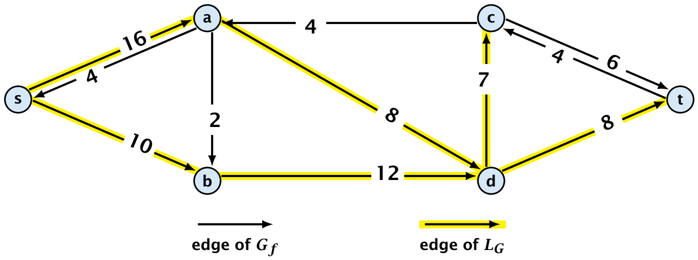

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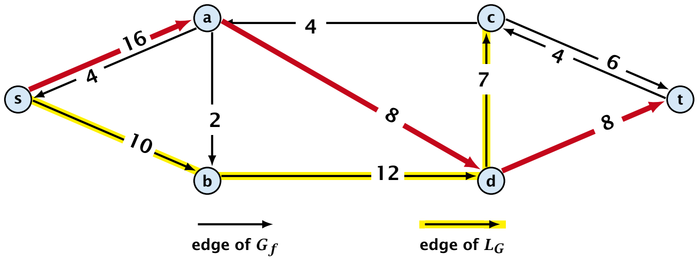

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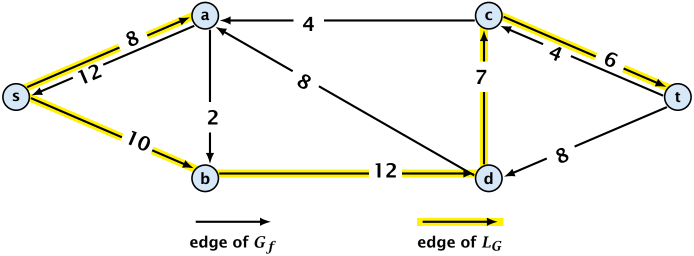

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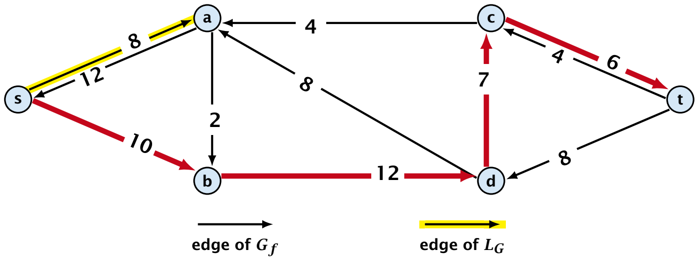

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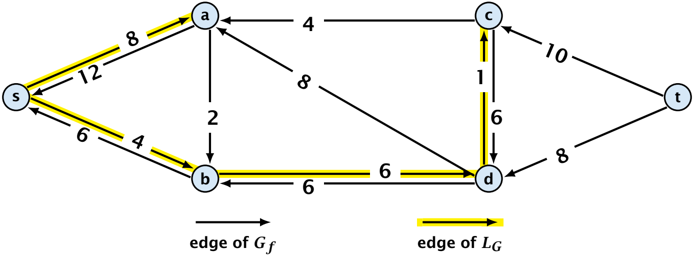

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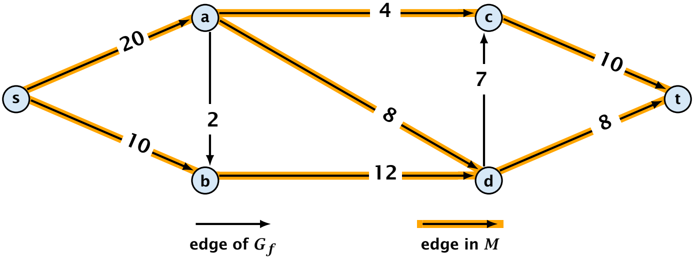

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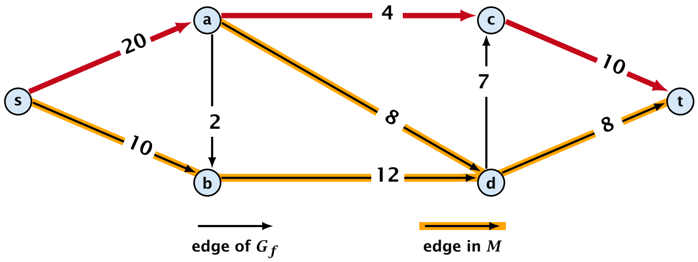

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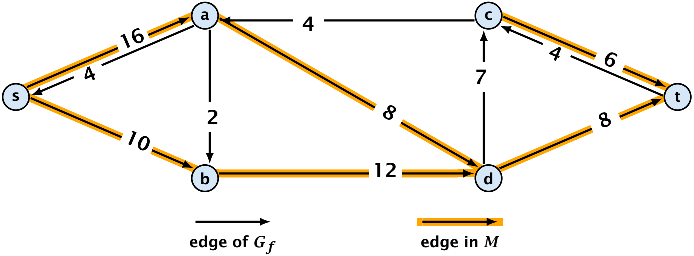

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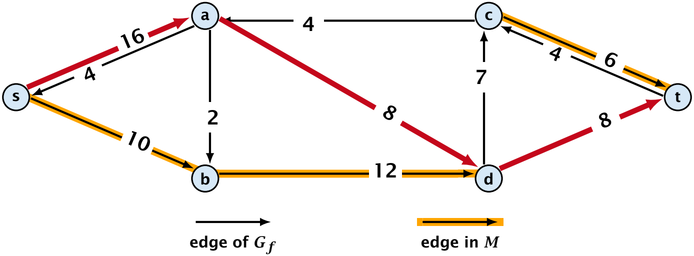

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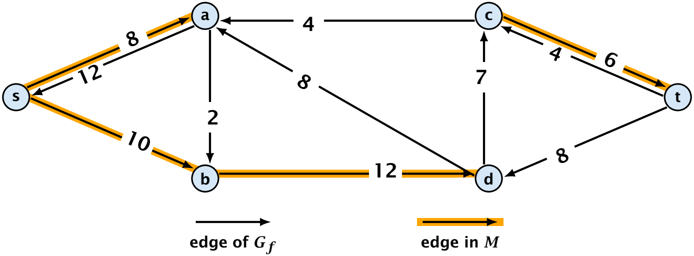

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Note:

There always exists a set of m augmentations that gives a maximum flow (why?).

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However, we can improve the running time to $\mathcal{O}(mn^2)$ by improving the running time for finding an augmenting path (currently we assume $\mathcal{O}(m)$ per augmentation for this).

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Note that M is not the set of edges of the level graph but a subset of level-graph edges.

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There are at most n phases. Hence, total cost is $\mathcal{O}(mn^2)$.

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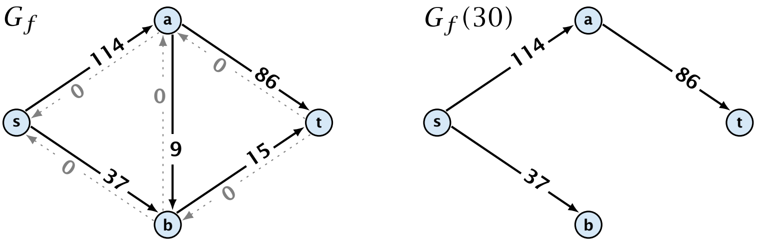
Intuition:

- Choosing a path with the highest bottleneck increases the flow as much as possible in a single step.
- Don't worry about finding the exact bottleneck.
- Maintain scaling parameter Δ.
- $G_f(\Delta)$ is a sub-graph of the residual graph G_f that contains only edges with capacity at least Δ.



Capacity Scaling

Algorithm 1 maxflow(G, s, t, c)

foreach $e \in E$ **do** $f_e \leftarrow 0$;
$\Delta \leftarrow 2^{\lceil \log_2 C \rceil}$
while $\Delta \geq 1$ **do**
    $G_f(\Delta) \leftarrow \Delta$-residual graph
    while there is augmenting path P in $G_f(\Delta)$ **do**
        $f \leftarrow$ augment(f, c, P)
        update$(G_f(\Delta))$
    $\Delta \leftarrow \Delta/2$
return f

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- this means we have a maximum flow.

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- In G_f this cut can have capacity at most $m\Delta$.
- This gives me an upper bound on the flow that I can still add.

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Theorem 50

We need $\mathcal{O}(m \log C)$ augmentations. The algorithm can be implemented in time $\mathcal{O}(m^2 \log C)$.