

6.2 Master Theorem

Note that the cases do not cover all possibilities.

Lemma 1

Let $a \geq 1$, $b \geq 1$ and $\epsilon > 0$ denote constants. Consider the recurrence

$$T(n) = aT\left(\frac{n}{b}\right) + f(n) .$$

Case 1.

If $f(n) = \mathcal{O}(n^{\log_b(a)-\epsilon})$ then $T(n) = \Theta(n^{\log_b a})$.

Case 2.

If $f(n) = \Theta(n^{\log_b(a)} \log^k n)$ then $T(n) = \Theta(n^{\log_b a} \log^{k+1} n)$,
 $k \geq 0$.

Case 3.

If $f(n) = \Omega(n^{\log_b(a)+\epsilon})$ and for sufficiently large n
 $af\left(\frac{n}{b}\right) \leq cf(n)$ for some constant $c < 1$ then $T(n) = \Theta(f(n))$.

6.2 Master Theorem

We prove the Master Theorem for the case that n is of the form b^ℓ , and we assume that the non-recursive case occurs for problem size 1 and incurs cost 1.

The Recursion Tree

The running time of a recursive algorithm can be visualized by a recursion tree:

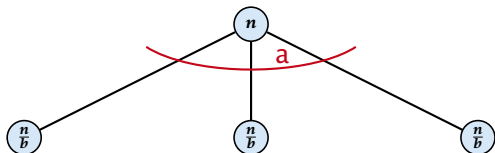
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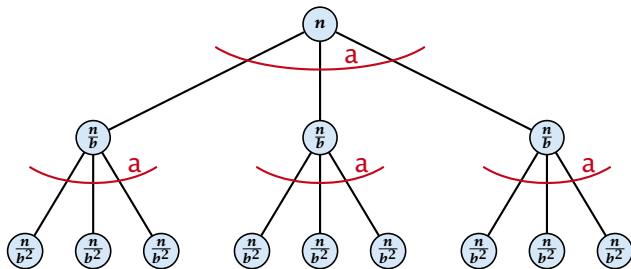
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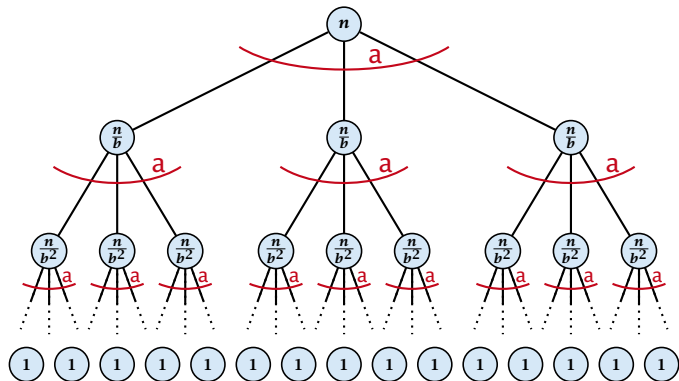
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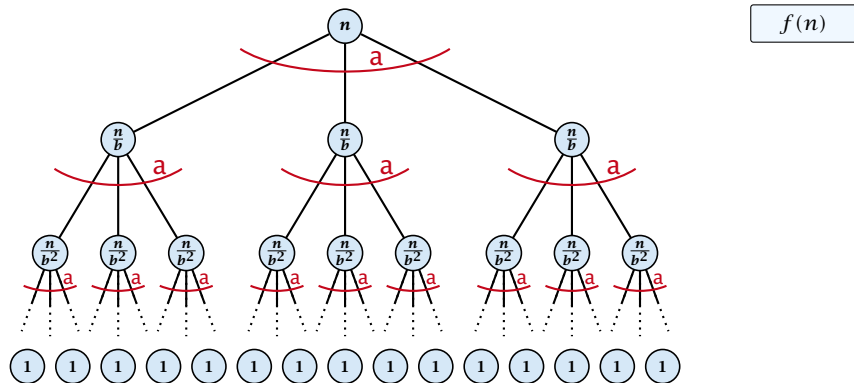
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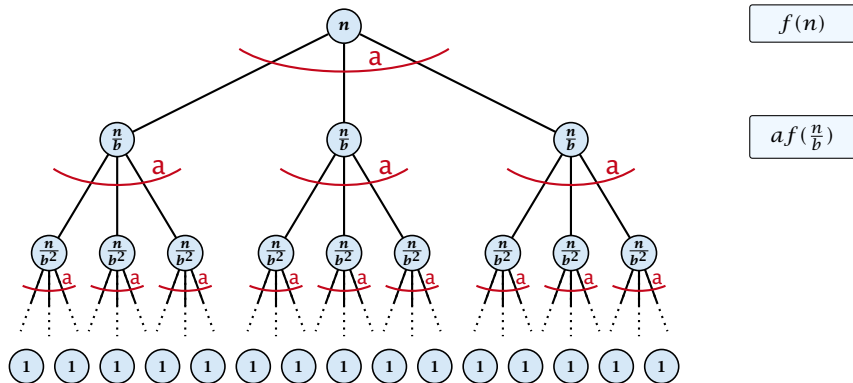
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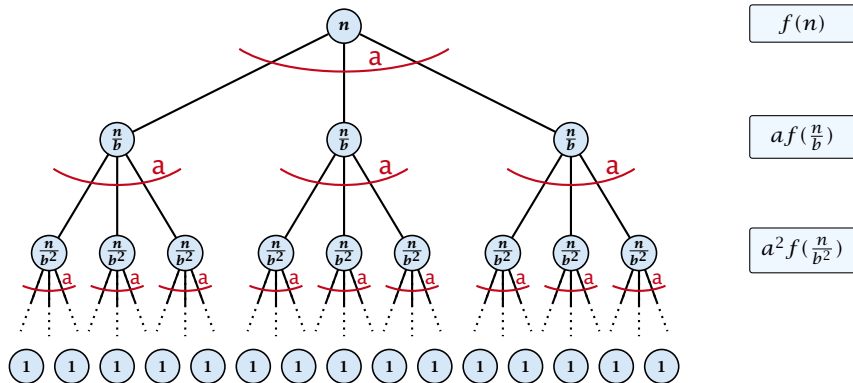
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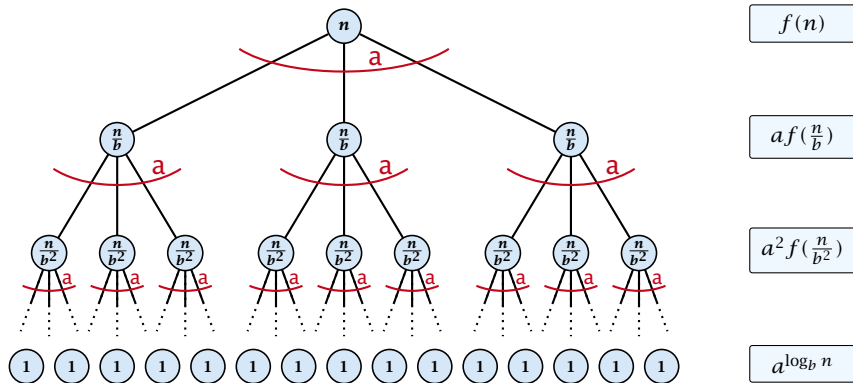
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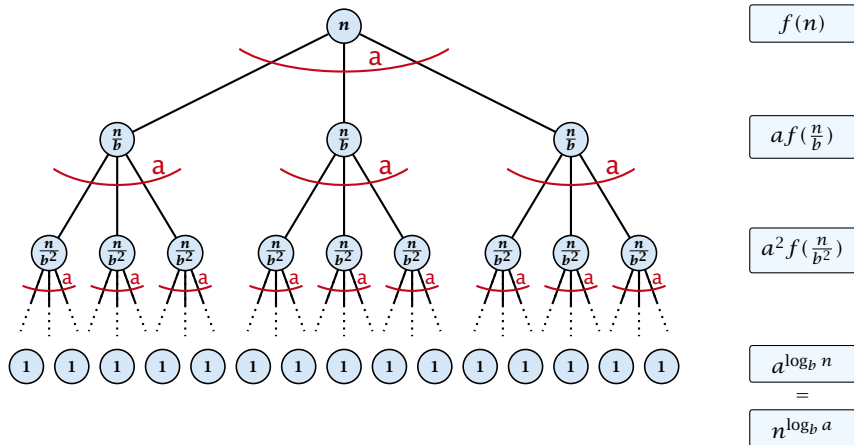
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The running time of a recursive algorithm can be visualized by a recursion tree:



The Recursion Tree

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6.2 Master Theorem

This gives

$$T(n) = n^{\log_b a} + \sum_{i=0}^{\log_b n - 1} a^i f\left(\frac{n}{b^i}\right).$$

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$$T(n) - n^{\log_b a} = \sum_{i=0}^{\log_b n - 1} a^i f\left(\frac{n}{b^i}\right)$$

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$$\begin{aligned} T(n) - n^{\log_b a} &= \sum_{i=0}^{\log_b n - 1} a^i f\left(\frac{n}{b^i}\right) \\ &\leq c \sum_{i=0}^{\log_b n - 1} a^i \left(\frac{n}{b^i}\right)^{\log_b a - \epsilon} \end{aligned}$$

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$$\boxed{\sum_{i=0}^k q^i = \frac{q^{k+1} - 1}{q - 1}} = cn^{\log_b a - \epsilon} (b^{\epsilon \log_b n} - 1) / (b^{\epsilon} - 1)$$

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Hence,

$$T(n) \leq \left(\frac{c}{b^{\epsilon} - 1} + 1 \right) n^{\log_b(a)}$$

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Hence,

$$T(n) \leq \left(\frac{c}{b^{\epsilon} - 1} + 1 \right) n^{\log_b(a)} \quad \Rightarrow T(n) = \mathcal{O}(n^{\log_b a}).$$

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Hence,

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$$= cn^{\log_b a} \sum_{i=1}^{\ell} i^k \approx \frac{1}{k} \ell^{k+1}$$

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$$\approx \frac{c}{k} n^{\log_b a} \ell^{k+1}$$

$$\Rightarrow T(n) = \mathcal{O}(n^{\log_b a} \log^{k+1} n).$$

Case 3. Now suppose that $f(n) \geq dn^{\log_b a + \epsilon}$, and that for sufficiently large n : $af(n/b) \leq cf(n)$, for $c < 1$.

Where did we use $f(n) \geq \Omega(n^{\log_b a + \epsilon})$?

Case 3. Now suppose that $f(n) \geq dn^{\log_b a + \epsilon}$, and that for sufficiently large n : $af(n/b) \leq cf(n)$, for $c < 1$.

From this we get $a^i f(n/b^i) \leq c^i f(n)$, where we assume that $n/b^{i-1} \geq n_0$ is still sufficiently large.

Where did we use $f(n) \geq \Omega(n^{\log_b a + \epsilon})$?

Case 3. Now suppose that $f(n) \geq dn^{\log_b a + \epsilon}$, and that for sufficiently large n : $af(n/b) \leq cf(n)$, for $c < 1$.

From this we get $a^i f(n/b^i) \leq c^i f(n)$, where we assume that $n/b^{i-1} \geq n_0$ is still sufficiently large.

$$T(n) - n^{\log_b a} = \sum_{i=0}^{\log_b n - 1} a^i f\left(\frac{n}{b^i}\right)$$

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$$\begin{aligned} T(n) - n^{\log_b a} &= \sum_{i=0}^{\log_b n - 1} a^i f\left(\frac{n}{b^i}\right) \\ &\leq \sum_{i=0}^{\log_b n - 1} c^i f(n) + \mathcal{O}(n^{\log_b a}) \end{aligned}$$

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$$q < 1 : \sum_{i=0}^n q^i = \frac{1-q^{n+1}}{1-q} \leq \frac{1}{1-q}$$

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Hence,

$$T(n) \leq \mathcal{O}(f(n))$$

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From this we get $a^i f(n/b^i) \leq c^i f(n)$, where we assume that $n/b^{i-1} \geq n_0$ is still sufficiently large.

$$\begin{aligned} T(n) - n^{\log_b a} &= \sum_{i=0}^{\log_b n - 1} a^i f\left(\frac{n}{b^i}\right) \\ &\leq \sum_{i=0}^{\log_b n - 1} c^i f(n) + \mathcal{O}(n^{\log_b a}) \\ &\leq \frac{1}{1-c} f(n) + \mathcal{O}(n^{\log_b a}) \end{aligned}$$

$$q < 1 : \sum_{i=0}^n q^i = \frac{1-q^{n+1}}{1-q} \leq \frac{1}{1-q}$$

Hence,

$$T(n) \leq \mathcal{O}(f(n))$$

$$\Rightarrow T(n) = \Theta(f(n)).$$

Where did we use $f(n) \geq \Omega(n^{\log_b a + \epsilon})$?

Example: Multiplying Two Integers

Suppose we want to multiply two n -bit Integers, but our registers can only perform operations on integers of constant size.

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$$\begin{array}{r} 1\ 1\ 0\ 1\ 1\ 0\ 1\ 0\ 1\ A \\ 1\ 0\ 0\ 0\ 1\ 0\ 0\ 1\ 1\ B \\ \hline \end{array}$$

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$$\begin{array}{r} 1\ 1\ 0\ 1\ 1\ 0\ 1\ 0\ 1 \\ 1\ 0\ 0\ 0\ 1\ 0\ 0\ 1\ 1 \\ \hline \end{array}$$

The diagram shows two 9-bit integers, A and B, aligned for addition. Integer A is represented by the red bits 1 1 0 1 1 0 1 0 1, and integer B is represented by the blue bits 1 0 0 0 1 0 0 1 1. A horizontal line is drawn under the bits of B. A vertical light blue box highlights the rightmost bit of A (the least significant bit) and the bit of B directly below it, indicating the starting point of the addition process.

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Suppose we want to multiply two n -bit Integers, but our registers can only perform operations on integers of constant size.

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$$\begin{array}{r} 1\ 1\ 0\ 1\ 1\ 0\ 1\ 0\ 1 \\ 1\ 0\ 0\ 0\ 1\ 0\ 0\ 1\ 1 \\ \hline 0 \end{array} \begin{array}{l} A \\ B \end{array}$$

The diagram shows the addition of two 9-bit integers, A and B. Integer A is 110110101 and integer B is 100010011. A horizontal line is drawn under the second row. A vertical box on the right contains the result of the addition, which is 0. A small '1' is written below the horizontal line at the 9th bit position, indicating a carry.

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For this we first need to be able to add two integers A and B :

$$\begin{array}{rcccccccc} 1 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & A \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & B \\ \hline & & & & & & & 1 & & \\ & & & & & & & & & 0 \end{array}$$

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For this we first need to be able to add two integers A and B :

$$\begin{array}{r} 1\ 1\ 0\ 1\ 1\ 0\ 1\ 0\ 1\ A \\ 1\ 0\ 0\ 0\ 1\ 0\ 0\ 1\ 1\ B \\ \hline 0\ 0 \end{array}$$

The diagram illustrates the addition of two 10-bit integers, A and B. The bits of A are 1, 1, 0, 1, 1, 0, 1, 0, 1 and the bits of B are 1, 0, 0, 0, 1, 0, 0, 1, 1. A horizontal line is drawn under the bits of B. The result of the addition is shown below the line, with the bits 0 and 0. A vertical box highlights the 8th and 9th bits of the result, which are 0 and 0. Small '1' characters are placed below the 7th and 8th bits of the result, indicating carry bits.

Example: Multiplying Two Integers

Suppose we want to multiply two n -bit Integers, but our registers can only perform operations on integers of constant size.

For this we first need to be able to add two integers A and B :

$$\begin{array}{rcccccccc} 1 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & A \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & B \\ \hline & & & & & & 1 & 1 & & \\ & & & & & & & & 0 & 0 \end{array}$$

Example: Multiplying Two Integers

Suppose we want to multiply two n -bit Integers, but our registers can only perform operations on integers of constant size.

For this we first need to be able to add two integers A and B :

$$\begin{array}{rcccccccc} 1 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & A \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & B \\ \hline & & & & & & 0 & 0 & 0 & \end{array}$$

The diagram illustrates the addition of two 9-bit integers, A and B. The bits of A are 1, 1, 0, 1, 1, 0, 1, 0, 1. The bits of B are 1, 0, 0, 0, 1, 0, 0, 1, 1. A horizontal line is drawn under the bits of B. The result of the addition is shown below the line, with the bits 0, 0, 0 aligned under the 7th, 8th, and 9th bits of the input. A vertical box highlights the 7th, 8th, and 9th bits of the input and the corresponding result bits. Small '1' characters are placed below the 6th, 7th, and 8th bits of the input, indicating carry bits.

Example: Multiplying Two Integers

Suppose we want to multiply two n -bit Integers, but our registers can only perform operations on integers of constant size.

For this we first need to be able to add two integers A and B :

1	1	0	1	1	0	1	0	1	A
1	0	0	0	1	0	0	1	1	B
<hr/>									
					1	1	1		
						0	0	0	

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For this we first need to be able to add two integers A and B :

1	1	0	1	1	0	1	0	1	A
1	0	0	0	1	0	0	1	1	B
					0	1	1	1	
					1	0	0	0	

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1	1	0	1	1	0	1	0	1	A
1	0	0	0	1	0	0	1	1	B
				1					
				0	1	1	1		
					1	0	0	0	

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1	0	0	0	1	0	0	1	1	B
				1	0	1	1	1	
-----				0	1	0	0	0	

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1	0	0	0	1	0	0	1	1	B
<hr/>									
			1	0	1	1	1		
				0	1	0	0	0	

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1	1	0	1	1	0	1	0	1		A
1	0	0	0	1	0	0	1	1		B
			0	0	1	0	0	0		

Carry bits: 1, 1, 0, 1, 1, 1, 1

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<hr/>									
			1	1	0	1	1	1	
									0 0 1 0 0 0

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For this we first need to be able to add two integers A and B :

$$\begin{array}{rcccccccc} 1 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & A \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & B \\ \hline & & 1 & 0 & 0 & 1 & 0 & 0 & 0 & \end{array}$$

The diagram illustrates the addition of two 9-bit integers, A and B. The bits of A are 1, 1, 0, 1, 1, 0, 1, 0, 1. The bits of B are 1, 0, 0, 0, 1, 0, 0, 1, 1. A horizontal line is drawn under the bits of B. The result of the addition is shown below the line: 1, 0, 0, 1, 0, 0, 0. A vertical box highlights the second bit position (index 2) from the right, which contains a 1. This 1 is the carry from the addition of the third bit positions (index 3) of A and B. The carry bits are indicated by small subscripts below the result: 0 under the first bit, 1 under the second bit, 1 under the third bit, 0 under the fourth bit, 1 under the fifth bit, 1 under the sixth bit, and 1 under the seventh bit.

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For this we first need to be able to add two integers A and B :

1	1	0	1	1	0	1	0	1	A
1	0	0	0	1	0	0	1	1	B
	0	1	1	0	1	1	1		
<hr/>									
		1	0	0	1	0	0	0	

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	0	0	1	1	0	1	1	1		
	1	1	0	0	1	0	0	0		

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	1	0	0	0	1	0	0	1	1	B
	0	0	1	1	0	1	1	1		
		1	1	0	0	1	0	0	0	

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	0	1	1	0	0	1	0	0	0	

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	<hr/>									
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$$\begin{array}{r} 1\ 1\ 0\ 1\ 1\ 0\ 1\ 0\ 1\ A \\ 1\ 0\ 0\ 0\ 1\ 0\ 0\ 1\ 1\ B \\ \hline 1\ 0\ 1\ 1\ 0\ 0\ 1\ 0\ 0\ 0 \end{array}$$

This gives that two n -bit integers can be added in time $\mathcal{O}(n)$.

Example: Multiplying Two Integers

Suppose that we want to multiply an n -bit integer A and an m -bit integer B ($m \leq n$).

- This is also known as the “school method” for multiplying integers.
- Note that the intermediate numbers that are generated can have at most $m + n \leq 2n$ bits.

Example: Multiplying Two Integers

Suppose that we want to multiply an n -bit integer A and an m -bit integer B ($m \leq n$).

$$\begin{array}{r} 10001 \\ \times 1011 \\ \hline \end{array}$$

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Suppose that we want to multiply an n -bit integer A and an m -bit integer B ($m \leq n$).

$$\begin{array}{r} 1\ 0\ 0\ 0\ 1 \times 1\ 0\ 1\ 1 \\ \hline 1\ 0\ 0\ 0\ 1 \\ 1\ 0\ 0\ 0\ 1\ 0 \end{array}$$

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Example: Multiplying Two Integers

Suppose that we want to multiply an n -bit integer A and an m -bit integer B ($m \leq n$).

$$\begin{array}{r} 10001 \times 1011 \\ \hline 10001 \\ 100010 \\ 0000000 \\ 00001000 \\ \hline 100011001 \end{array}$$

- This is also known as the “school method” for multiplying integers.
- Note that the intermediate numbers that are generated can have at most $m + n \leq 2n$ bits.

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$$\begin{array}{r} 1\ 0\ 0\ 0\ 1 \times 1\ 0\ 1\ 1 \\ \hline 1\ 0\ 0\ 0\ 1 \\ 1\ 0\ 0\ 0\ 1\ 0 \\ 0\ 0\ 0\ 0\ 0\ 0\ 0 \\ 1\ 0\ 0\ 0\ 1\ 0\ 0\ 0 \\ \hline 1\ 0\ 1\ 1\ 1\ 0\ 1\ 1 \end{array}$$

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$$\begin{array}{r} 10001 \times 1011 \\ \hline 10001 \\ 100010 \\ 0000000 \\ 10001000 \\ \hline 10111011 \end{array}$$

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Time requirement:

Example: Multiplying Two Integers

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- This is also known as the “school method” for multiplying integers.
- Note that the intermediate numbers that are generated can have at most $m + n \leq 2n$ bits.

Time requirement:

- ▶ Computing intermediate results: $\mathcal{O}(nm)$.

Example: Multiplying Two Integers

Suppose that we want to multiply an n -bit integer A and an m -bit integer B ($m \leq n$).

$$\begin{array}{r} 1\ 0\ 0\ 0\ 1 \times 1\ 0\ 1\ 1 \\ \hline 1\ 0\ 0\ 0\ 1 \\ 1\ 0\ 0\ 0\ 1\ 0 \\ 0\ 0\ 0\ 0\ 0\ 0\ 0 \\ 1\ 0\ 0\ 0\ 1\ 0\ 0\ 0 \\ \hline 1\ 0\ 1\ 1\ 1\ 0\ 1\ 1 \end{array}$$

- This is also known as the “school method” for multiplying integers.
- Note that the intermediate numbers that are generated can have at most $m + n \leq 2n$ bits.

Time requirement:

- ▶ Computing intermediate results: $\mathcal{O}(nm)$.
- ▶ Adding m numbers of length $\leq 2n$: $\mathcal{O}((m + n)m) = \mathcal{O}(nm)$.

Example: Multiplying Two Integers

A recursive approach:

Suppose that integers A and B are of length $n = 2^k$, for some k .

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Suppose that integers A and B are of length $n = 2^k$, for some k .

$$\boxed{b_{n-1} \quad \dots \quad b_0} \times \boxed{a_{n-1} \quad \dots \quad a_0}$$

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A recursive approach:

Suppose that integers A and B are of length $n = 2^k$, for some k .

$$\boxed{b_{n-1} \quad \cdots \quad b_{\frac{n}{2}} \quad b_{\frac{n}{2}-1} \quad \cdots \quad b_0} \times \boxed{a_{n-1} \quad \cdots \quad a_{\frac{n}{2}} \quad a_{\frac{n}{2}-1} \quad \cdots \quad a_0}$$

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Then it holds that

$$A = A_1 \cdot 2^{\frac{n}{2}} + A_0 \text{ and } B = B_1 \cdot 2^{\frac{n}{2}} + B_0$$

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A recursive approach:

Suppose that integers A and B are of length $n = 2^k$, for some k .



Then it holds that

$$A = A_1 \cdot 2^{\frac{n}{2}} + A_0 \text{ and } B = B_1 \cdot 2^{\frac{n}{2}} + B_0$$

Hence,

$$A \cdot B = A_1 B_1 \cdot 2^n + (A_1 B_0 + A_0 B_1) \cdot 2^{\frac{n}{2}} + A_0 B_0$$

Example: Multiplying Two Integers

Algorithm 3 $\text{mult}(A, B)$

```
1: if  $|A| = |B| = 1$  then  
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Example: Multiplying Two Integers

Algorithm 3 $\text{mult}(A, B)$

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We get the following recurrence:

$$T(n) = 4T\left(\frac{n}{2}\right) + \mathcal{O}(n) .$$

Example: Multiplying Two Integers

Master Theorem: Recurrence: $T[n] = aT(\frac{n}{b}) + f(n)$.

- ▶ Case 1: $f(n) = \mathcal{O}(n^{\log_b a - \epsilon})$ $T(n) = \Theta(n^{\log_b a})$
- ▶ Case 2: $f(n) = \Theta(n^{\log_b a} \log^k n)$ $T(n) = \Theta(n^{\log_b a} \log^{k+1} n)$
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In our case $a = 4$, $b = 2$, and $f(n) = \Theta(n)$. Hence, we are in Case 1, since $n = \mathcal{O}(n^{2-\epsilon}) = \mathcal{O}(n^{\log_b a - \epsilon})$.

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We get a running time of $\mathcal{O}(n^2)$ for our algorithm.

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⇒ Not better than the “school method”.

Example: Multiplying Two Integers

We can use the following identity to compute Z_1 :

A more precise
(correct) analysis
would say that
computing Z_1
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- ▶ Case 3: $f(n) = \Omega(n^{\log_b a + \epsilon})$ $T(n) = \Theta(f(n))$

Again we are in Case 1. We get a running time of $\Theta(n^{\log_2 3}) \approx \Theta(n^{1.59})$.

Example: Multiplying Two Integers

We get the following recurrence:

$$T(n) = 3T\left(\frac{n}{2}\right) + \mathcal{O}(n) .$$

Master Theorem: Recurrence: $T[n] = aT\left(\frac{n}{b}\right) + f(n)$.

- ▶ Case 1: $f(n) = \mathcal{O}(n^{\log_b a - \epsilon})$ $T(n) = \Theta(n^{\log_b a})$
- ▶ Case 2: $f(n) = \Theta(n^{\log_b a} \log^k n)$ $T(n) = \Theta(n^{\log_b a} \log^{k+1} n)$
- ▶ Case 3: $f(n) = \Omega(n^{\log_b a + \epsilon})$ $T(n) = \Theta(f(n))$

Again we are in Case 1. We get a running time of $\Theta(n^{\log_2 3}) \approx \Theta(n^{1.59})$.

A huge improvement over the “school method”.