

## 13.2 Relabel to Front

### Algorithm 1 relabel-to-front( $G, s, t$ )

```
1: initialize preflow
2: initialize node list  $L$  containing  $V \setminus \{s, t\}$  in any order
3: foreach  $u \in V \setminus \{s, t\}$  do
4:    $u.current\text{-neighbour} \leftarrow u.neighbour\text{-list-head}$ 
5:  $u \leftarrow L.head$ 
6: while  $u \neq \text{null}$  do
7:    $old\text{-height} \leftarrow \ell(u)$ 
8:   discharge( $u$ )
9:   if  $\ell(u) > old\text{-height}$  then // relabel happened
10:    move  $u$  to the front of  $L$ 
11:    $u \leftarrow u.next$ 
```

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### Lemma 6 (Invariant)

In Line 6 of the relabel-to-front algorithm the following invariant holds.

1. The sequence  $L$  is topologically sorted w.r.t. the set of admissible edges; this means for an admissible edge  $(x, y)$  the node  $x$  appears before  $y$  in sequence  $L$ .
2. No node before  $u$  in the list  $L$  is active.

### Proof:

#### ► Initialization:

1. In the beginning  $s$  has label  $n \geq 2$ , and all other nodes have label 0. Hence, no edge is admissible, which means that any ordering  $L$  is permitted.
2. We start with  $u$  being the head of the list; hence no node before  $u$  can be active

#### ► Maintenance:

1.
  - Pushes do not create any new admissible edges. Therefore, if discharge() does not relabel  $u$ ,  $L$  is still topologically sorted.
  - After relabeling,  $u$  cannot have admissible incoming edges as such an edge  $(x, u)$  would have had a difference  $\ell(x) - \ell(u) \geq 2$  before the re-labeling (such edges do not exist in the residual graph). Hence, moving  $u$  to the front does not violate the sorting property for any edge; however it fixes this property for all admissible edges leaving  $u$  that were generated by the relabeling.

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### Proof:

#### ► Maintenance:

2. If we do a relabel there is nothing to prove because the only node before  $u'$  ( $u$  in the next iteration) will be the current  $u$ ; the discharge( $u$ ) operation only terminates when  $u$  is not active anymore.

For the case that we do not relabel, observe that the only way a predecessor could be active is that we push flow to it via an admissible arc. However, all admissible arcs point to successors of  $u$ .

Note that the invariant means that for  $u = \text{null}$  we have a preflow with a valid labelling that does not have active nodes. This means we have a maximum flow.

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### Lemma 7

There are at most  $\mathcal{O}(n^3)$  calls to  $\text{discharge}(u)$ .

Every discharge operation without a relabel advances  $u$  (the current node within list  $L$ ). Hence, if we have  $n$  discharge operations without a relabel we have  $u = \text{null}$  and the algorithm terminates.

Therefore, the number of calls to discharge is at most  $n(\# \text{relabels} + 1) = \mathcal{O}(n^3)$ .

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### Lemma 8

The cost for all relabel-operations is only  $\mathcal{O}(n^2)$ .

A relabel-operation at a node is constant time (increasing the label and resetting  $u.\text{current-neighbour}$ ). In total we have  $\mathcal{O}(n^2)$  relabel-operations.

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Recall that a saturating push operation ( $\min\{c_f(e), f(u)\} = c_f(e)$ ) can also be a deactivating push operation ( $\min\{c_f(e), f(u)\} = f(u)$ ).

### Lemma 9

The cost for all saturating push-operations that are **not** deactivating is only  $\mathcal{O}(mn)$ .

Note that such a push-operation leaves the node  $u$  active but makes the edge  $e$  disappear from the residual graph. Therefore the push-operation is immediately followed by an increase of the pointer  $u.\text{current-neighbour}$ .

This pointer can traverse the neighbour-list at most  $\mathcal{O}(n)$  times (upper bound on number of relabels) and the neighbour-list has only  $\text{degree}(u) + 1$  many entries (+1 for null-entry).

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### Lemma 10

The cost for all deactivating push-operations is only  $\mathcal{O}(n^3)$ .

A deactivating push-operation takes constant time and ends the current call to  $\text{discharge}()$ . Hence, there are only  $\mathcal{O}(n^3)$  such operations.

### Theorem 11

The push-relabel algorithm with the rule relabel-to-front takes time  $\mathcal{O}(n^3)$ .