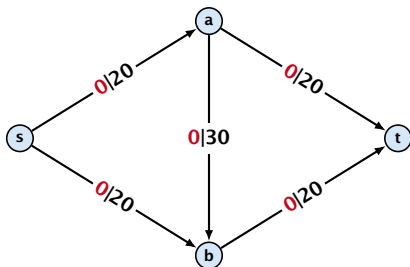


## Greedy-algorithm:

- ▶ start with  $f(e) = 0$  everywhere
- ▶ find an  $s$ - $t$  path with  $f(e) < c(e)$  on every edge
- ▶ augment flow along the path
- ▶ repeat as long as possible

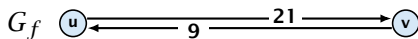
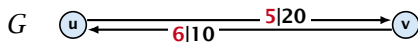


fflowvalue: 00

# The Residual Graph

From the graph  $G = (V, E, c)$  and the current flow  $f$  we construct an auxiliary graph  $G_f = (V, E_f, c_f)$  (the residual graph):

- ▶ Suppose the original graph has edges  $e_1 = (u, v)$ , and  $e_2 = (v, u)$  between  $u$  and  $v$ .
- ▶  $G_f$  has edge  $e'_1$  with capacity  $\max\{0, c(e_1) - f(e_1) + f(e_2)\}$  and  $e'_2$  with with capacity  $\max\{0, c(e_2) - f(e_2) + f(e_1)\}$ .



# Augmenting Path Algorithm

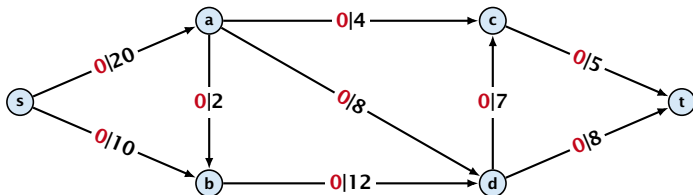
## Definition 4

An **augmenting path** with respect to flow  $f$ , is a path from  $s$  to  $t$  in the auxiliary graph  $G_f$  that contains only edges with non-zero capacity.

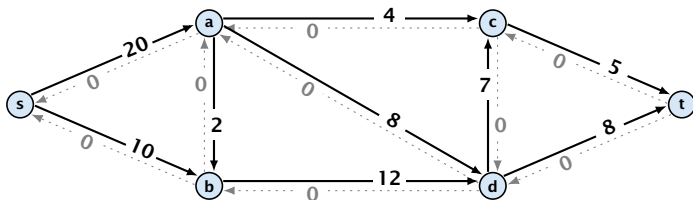
### Algorithm 1 FordFulkerson( $G = (V, E, c)$ )

- 1: Initialize  $f(e) \leftarrow 0$  for all edges.
- 2: **while**  $\exists$  augmenting path  $p$  in  $G_f$  **do**
- 3:     augment as much flow along  $p$  as possible.

# Augmenting Paths



flow value: 0



# Augmenting Path Algorithm

## Theorem 5

A flow  $f$  is a maximum flow **iff** there are no augmenting paths.

## Theorem 6

The value of a maximum flow is equal to the value of a minimum cut.

## Proof.

Let  $f$  be a flow. The following are equivalent:

1. There exists a cut  $A$  such that  $\text{val}(f) = \text{cap}(A, V \setminus A)$ .
2. Flow  $f$  is a maximum flow.
3. There is no augmenting path w.r.t.  $f$ .



# Augmenting Path Algorithm

1.  $\Rightarrow$  2.

This we already showed.

2.  $\Rightarrow$  3.

If there were an augmenting path, we could improve the flow.  
Contradiction.

3.  $\Rightarrow$  1.

- ▶ Let  $f$  be a flow with no augmenting paths.
- ▶ Let  $A$  be the set of vertices reachable from  $s$  in the residual graph along non-zero capacity edges.
- ▶ Since there is no augmenting path we have  $s \in A$  and  $t \notin A$ .

# Augmenting Path Algorithm

$$\begin{aligned}\text{val}(f) &= \sum_{e \in \text{out}(A)} f(e) - \sum_{e \in \text{into}(A)} f(e) \\ &= \sum_{e \in \text{out}(A)} c(e) \\ &= \text{cap}(A, V \setminus A)\end{aligned}$$

This finishes the proof.

Here the first equality uses the flow value lemma, and the second exploits the fact that the flow along incoming edges must be 0 as the residual graph does not have edges leaving  $A$ .

**Assumption:**

All capacities are integers between 1 and  $C$ .

**Invariant:**

Every flow value  $f(e)$  and every residual capacity  $c_f(e)$  remains integral throughout the algorithm.



## Lemma 7

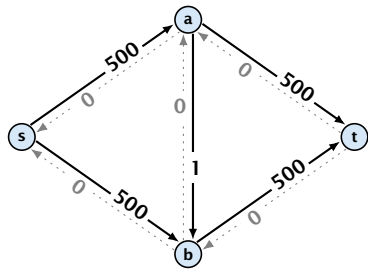
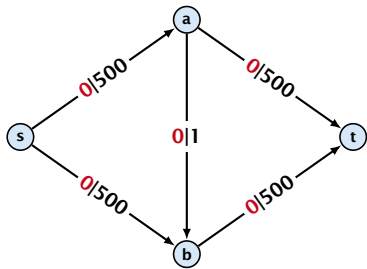
The algorithm terminates in at most  $\text{val}(f^*) \leq nC$  iterations, where  $f^*$  denotes the maximum flow. Each iteration can be implemented in time  $\mathcal{O}(m)$ . This gives a total running time of  $\mathcal{O}(nmC)$ .

## Theorem 8

If all capacities are integers, then there exists a maximum flow for which every flow value  $f(e)$  is integral.

# A Bad Input

**Problem:** The running time may not be polynomial



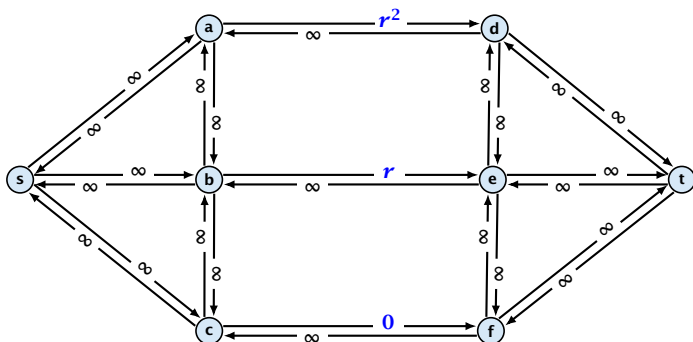
**flow value: 1**

**Question:**

Can we tweak the algorithm so that the running time is polynomial in the input length?

# A Pathological Input

Let  $r = \frac{1}{2}(\sqrt{5} - 1)$ . Then  $r^{n+2} = r^n - r^{n+1}$ .



flow value:  $r^0 r^3 r^4$

Running time may be infinite!!!

## How to choose augmenting paths?

- ▶ We need to find paths efficiently.
- ▶ We want to guarantee a small number of iterations.

### Several possibilities:

- ▶ Choose path with maximum bottleneck capacity.
- ▶ Choose path with sufficiently large bottleneck capacity.
- ▶ Choose the shortest augmenting path.