

## 7.7 Hashing

### Dictionary:

- ▶  **$S.insert(x)$** : Insert an element  $x$ .
- ▶  **$S.delete(x)$** : Delete the element pointed to by  $x$ .
- ▶  **$S.search(k)$** : Return a pointer to an element  $e$  with  $key[e] = k$  in  $S$  if it exists; otherwise return **null**.

## 7.7 Hashing

### Dictionary:

- ▶  **$S.insert(x)$** : Insert an element  $x$ .
- ▶  **$S.delete(x)$** : Delete the element pointed to by  $x$ .
- ▶  **$S.search(k)$** : Return a pointer to an element  $e$  with  $key[e] = k$  in  $S$  if it exists; otherwise return **null**.

So far we have implemented the search for a key by carefully choosing split-elements.

## 7.7 Hashing

### Dictionary:

- ▶  **$S$ .insert( $x$ )**: Insert an element  $x$ .
- ▶  **$S$ .delete( $x$ )**: Delete the element pointed to by  $x$ .
- ▶  **$S$ .search( $k$ )**: Return a pointer to an element  $e$  with  $\text{key}[e] = k$  in  $S$  if it exists; otherwise return **null**.

So far we have implemented the search for a key by carefully choosing split-elements.

Then the memory location of an object  $x$  with key  $k$  is determined by successively comparing  $k$  to split-elements.

## 7.7 Hashing

### Dictionary:

- ▶  **$S.insert(x)$** : Insert an element  $x$ .
- ▶  **$S.delete(x)$** : Delete the element pointed to by  $x$ .
- ▶  **$S.search(k)$** : Return a pointer to an element  $e$  with  $key[e] = k$  in  $S$  if it exists; otherwise return **null**.

So far we have implemented the search for a key by carefully choosing split-elements.

Then the memory location of an object  $x$  with key  $k$  is determined by successively comparing  $k$  to split-elements.

**Hashing** tries to **directly** compute the memory location from the given key. The goal is to have constant search time.

## 7.7 Hashing

### Definitions:

- ▶ Universe  $U$  of keys, e.g.,  $U \subseteq \mathbb{N}_0$ .  $U$  very large.

## 7.7 Hashing

### Definitions:

- ▶ Universe  $U$  of keys, e.g.,  $U \subseteq \mathbb{N}_0$ .  $U$  very large.
- ▶ Set  $S \subseteq U$  of keys,  $|S| = m \leq |U|$ .

## 7.7 Hashing

### Definitions:

- ▶ Universe  $U$  of keys, e.g.,  $U \subseteq \mathbb{N}_0$ .  $U$  very large.
- ▶ Set  $S \subseteq U$  of keys,  $|S| = m \leq |U|$ .
- ▶ Array  $T[0, \dots, n - 1]$  hash-table.

## 7.7 Hashing

### Definitions:

- ▶ Universe  $U$  of keys, e.g.,  $U \subseteq \mathbb{N}_0$ .  $U$  very large.
- ▶ Set  $S \subseteq U$  of keys,  $|S| = m \leq |U|$ .
- ▶ Array  $T[0, \dots, n - 1]$  hash-table.
- ▶ Hash function  $h : U \rightarrow [0, \dots, n - 1]$ .



## 7.7 Hashing

### Definitions:

- ▶ Universe  $U$  of keys, e.g.,  $U \subseteq \mathbb{N}_0$ .  $U$  very large.
- ▶ Set  $S \subseteq U$  of keys,  $|S| = m \leq |U|$ .
- ▶ Array  $T[0, \dots, n-1]$  hash-table.
- ▶ Hash function  $h : U \rightarrow [0, \dots, n-1]$ .

The hash-function  $h$  should fulfill:

## 7.7 Hashing

### Definitions:

- ▶ Universe  $U$  of keys, e.g.,  $U \subseteq \mathbb{N}_0$ .  $U$  very large.
- ▶ Set  $S \subseteq U$  of keys,  $|S| = m \leq |U|$ .
- ▶ Array  $T[0, \dots, n-1]$  hash-table.
- ▶ Hash function  $h : U \rightarrow [0, \dots, n-1]$ .

### The hash-function $h$ should fulfill:

- ▶ Fast to evaluate.

## 7.7 Hashing

### Definitions:

- ▶ Universe  $U$  of keys, e.g.,  $U \subseteq \mathbb{N}_0$ .  $U$  very large.
- ▶ Set  $S \subseteq U$  of keys,  $|S| = m \leq |U|$ .
- ▶ Array  $T[0, \dots, n-1]$  hash-table.
- ▶ Hash function  $h : U \rightarrow [0, \dots, n-1]$ .

### The hash-function $h$ should fulfill:

- ▶ Fast to evaluate.
- ▶ Small storage requirement.

## 7.7 Hashing

### Definitions:

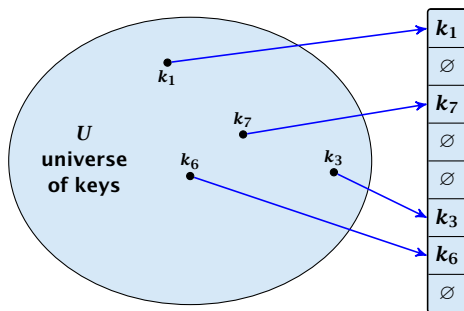
- ▶ Universe  $U$  of keys, e.g.,  $U \subseteq \mathbb{N}_0$ .  $U$  very large.
- ▶ Set  $S \subseteq U$  of keys,  $|S| = m \leq |U|$ .
- ▶ Array  $T[0, \dots, n-1]$  hash-table.
- ▶ Hash function  $h : U \rightarrow [0, \dots, n-1]$ .

### The hash-function $h$ should fulfill:

- ▶ Fast to evaluate.
- ▶ Small storage requirement.
- ▶ Good distribution of elements over the whole table.

# Direct Addressing

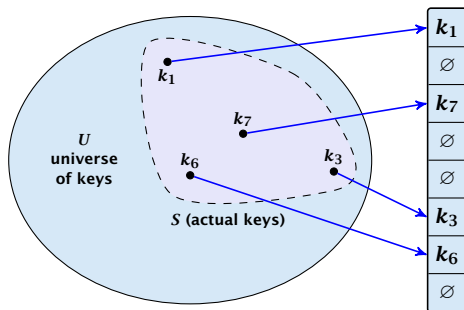
Ideally the hash function maps **all** keys to different memory locations.



This special case is known as **Direct Addressing**. It is usually very unrealistic as the universe of keys typically is quite large, and in particular larger than the available memory.

# Perfect Hashing

Suppose that we **know** the set  $S$  of actual keys (no insert/no delete). Then we may want to design a **simple** hash-function that maps all these keys to different memory locations.



Such a hash function  $h$  is called a **perfect hash function** for set  $S$ .

# Collisions

If we do not know the keys in advance, the best we can hope for is that the hash function distributes keys evenly across the table.

# Collisions

If we do not know the keys in advance, the best we can hope for is that the hash function distributes keys evenly across the table.

## **Problem: Collisions**

Usually the universe  $U$  is much larger than the table-size  $n$ .



# Collisions

If we do not know the keys in advance, the best we can hope for is that the hash function distributes keys evenly across the table.

## Problem: Collisions

Usually the universe  $U$  is much larger than the table-size  $n$ .

Hence, there may be two elements  $k_1, k_2$  from the set  $S$  that map to the same memory location (i.e.,  $h(k_1) = h(k_2)$ ). This is called a **collision**.

# Collisions

Typically, collisions do not appear once the size of the set  $S$  of actual keys gets close to  $n$ , but already when  $|S| \geq \omega(\sqrt{n})$ .

# Collisions

Typically, collisions do not appear once the size of the set  $S$  of actual keys gets close to  $n$ , but already when  $|S| \geq \omega(\sqrt{n})$ .

## Lemma 1

*The probability of having a collision when hashing  $m$  elements into a table of size  $n$  under uniform hashing is at least*

$$1 - e^{-\frac{m(m-1)}{2n}} \approx 1 - e^{-\frac{m^2}{2n}} .$$

# Collisions

Typically, collisions do not appear once the size of the set  $S$  of actual keys gets close to  $n$ , but already when  $|S| \geq \omega(\sqrt{n})$ .

## Lemma 1

*The probability of having a collision when hashing  $m$  elements into a table of size  $n$  under uniform hashing is at least*

$$1 - e^{-\frac{m(m-1)}{2n}} \approx 1 - e^{-\frac{m^2}{2n}} .$$

## Uniform hashing:

Choose a hash function uniformly at random from all functions  $f : U \rightarrow [0, \dots, n-1]$ .

# Collisions

## Proof.

Let  $A_{m,n}$  denote the event that inserting  $m$  keys into a table of size  $n$  does **not** generate a collision. Then

# Collisions

## Proof.

Let  $A_{m,n}$  denote the event that inserting  $m$  keys into a table of size  $n$  does **not** generate a collision. Then

$$\Pr[A_{m,n}]$$

# Collisions

## Proof.

Let  $A_{m,n}$  denote the event that inserting  $m$  keys into a table of size  $n$  does **not** generate a collision. Then

$$\Pr[A_{m,n}] = \prod_{\ell=1}^m \frac{n - \ell + 1}{n}$$

# Collisions

## Proof.

Let  $A_{m,n}$  denote the event that inserting  $m$  keys into a table of size  $n$  does **not** generate a collision. Then

$$\Pr[A_{m,n}] = \prod_{\ell=1}^m \frac{n - \ell + 1}{n} = \prod_{j=0}^{m-1} \left(1 - \frac{j}{n}\right)$$



# Collisions

## Proof.

Let  $A_{m,n}$  denote the event that inserting  $m$  keys into a table of size  $n$  does **not** generate a collision. Then

$$\begin{aligned}\Pr[A_{m,n}] &= \prod_{\ell=1}^m \frac{n - \ell + 1}{n} = \prod_{j=0}^{m-1} \left(1 - \frac{j}{n}\right) \\ &\leq \prod_{j=0}^{m-1} e^{-j/n}\end{aligned}$$

# Collisions

## Proof.

Let  $A_{m,n}$  denote the event that inserting  $m$  keys into a table of size  $n$  does **not** generate a collision. Then

$$\begin{aligned}\Pr[A_{m,n}] &= \prod_{\ell=1}^m \frac{n - \ell + 1}{n} = \prod_{j=0}^{m-1} \left(1 - \frac{j}{n}\right) \\ &\leq \prod_{j=0}^{m-1} e^{-j/n} = e^{-\sum_{j=0}^{m-1} \frac{j}{n}}\end{aligned}$$

# Collisions

## Proof.

Let  $A_{m,n}$  denote the event that inserting  $m$  keys into a table of size  $n$  does **not** generate a collision. Then

$$\begin{aligned}\Pr[A_{m,n}] &= \prod_{\ell=1}^m \frac{n - \ell + 1}{n} = \prod_{j=0}^{m-1} \left(1 - \frac{j}{n}\right) \\ &\leq \prod_{j=0}^{m-1} e^{-j/n} = e^{-\sum_{j=0}^{m-1} \frac{j}{n}} = e^{-\frac{m(m-1)}{2n}} .\end{aligned}$$

# Collisions

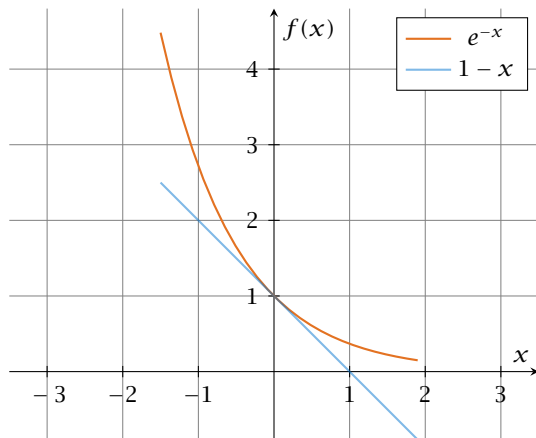
## Proof.

Let  $A_{m,n}$  denote the event that inserting  $m$  keys into a table of size  $n$  does **not** generate a collision. Then

$$\begin{aligned}\Pr[A_{m,n}] &= \prod_{\ell=1}^m \frac{n - \ell + 1}{n} = \prod_{j=0}^{m-1} \left(1 - \frac{j}{n}\right) \\ &\leq \prod_{j=0}^{m-1} e^{-j/n} = e^{-\sum_{j=0}^{m-1} \frac{j}{n}} = e^{-\frac{m(m-1)}{2n}} .\end{aligned}$$

Here the first equality follows since the  $\ell$ -th element that is hashed has a probability of  $\frac{n-\ell+1}{n}$  to not generate a collision under the condition that the previous elements did not induce collisions. □

# Collisions



The inequality  $1 - x \leq e^{-x}$  is derived by stopping the Taylor-expansion of  $e^{-x}$  after the second term.

# Resolving Collisions

The methods for dealing with collisions can be classified into the two main types

- ▶ **open addressing**, aka. closed hashing
- ▶ **hashing with chaining**, aka. closed addressing, open hashing.

# Resolving Collisions

The methods for dealing with collisions can be classified into the two main types

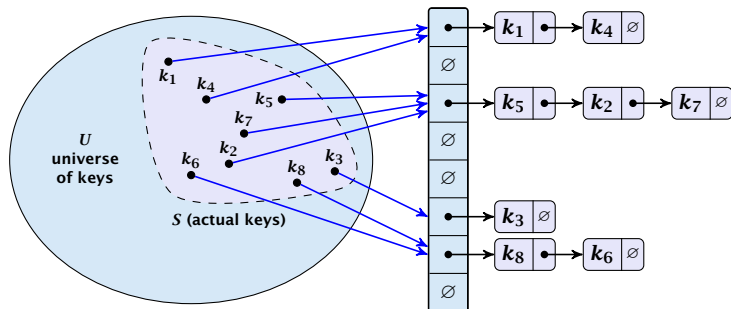
- ▶ **open addressing**, aka. closed hashing
- ▶ **hashing with chaining**, aka. closed addressing, open hashing.

There are applications e.g. computer chess where you do not resolve collisions at all.

# Hashing with Chaining

Arrange elements that map to the same position in a linear list.

- ▶ Access: compute  $h(x)$  and search list for  $\text{key}[x]$ .
- ▶ Insert: insert at the front of the list.





# Hashing with Chaining

Let  $A$  denote a strategy for resolving collisions. We use the following notation:

# Hashing with Chaining

Let  $A$  denote a strategy for resolving collisions. We use the following notation:

- ▶  $A^+$  denotes the average time for a **successful** search when using  $A$ ;

# Hashing with Chaining

Let  $A$  denote a strategy for resolving collisions. We use the following notation:

- ▶  $A^+$  denotes the average time for a **successful** search when using  $A$ ;
- ▶  $A^-$  denotes the average time for an **unsuccessful** search when using  $A$ ;

# Hashing with Chaining

Let  $A$  denote a strategy for resolving collisions. We use the following notation:

- ▶  $A^+$  denotes the average time for a **successful** search when using  $A$ ;
- ▶  $A^-$  denotes the average time for an **unsuccessful** search when using  $A$ ;
- ▶ We parameterize the complexity results in terms of  $\alpha := \frac{m}{n}$ , the so-called **fill factor** of the hash-table.

# Hashing with Chaining

Let  $A$  denote a strategy for resolving collisions. We use the following notation:

- ▶  $A^+$  denotes the average time for a **successful** search when using  $A$ ;
- ▶  $A^-$  denotes the average time for an **unsuccessful** search when using  $A$ ;
- ▶ We parameterize the complexity results in terms of  $\alpha := \frac{m}{n}$ , the so-called **fill factor** of the hash-table.

We assume **uniform hashing** for the following analysis.

# Hashing with Chaining

The time required for an unsuccessful search is 1 plus the length of the list that is examined.

# Hashing with Chaining

The time required for an unsuccessful search is 1 plus the length of the list that is examined. The average length of a list is  $\alpha = \frac{m}{n}$ .

# Hashing with Chaining

The time required for an unsuccessful search is 1 plus the length of the list that is examined. The average length of a list is  $\alpha = \frac{m}{n}$ . Hence, if  $A$  is the collision resolving strategy “Hashing with Chaining” we have

$$A^- = 1 + \alpha .$$



# Hashing with Chaining

For a successful search observe that we do **not** choose a list at random, but we consider a random key  $k$  in the hash-table and ask for the search-time for  $k$ .

## Hashing with Chaining

For a successful search observe that we do **not** choose a list at random, but we consider a random key  $k$  in the hash-table and ask for the search-time for  $k$ .

This is 1 plus the number of elements that lie before  $k$  in  $k$ 's list.

## Hashing with Chaining

For a successful search observe that we do **not** choose a list at random, but we consider a random key  $k$  in the hash-table and ask for the search-time for  $k$ .

This is 1 plus the number of elements that lie before  $k$  in  $k$ 's list.

Let  $k_\ell$  denote the  $\ell$ -th key inserted into the table.

## Hashing with Chaining

For a successful search observe that we do **not** choose a list at random, but we consider a random key  $k$  in the hash-table and ask for the search-time for  $k$ .

This is 1 plus the number of elements that lie before  $k$  in  $k$ 's list.

Let  $k_\ell$  denote the  $\ell$ -th key inserted into the table.

Let for two keys  $k_i$  and  $k_j$ ,  $X_{ij}$  denote the indicator variable for the event that  $k_i$  and  $k_j$  hash to the same position. Clearly,  $\Pr[X_{ij} = 1] = 1/n$  for uniform hashing.

## Hashing with Chaining

For a successful search observe that we do **not** choose a list at random, but we consider a random key  $k$  in the hash-table and ask for the search-time for  $k$ .

This is 1 plus the number of elements that lie before  $k$  in  $k$ 's list.

Let  $k_\ell$  denote the  $\ell$ -th key inserted into the table.

Let for two keys  $k_i$  and  $k_j$ ,  $X_{ij}$  denote the indicator variable for the event that  $k_i$  and  $k_j$  hash to the same position. Clearly,  $\Pr[X_{ij} = 1] = 1/n$  for uniform hashing.

The expected successful search cost is

$$E \left[ \frac{1}{m} \sum_{i=1}^m \left( 1 + \sum_{j=i+1}^m X_{ij} \right) \right]$$

## Hashing with Chaining

For a successful search observe that we do **not** choose a list at random, but we consider a random key  $k$  in the hash-table and ask for the search-time for  $k$ .

This is 1 plus the number of elements that lie before  $k$  in  $k$ 's list.

Let  $k_\ell$  denote the  $\ell$ -th key inserted into the table.

Let for two keys  $k_i$  and  $k_j$ ,  $X_{ij}$  denote the indicator variable for the event that  $k_i$  and  $k_j$  hash to the same position. Clearly,  $\Pr[X_{ij} = 1] = 1/n$  for uniform hashing.

The expected successful search cost is

$$E \left[ \frac{1}{m} \sum_{i=1}^m \left( 1 + \sum_{j=i+1}^m X_{ij} \right) \right]$$

keys before  $k_i$

## Hashing with Chaining

For a successful search observe that we do **not** choose a list at random, but we consider a random key  $k$  in the hash-table and ask for the search-time for  $k$ .

This is 1 plus the number of elements that lie before  $k$  in  $k$ 's list.

Let  $k_\ell$  denote the  $\ell$ -th key inserted into the table.

Let for two keys  $k_i$  and  $k_j$ ,  $X_{ij}$  denote the indicator variable for the event that  $k_i$  and  $k_j$  hash to the same position. Clearly,  $\Pr[X_{ij} = 1] = 1/n$  for uniform hashing.

The expected successful search cost is

$$\mathbb{E} \left[ \frac{1}{m} \sum_{i=1}^m \left( 1 + \sum_{j=i+1}^m X_{ij} \right) \right]$$

cost for key  $k_i$

# Hashing with Chaining

$$E \left[ \frac{1}{m} \sum_{i=1}^m \left( 1 + \sum_{j=i+1}^m X_{ij} \right) \right]$$



# Hashing with Chaining

$$\mathbb{E} \left[ \frac{1}{m} \sum_{i=1}^m \left( 1 + \sum_{j=i+1}^m X_{ij} \right) \right] = \frac{1}{m} \sum_{i=1}^m \left( 1 + \sum_{j=i+1}^m \mathbb{E}[X_{ij}] \right)$$

# Hashing with Chaining

$$\begin{aligned} E \left[ \frac{1}{m} \sum_{i=1}^m \left( 1 + \sum_{j=i+1}^m X_{ij} \right) \right] &= \frac{1}{m} \sum_{i=1}^m \left( 1 + \sum_{j=i+1}^m E[X_{ij}] \right) \\ &= \frac{1}{m} \sum_{i=1}^m \left( 1 + \sum_{j=i+1}^m \frac{1}{n} \right) \end{aligned}$$

# Hashing with Chaining

$$\begin{aligned} \mathbb{E} \left[ \frac{1}{m} \sum_{i=1}^m \left( 1 + \sum_{j=i+1}^m X_{ij} \right) \right] &= \frac{1}{m} \sum_{i=1}^m \left( 1 + \sum_{j=i+1}^m \mathbb{E}[X_{ij}] \right) \\ &= \frac{1}{m} \sum_{i=1}^m \left( 1 + \sum_{j=i+1}^m \frac{1}{n} \right) \\ &= 1 + \frac{1}{mn} \sum_{i=1}^m (m - i) \end{aligned}$$

# Hashing with Chaining

$$\begin{aligned} E \left[ \frac{1}{m} \sum_{i=1}^m \left( 1 + \sum_{j=i+1}^m X_{ij} \right) \right] &= \frac{1}{m} \sum_{i=1}^m \left( 1 + \sum_{j=i+1}^m E[X_{ij}] \right) \\ &= \frac{1}{m} \sum_{i=1}^m \left( 1 + \sum_{j=i+1}^m \frac{1}{n} \right) \\ &= 1 + \frac{1}{mn} \sum_{i=1}^m (m - i) \\ &= 1 + \frac{1}{mn} \left( m^2 - \frac{m(m+1)}{2} \right) \end{aligned}$$

# Hashing with Chaining

$$\begin{aligned} E \left[ \frac{1}{m} \sum_{i=1}^m \left( 1 + \sum_{j=i+1}^m X_{ij} \right) \right] &= \frac{1}{m} \sum_{i=1}^m \left( 1 + \sum_{j=i+1}^m E[X_{ij}] \right) \\ &= \frac{1}{m} \sum_{i=1}^m \left( 1 + \sum_{j=i+1}^m \frac{1}{n} \right) \\ &= 1 + \frac{1}{mn} \sum_{i=1}^m (m - i) \\ &= 1 + \frac{1}{mn} \left( m^2 - \frac{m(m+1)}{2} \right) \\ &= 1 + \frac{m-1}{2n} \end{aligned}$$

# Hashing with Chaining

$$\begin{aligned} \mathbb{E} \left[ \frac{1}{m} \sum_{i=1}^m \left( 1 + \sum_{j=i+1}^m X_{ij} \right) \right] &= \frac{1}{m} \sum_{i=1}^m \left( 1 + \sum_{j=i+1}^m \mathbb{E}[X_{ij}] \right) \\ &= \frac{1}{m} \sum_{i=1}^m \left( 1 + \sum_{j=i+1}^m \frac{1}{n} \right) \\ &= 1 + \frac{1}{mn} \sum_{i=1}^m (m - i) \\ &= 1 + \frac{1}{mn} \left( m^2 - \frac{m(m+1)}{2} \right) \\ &= 1 + \frac{m-1}{2n} = 1 + \frac{\alpha}{2} - \frac{\alpha}{2m} . \end{aligned}$$

# Hashing with Chaining

$$\begin{aligned} E \left[ \frac{1}{m} \sum_{i=1}^m \left( 1 + \sum_{j=i+1}^m X_{ij} \right) \right] &= \frac{1}{m} \sum_{i=1}^m \left( 1 + \sum_{j=i+1}^m E[X_{ij}] \right) \\ &= \frac{1}{m} \sum_{i=1}^m \left( 1 + \sum_{j=i+1}^m \frac{1}{n} \right) \\ &= 1 + \frac{1}{mn} \sum_{i=1}^m (m - i) \\ &= 1 + \frac{1}{mn} \left( m^2 - \frac{m(m+1)}{2} \right) \\ &= 1 + \frac{m-1}{2n} = 1 + \frac{\alpha}{2} - \frac{\alpha}{2m} . \end{aligned}$$

Hence, the expected cost for a successful search is  $A^+ \leq 1 + \frac{\alpha}{2}$ .

# Hashing with Chaining

## Disadvantages:

- ▶ pointers increase memory requirements
- ▶ pointers may lead to bad cache efficiency

## Advantages:

- ▶ no à priori limit on the number of elements
- ▶ deletion can be implemented efficiently
- ▶ by using balanced trees instead of linked list one can also obtain worst-case guarantees.



# Open Addressing

# Open Addressing

All objects are stored in the table itself.

# Open Addressing

All objects are stored in the table itself.

Define a function  $h(k, j)$  that determines the table-position to be examined in the  $j$ -th step. The values  $h(k, 0), \dots, h(k, n - 1)$  must form a permutation of  $0, \dots, n - 1$ .

# Open Addressing

All objects are stored in the table itself.

Define a function  $h(k, j)$  that determines the table-position to be examined in the  $j$ -th step. The values  $h(k, 0), \dots, h(k, n - 1)$  must form a permutation of  $0, \dots, n - 1$ .

**Search( $k$ ):** Try position  $h(k, 0)$ ; if it is empty your search fails; otw. continue with  $h(k, 1), h(k, 2), \dots$

# Open Addressing

All objects are stored in the table itself.

Define a function  $h(k, j)$  that determines the table-position to be examined in the  $j$ -th step. The values  $h(k, 0), \dots, h(k, n - 1)$  must form a permutation of  $0, \dots, n - 1$ .

**Search( $k$ ):** Try position  $h(k, 0)$ ; if it is empty your search fails; otw. continue with  $h(k, 1), h(k, 2), \dots$

**Insert( $x$ ):** Search until you find an empty slot; insert your element there. If your search reaches  $h(k, n - 1)$ , and this slot is non-empty then your table is full.

# Open Addressing

Choices for  $h(k, j)$ :

- ▶ **Linear probing:**

$$h(k, i) = h(k) + i \bmod n$$

(sometimes:  $h(k, i) = h(k) + ci \bmod n$ ).

# Open Addressing

Choices for  $h(k, j)$ :

- ▶ **Linear probing:**

$$h(k, i) = h(k) + i \bmod n$$

(sometimes:  $h(k, i) = h(k) + ci \bmod n$ ).

- ▶ **Quadratic probing:**

$$h(k, i) = h(k) + c_1i + c_2i^2 \bmod n.$$

# Open Addressing

Choices for  $h(k, j)$ :

- ▶ **Linear probing:**

$$h(k, i) = h(k) + i \pmod n$$

(sometimes:  $h(k, i) = h(k) + ci \pmod n$ ).

- ▶ **Quadratic probing:**

$$h(k, i) = h(k) + c_1i + c_2i^2 \pmod n.$$

- ▶ **Double hashing:**

$$h(k, i) = h_1(k) + ih_2(k) \pmod n.$$



# Open Addressing

Choices for  $h(k, j)$ :

▶ **Linear probing:**

$$h(k, i) = h(k) + i \bmod n$$

(sometimes:  $h(k, i) = h(k) + ci \bmod n$ ).

▶ **Quadratic probing:**

$$h(k, i) = h(k) + c_1i + c_2i^2 \bmod n.$$

▶ **Double hashing:**

$$h(k, i) = h_1(k) + ih_2(k) \bmod n.$$

For quadratic probing and double hashing one has to ensure that the search covers all positions in the table (i.e., for double hashing  $h_2(k)$  must be relatively prime to  $n$  (**teilerfremd**); for quadratic probing  $c_1$  and  $c_2$  have to be chosen carefully).

# Linear Probing

- ▶ Advantage: **Cache-efficiency**. The new probe position is very likely to be in the cache.

# Linear Probing

- ▶ Advantage: **Cache-efficiency**. The new probe position is very likely to be in the cache.
- ▶ Disadvantage: **Primary clustering**. Long sequences of occupied table-positions get longer as they have a larger probability to be hit. Furthermore, they can merge forming larger sequences.

# Linear Probing

- ▶ Advantage: **Cache-efficiency**. The new probe position is very likely to be in the cache.
- ▶ Disadvantage: **Primary clustering**. Long sequences of occupied table-positions get longer as they have a larger probability to be hit. Furthermore, they can merge forming larger sequences.

## Lemma 2

Let  $L$  be the method of linear probing for resolving collisions:

$$L^+ \approx \frac{1}{2} \left( 1 + \frac{1}{1 - \alpha} \right)$$

$$L^- \approx \frac{1}{2} \left( 1 + \frac{1}{(1 - \alpha)^2} \right)$$

# Quadratic Probing

- ▶ Not as cache-efficient as Linear Probing.
- ▶ **Secondary clustering**: caused by the fact that all keys mapped to the same position have the same probe sequence.

# Quadratic Probing

- ▶ Not as cache-efficient as Linear Probing.
- ▶ **Secondary clustering**: caused by the fact that all keys mapped to the same position have the same probe sequence.

## Lemma 3

Let  $Q$  be the method of quadratic probing for resolving collisions:

$$Q^+ \approx 1 + \ln\left(\frac{1}{1-\alpha}\right) - \frac{\alpha}{2}$$

$$Q^- \approx \frac{1}{1-\alpha} + \ln\left(\frac{1}{1-\alpha}\right) - \alpha$$

# Double Hashing

- ▶ Any probe into the hash-table usually creates a cache-miss.

# Double Hashing

- ▶ Any probe into the hash-table usually creates a cache-miss.

## Lemma 4

Let  $D$  be the method of double hashing for resolving collisions:

$$D^+ \approx \frac{1}{\alpha} \ln \left( \frac{1}{1 - \alpha} \right)$$

$$D^- \approx \frac{1}{1 - \alpha}$$

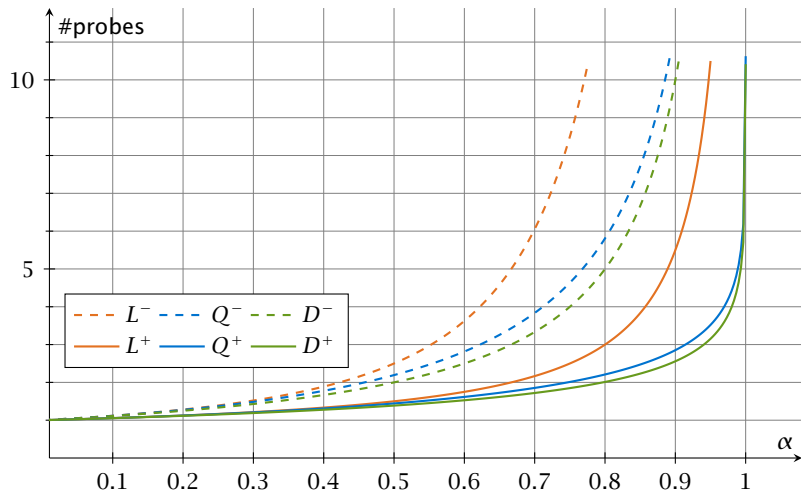


# Open Addressing

Some values:

$\alpha$	<i>Linear Probing</i>		<i>Quadratic Probing</i>		<i>Double Hashing</i>	
	$L^+$	$L^-$	$Q^+$	$Q^-$	$D^+$	$D^-$
0.5	1.5	2.5	1.44	2.19	1.39	2
0.9	5.5	50.5	2.85	11.40	2.55	10
0.95	10.5	200.5	3.52	22.05	3.15	20

# Open Addressing



# Analysis of Idealized Open Address Hashing

We analyze the time for a search in a very idealized Open Addressing scheme.

- ▶ The probe sequence  $h(k, 0), h(k, 1), h(k, 2), \dots$  is equally likely to be any permutation of  $\langle 0, 1, \dots, n - 1 \rangle$ .

# Analysis of Idealized Open Address Hashing

# Analysis of Idealized Open Address Hashing

Let  $X$  denote a random variable describing the number of probes in an **unsuccessful** search.

# Analysis of Idealized Open Address Hashing

Let  $X$  denote a random variable describing the number of probes in an **unsuccessful** search.

Let  $A_i$  denote the event that the  $i$ -th probe **occurs** and is to a non-empty slot.

$$\Pr[A_1 \cap A_2 \cap \dots \cap A_{i-1}]$$

# Analysis of Idealized Open Address Hashing

Let  $X$  denote a random variable describing the number of probes in an **unsuccessful** search.

Let  $A_i$  denote the event that the  $i$ -th probe **occurs** and is to a non-empty slot.

$$\begin{aligned} & \Pr[A_1 \cap A_2 \cap \dots \cap A_{i-1}] \\ &= \Pr[A_1] \cdot \Pr[A_2 \mid A_1] \cdot \Pr[A_3 \mid A_1 \cap A_2] \cdot \\ & \quad \dots \cdot \Pr[A_{i-1} \mid A_1 \cap \dots \cap A_{i-2}] \end{aligned}$$

# Analysis of Idealized Open Address Hashing

Let  $X$  denote a random variable describing the number of probes in an **unsuccessful** search.

Let  $A_i$  denote the event that the  $i$ -th probe **occurs** and is to a non-empty slot.

$$\begin{aligned}\Pr[A_1 \cap A_2 \cap \dots \cap A_{i-1}] \\ &= \Pr[A_1] \cdot \Pr[A_2 \mid A_1] \cdot \Pr[A_3 \mid A_1 \cap A_2] \cdot \\ &\quad \dots \cdot \Pr[A_{i-1} \mid A_1 \cap \dots \cap A_{i-2}]\end{aligned}$$

$$\Pr[X \geq i]$$



# Analysis of Idealized Open Address Hashing

Let  $X$  denote a random variable describing the number of probes in an **unsuccessful** search.

Let  $A_i$  denote the event that the  $i$ -th probe **occurs** and is to a non-empty slot.

$$\begin{aligned}\Pr[A_1 \cap A_2 \cap \dots \cap A_{i-1}] \\ &= \Pr[A_1] \cdot \Pr[A_2 \mid A_1] \cdot \Pr[A_3 \mid A_1 \cap A_2] \cdot \\ &\quad \dots \cdot \Pr[A_{i-1} \mid A_1 \cap \dots \cap A_{i-2}]\end{aligned}$$

$$\Pr[X \geq i] = \frac{m}{n} \cdot \frac{m-1}{n-1} \cdot \frac{m-2}{n-2} \cdot \dots \cdot \frac{m-i+2}{n-i+2}$$

# Analysis of Idealized Open Address Hashing

Let  $X$  denote a random variable describing the number of probes in an **unsuccessful** search.

Let  $A_i$  denote the event that the  $i$ -th probe **occurs** and is to a non-empty slot.

$$\begin{aligned}\Pr[A_1 \cap A_2 \cap \dots \cap A_{i-1}] \\ &= \Pr[A_1] \cdot \Pr[A_2 \mid A_1] \cdot \Pr[A_3 \mid A_1 \cap A_2] \cdot \\ &\quad \dots \cdot \Pr[A_{i-1} \mid A_1 \cap \dots \cap A_{i-2}]\end{aligned}$$

$$\begin{aligned}\Pr[X \geq i] &= \frac{m}{n} \cdot \frac{m-1}{n-1} \cdot \frac{m-2}{n-2} \cdot \dots \cdot \frac{m-i+2}{n-i+2} \\ &\leq \left(\frac{m}{n}\right)^{i-1}\end{aligned}$$

# Analysis of Idealized Open Address Hashing

Let  $X$  denote a random variable describing the number of probes in an **unsuccessful** search.

Let  $A_i$  denote the event that the  $i$ -th probe **occurs** and is to a non-empty slot.

$$\begin{aligned}\Pr[A_1 \cap A_2 \cap \dots \cap A_{i-1}] \\ &= \Pr[A_1] \cdot \Pr[A_2 \mid A_1] \cdot \Pr[A_3 \mid A_1 \cap A_2] \cdot \\ &\quad \dots \cdot \Pr[A_{i-1} \mid A_1 \cap \dots \cap A_{i-2}]\end{aligned}$$

$$\begin{aligned}\Pr[X \geq i] &= \frac{m}{n} \cdot \frac{m-1}{n-1} \cdot \frac{m-2}{n-2} \cdot \dots \cdot \frac{m-i+2}{n-i+2} \\ &\leq \left(\frac{m}{n}\right)^{i-1} = \alpha^{i-1} .\end{aligned}$$

# Analysis of Idealized Open Address Hashing

$E[X]$

# Analysis of Idealized Open Address Hashing

$$E[X] = \sum_{i=1}^{\infty} \Pr[X \geq i]$$

# Analysis of Idealized Open Address Hashing

$$E[X] = \sum_{i=1}^{\infty} \Pr[X \geq i] \leq \sum_{i=1}^{\infty} \alpha^{i-1}$$

# Analysis of Idealized Open Address Hashing

$$E[X] = \sum_{i=1}^{\infty} \Pr[X \geq i] \leq \sum_{i=1}^{\infty} \alpha^{i-1} = \sum_{i=0}^{\infty} \alpha^i$$

# Analysis of Idealized Open Address Hashing

$$E[X] = \sum_{i=1}^{\infty} \Pr[X \geq i] \leq \sum_{i=1}^{\infty} \alpha^{i-1} = \sum_{i=0}^{\infty} \alpha^i = \frac{1}{1-\alpha} .$$

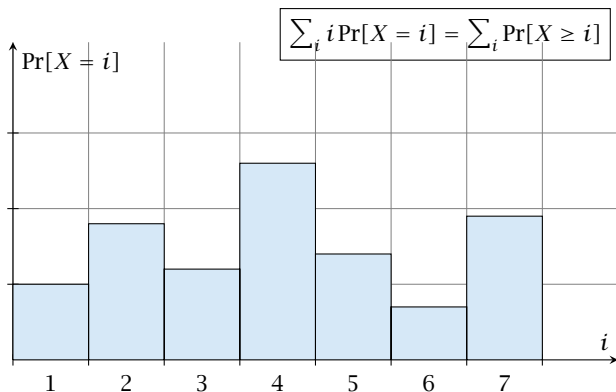


# Analysis of Idealized Open Address Hashing

$$E[X] = \sum_{i=1}^{\infty} \Pr[X \geq i] \leq \sum_{i=1}^{\infty} \alpha^{i-1} = \sum_{i=0}^{\infty} \alpha^i = \frac{1}{1-\alpha} .$$

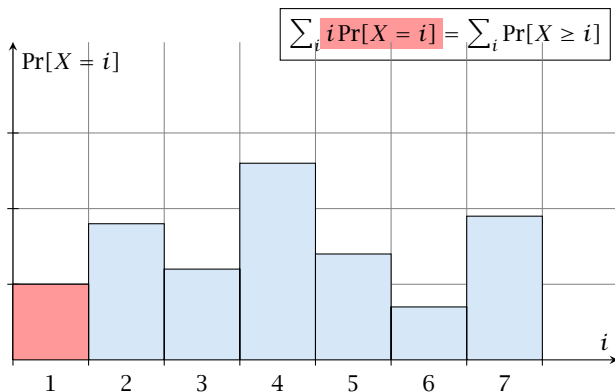
$$\frac{1}{1-\alpha} = 1 + \alpha + \alpha^2 + \alpha^3 + \dots$$

# Analysis of Idealized Open Address Hashing



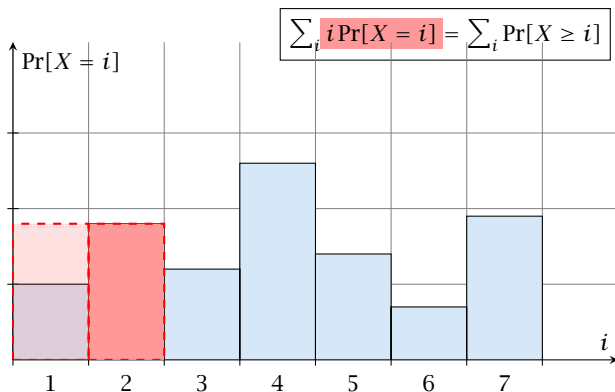
# Analysis of Idealized Open Address Hashing

$i = 1$



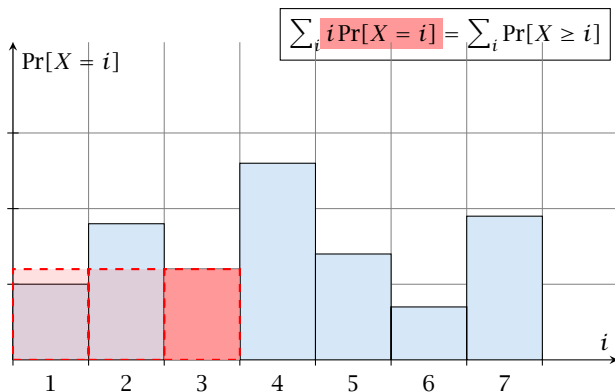
# Analysis of Idealized Open Address Hashing

$i = 2$



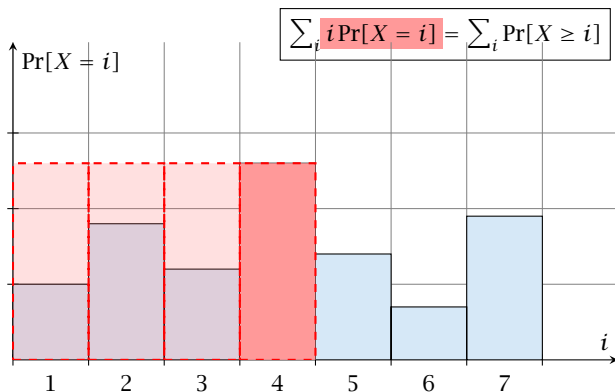
# Analysis of Idealized Open Address Hashing

$i = 3$



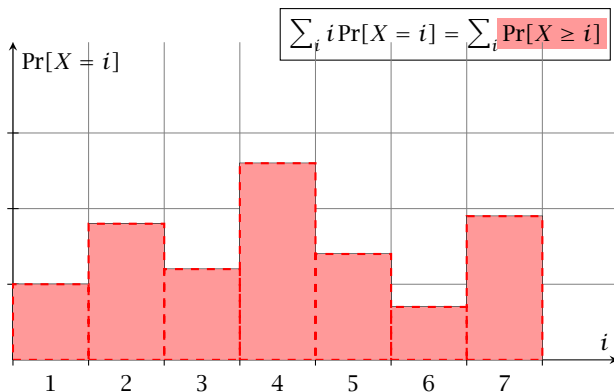
# Analysis of Idealized Open Address Hashing

$i = 4$



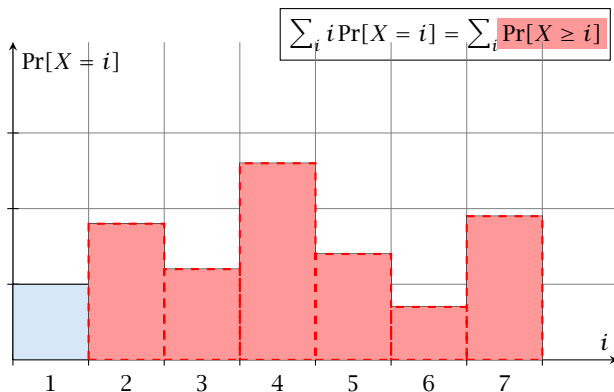
# Analysis of Idealized Open Address Hashing

$i = 1$



# Analysis of Idealized Open Address Hashing

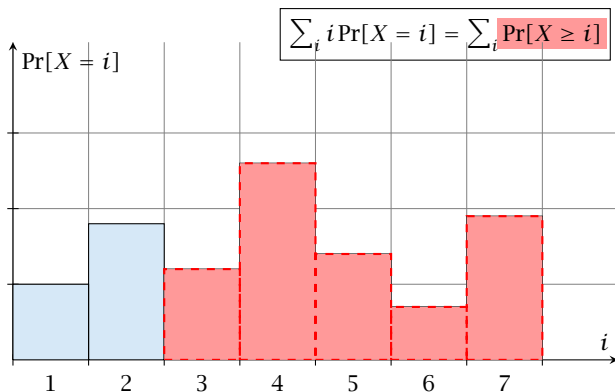
$i = 2$





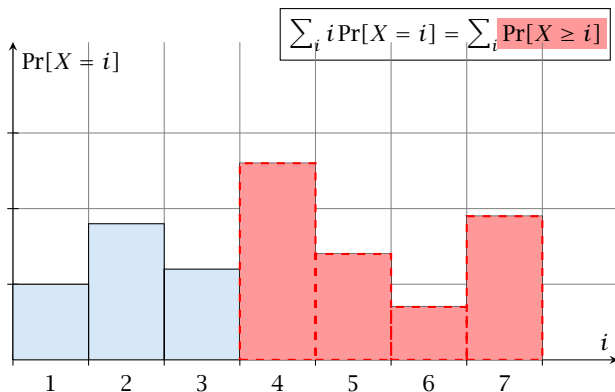
# Analysis of Idealized Open Address Hashing

$i = 3$

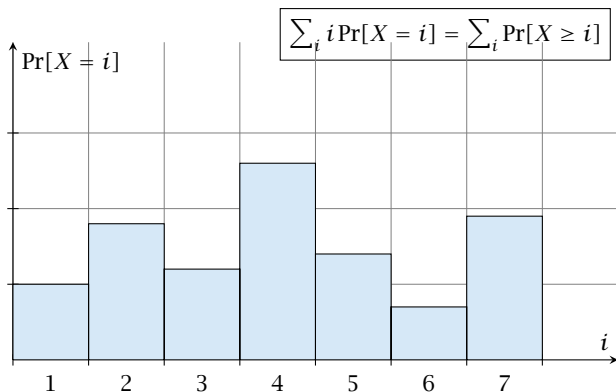


# Analysis of Idealized Open Address Hashing

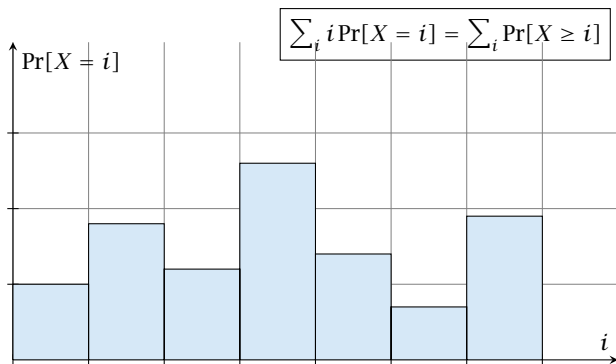
$i = 4$



# Analysis of Idealized Open Address Hashing



# Analysis of Idealized Open Address Hashing



The  $j$ -th rectangle appears in both sums  $j$  times. ( $j$  times in the first due to multiplication with  $j$ ; and  $j$  times in the second for summands  $i = 1, 2, \dots, j$ )

# Analysis of Idealized Open Address Hashing

# Analysis of Idealized Open Address Hashing

The number of probes in a **successful** search for  $k$  is equal to the number of probes made in an unsuccessful search for  $k$  at the time that  $k$  is inserted.

# Analysis of Idealized Open Address Hashing

The number of probes in a **successful** search for  $k$  is equal to the number of probes made in an unsuccessful search for  $k$  at the time that  $k$  is inserted.

Let  $k$  be the  $i + 1$ -st element. The expected time for a search for  $k$  is at most  $\frac{1}{1-i/n} = \frac{n}{n-i}$ .

# Analysis of Idealized Open Address Hashing

The number of probes in a **successful** search for  $k$  is equal to the number of probes made in an unsuccessful search for  $k$  at the time that  $k$  is inserted.

Let  $k$  be the  $i + 1$ -st element. The expected time for a search for  $k$  is at most  $\frac{1}{1-i/n} = \frac{n}{n-i}$ .

$$\frac{1}{m} \sum_{i=0}^{m-1} \frac{n}{n-i}$$



# Analysis of Idealized Open Address Hashing

The number of probes in a **successful** search for  $k$  is equal to the number of probes made in an unsuccessful search for  $k$  at the time that  $k$  is inserted.

Let  $k$  be the  $i + 1$ -st element. The expected time for a search for  $k$  is at most  $\frac{1}{1-i/n} = \frac{n}{n-i}$ .

$$\frac{1}{m} \sum_{i=0}^{m-1} \frac{n}{n-i} = \frac{n}{m} \sum_{i=0}^{m-1} \frac{1}{n-i}$$

# Analysis of Idealized Open Address Hashing

The number of probes in a **successful** search for  $k$  is equal to the number of probes made in an unsuccessful search for  $k$  at the time that  $k$  is inserted.

Let  $k$  be the  $i + 1$ -st element. The expected time for a search for  $k$  is at most  $\frac{1}{1-i/n} = \frac{n}{n-i}$ .

$$\frac{1}{m} \sum_{i=0}^{m-1} \frac{n}{n-i} = \frac{n}{m} \sum_{i=0}^{m-1} \frac{1}{n-i} = \frac{1}{\alpha} \sum_{k=n-m+1}^n \frac{1}{k}$$

# Analysis of Idealized Open Address Hashing

The number of probes in a **successful** search for  $k$  is equal to the number of probes made in an unsuccessful search for  $k$  at the time that  $k$  is inserted.

Let  $k$  be the  $i + 1$ -st element. The expected time for a search for  $k$  is at most  $\frac{1}{1-i/n} = \frac{n}{n-i}$ .

$$\begin{aligned}\frac{1}{m} \sum_{i=0}^{m-1} \frac{n}{n-i} &= \frac{n}{m} \sum_{i=0}^{m-1} \frac{1}{n-i} = \frac{1}{\alpha} \sum_{k=n-m+1}^n \frac{1}{k} \\ &\leq \frac{1}{\alpha} \int_{n-m}^n \frac{1}{x} dx\end{aligned}$$

# Analysis of Idealized Open Address Hashing

The number of probes in a **successful** search for  $k$  is equal to the number of probes made in an unsuccessful search for  $k$  at the time that  $k$  is inserted.

Let  $k$  be the  $i + 1$ -st element. The expected time for a search for  $k$  is at most  $\frac{1}{1-i/n} = \frac{n}{n-i}$ .

$$\begin{aligned}\frac{1}{m} \sum_{i=0}^{m-1} \frac{n}{n-i} &= \frac{n}{m} \sum_{i=0}^{m-1} \frac{1}{n-i} = \frac{1}{\alpha} \sum_{k=n-m+1}^n \frac{1}{k} \\ &\leq \frac{1}{\alpha} \int_{n-m}^n \frac{1}{x} dx = \frac{1}{\alpha} \ln \frac{n}{n-m}\end{aligned}$$

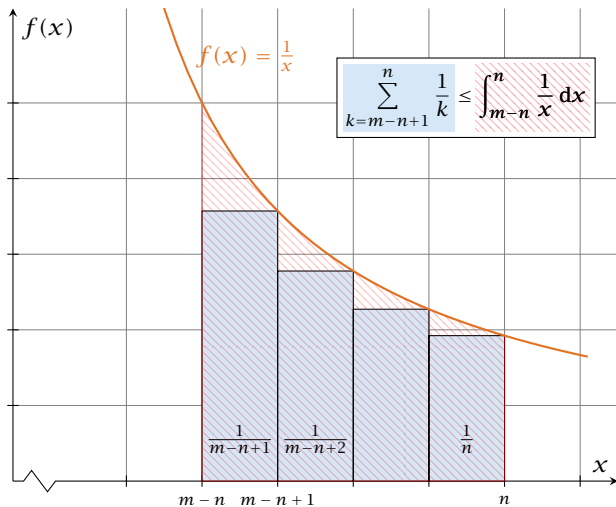
# Analysis of Idealized Open Address Hashing

The number of probes in a **successful** search for  $k$  is equal to the number of probes made in an unsuccessful search for  $k$  at the time that  $k$  is inserted.

Let  $k$  be the  $i + 1$ -st element. The expected time for a search for  $k$  is at most  $\frac{1}{1-i/n} = \frac{n}{n-i}$ .

$$\begin{aligned} \frac{1}{m} \sum_{i=0}^{m-1} \frac{n}{n-i} &= \frac{n}{m} \sum_{i=0}^{m-1} \frac{1}{n-i} = \frac{1}{\alpha} \sum_{k=n-m+1}^n \frac{1}{k} \\ &\leq \frac{1}{\alpha} \int_{n-m}^n \frac{1}{x} dx = \frac{1}{\alpha} \ln \frac{n}{n-m} = \frac{1}{\alpha} \ln \frac{1}{1-\alpha} . \end{aligned}$$

# Analysis of Idealized Open Address Hashing



## How do we delete in a hash-table?

- ▶ For hashing with chaining this is not a problem. Simply search for the key, and delete the item in the corresponding list.

## How do we delete in a hash-table?

- ▶ For hashing with chaining this is not a problem. Simply search for the key, and delete the item in the corresponding list.
- ▶ For open addressing this is difficult.



# Deletions in Hashtables

- ▶ Simply removing a key might interrupt the probe sequence of other keys which then cannot be found anymore.

# Deletions in Hashtables

- ▶ Simply removing a key might interrupt the probe sequence of other keys which then cannot be found anymore.
- ▶ One can delete an element by replacing it with a **deleted**-marker.

## Deletions in Hashtables

- ▶ Simply removing a key might interrupt the probe sequence of other keys which then cannot be found anymore.
- ▶ One can delete an element by replacing it with a **deleted**-marker.
  - ▶ During an insertion if a **deleted**-marker is encountered an element can be inserted there.

# Deletions in Hashtables

- ▶ Simply removing a key might interrupt the probe sequence of other keys which then cannot be found anymore.
- ▶ One can delete an element by replacing it with a **deleted**-marker.
  - ▶ During an insertion if a **deleted**-marker is encountered an element can be inserted there.
  - ▶ During a search a **deleted**-marker must not be used to terminate the probe sequence.

# Deletions in Hashtables

- ▶ Simply removing a key might interrupt the probe sequence of other keys which then cannot be found anymore.
- ▶ One can delete an element by replacing it with a **deleted**-marker.
  - ▶ During an insertion if a **deleted**-marker is encountered an element can be inserted there.
  - ▶ During a search a **deleted**-marker must not be used to terminate the probe sequence.
- ▶ The table could fill up with **deleted**-markers leading to bad performance.

# Deletions in Hashtables

- ▶ Simply removing a key might interrupt the probe sequence of other keys which then cannot be found anymore.
- ▶ One can delete an element by replacing it with a **deleted**-marker.
  - ▶ During an insertion if a **deleted**-marker is encountered an element can be inserted there.
  - ▶ During a search a **deleted**-marker must not be used to terminate the probe sequence.
- ▶ The table could fill up with **deleted**-markers leading to bad performance.
- ▶ If a table contains many deleted-markers (linear fraction of the keys) one can rehash the whole table and amortize the cost for this rehash against the cost for the deletions.

# Deletions for Linear Probing

- ▶ For Linear Probing one can delete elements without using **deletion**-markers.

# Deletions for Linear Probing

- ▶ For Linear Probing one can delete elements without using **deletion**-markers.
- ▶ Upon a deletion elements that are further down in the probe-sequence may be moved to guarantee that they are still found during a search.



## Deletions for Linear Probing

### Algorithm 37 delete( $p$ )

```
1:  $T[p] \leftarrow \text{null}$ 
2:  $p \leftarrow \text{succ}(p)$ 
3: while  $T[p] \neq \text{null}$  do
4:    $y \leftarrow T[p]$ 
5:    $T[p] \leftarrow \text{null}$ 
6:    $p \leftarrow \text{succ}(p)$ 
7:    $\text{insert}(y)$ 
```

$p$  is the index into the table-cell that contains the object to be deleted.

## Deletions for Linear Probing

### Algorithm 37 delete( $p$ )

```
1:  $T[p] \leftarrow \text{null}$ 
2:  $p \leftarrow \text{succ}(p)$ 
3: while  $T[p] \neq \text{null}$  do
4:    $y \leftarrow T[p]$ 
5:    $T[p] \leftarrow \text{null}$ 
6:    $p \leftarrow \text{succ}(p)$ 
7:   insert( $y$ )
```

$p$  is the index into the table-cell that contains the object to be deleted.

Pointers into the hash-table become invalid.

# Universal Hashing

# Universal Hashing

Regardless, of the choice of hash-function there is always an input (a set of keys) that has a very poor worst-case behaviour.

# Universal Hashing

Regardless, of the choice of hash-function there is always an input (a set of keys) that has a very poor worst-case behaviour.

Therefore, so far we assumed that the hash-function is random so that regardless of the input the average case behaviour is good.

# Universal Hashing

Regardless, of the choice of hash-function there is always an input (a set of keys) that has a very poor worst-case behaviour.

Therefore, so far we assumed that the hash-function is random so that regardless of the input the average case behaviour is good.

However, the assumption of uniform hashing that  $h$  is chosen randomly from all functions  $f : U \rightarrow [0, \dots, n - 1]$  is clearly unrealistic as there are  $n^{|U|}$  such functions. Even writing down such a function would take  $|U| \log n$  bits.

# Universal Hashing

Regardless, of the choice of hash-function there is always an input (a set of keys) that has a very poor worst-case behaviour.

Therefore, so far we assumed that the hash-function is random so that regardless of the input the average case behaviour is good.

However, the assumption of uniform hashing that  $h$  is chosen randomly from all functions  $f : U \rightarrow [0, \dots, n - 1]$  is clearly unrealistic as there are  $n^{|U|}$  such functions. Even writing down such a function would take  $|U| \log n$  bits.

Universal hashing tries to define a set  $\mathcal{H}$  of functions that is much smaller but still leads to good average case behaviour when selecting a hash-function uniformly at random from  $\mathcal{H}$ .

# Universal Hashing

## Definition 5

A class  $\mathcal{H}$  of hash-functions from the universe  $U$  into the set  $\{0, \dots, n-1\}$  is called **universal** if for all  $u_1, u_2 \in U$  with  $u_1 \neq u_2$

$$\Pr[h(u_1) = h(u_2)] \leq \frac{1}{n} ,$$

where the probability is w. r. t. the choice of a random hash-function from set  $\mathcal{H}$ .



# Universal Hashing

## Definition 5

A class  $\mathcal{H}$  of hash-functions from the universe  $U$  into the set  $\{0, \dots, n-1\}$  is called **universal** if for all  $u_1, u_2 \in U$  with  $u_1 \neq u_2$

$$\Pr[h(u_1) = h(u_2)] \leq \frac{1}{n} ,$$

where the probability is w. r. t. the choice of a random hash-function from set  $\mathcal{H}$ .

Note that this means that the probability of a collision between two arbitrary elements is at most  $\frac{1}{n}$ .

# Universal Hashing

## Definition 6

A class  $\mathcal{H}$  of hash-functions from the universe  $U$  into the set  $\{0, \dots, n-1\}$  is called **2-independent** (pairwise independent) if the following two conditions hold

- ▶ For any key  $u \in U$ , and  $t \in \{0, \dots, n-1\}$   $\Pr[h(u) = t] = \frac{1}{n}$ ,  
i.e., a key is distributed uniformly within the hash-table.
- ▶ For all  $u_1, u_2 \in U$  with  $u_1 \neq u_2$ , and for any two hash-positions  $t_1, t_2$ :

$$\Pr[h(u_1) = t_1 \wedge h(u_2) = t_2] \leq \frac{1}{n^2} .$$

# Universal Hashing

## Definition 6

A class  $\mathcal{H}$  of hash-functions from the universe  $U$  into the set  $\{0, \dots, n-1\}$  is called **2-independent** (pairwise independent) if the following two conditions hold

- ▶ For any key  $u \in U$ , and  $t \in \{0, \dots, n-1\}$   $\Pr[h(u) = t] = \frac{1}{n}$ , i.e., a key is distributed uniformly within the hash-table.
- ▶ For all  $u_1, u_2 \in U$  with  $u_1 \neq u_2$ , and for any two hash-positions  $t_1, t_2$ :

$$\Pr[h(u_1) = t_1 \wedge h(u_2) = t_2] \leq \frac{1}{n^2} .$$

This requirement clearly implies a universal hash-function.

# Universal Hashing

## Definition 7

A class  $\mathcal{H}$  of hash-functions from the universe  $U$  into the set  $\{0, \dots, n-1\}$  is called  **$k$ -independent** if for any choice of  $\ell \leq k$  distinct keys  $u_1, \dots, u_\ell \in U$ , and for any set of  $\ell$  not necessarily distinct hash-positions  $t_1, \dots, t_\ell$ :

$$\Pr[h(u_1) = t_1 \wedge \dots \wedge h(u_\ell) = t_\ell] \leq \frac{1}{n^\ell} ,$$

where the probability is w. r. t. the choice of a random hash-function from set  $\mathcal{H}$ .

## Definition 8

A class  $\mathcal{H}$  of hash-functions from the universe  $U$  into the set  $\{0, \dots, n-1\}$  is called  $(\mu, k)$ -independent if for any choice of  $\ell \leq k$  distinct keys  $u_1, \dots, u_\ell \in U$ , and for any set of  $\ell$  not necessarily distinct hash-positions  $t_1, \dots, t_\ell$ :

$$\Pr[h(u_1) = t_1 \wedge \dots \wedge h(u_\ell) = t_\ell] \leq \frac{\mu}{n^\ell},$$

where the probability is w. r. t. the choice of a random hash-function from set  $\mathcal{H}$ .

# Universal Hashing

# Universal Hashing

Let  $U := \{0, \dots, p - 1\}$  for a prime  $p$ . Let  $\mathbb{Z}_p := \{0, \dots, p - 1\}$ , and let  $\mathbb{Z}_p^* := \{1, \dots, p - 1\}$  denote the set of invertible elements in  $\mathbb{Z}_p$ .

# Universal Hashing

Let  $U := \{0, \dots, p - 1\}$  for a prime  $p$ . Let  $\mathbb{Z}_p := \{0, \dots, p - 1\}$ , and let  $\mathbb{Z}_p^* := \{1, \dots, p - 1\}$  denote the set of invertible elements in  $\mathbb{Z}_p$ .

Define

$$h_{a,b}(x) := (ax + b \bmod p) \bmod n$$



# Universal Hashing

Let  $U := \{0, \dots, p-1\}$  for a prime  $p$ . Let  $\mathbb{Z}_p := \{0, \dots, p-1\}$ , and let  $\mathbb{Z}_p^* := \{1, \dots, p-1\}$  denote the set of invertible elements in  $\mathbb{Z}_p$ .

Define

$$h_{a,b}(x) := (ax + b \bmod p) \bmod n$$

## Lemma 9

*The class*

$$\mathcal{H} = \{h_{a,b} \mid a \in \mathbb{Z}_p^*, b \in \mathbb{Z}_p\}$$

*is a universal class of hash-functions from  $U$  to  $\{0, \dots, n-1\}$ .*

# Universal Hashing

# Universal Hashing

## Proof.

Let  $x, y \in U$  be two distinct keys. We have to show that the probability of a collision is only  $1/n$ .

# Universal Hashing

## Proof.

Let  $x, y \in U$  be two distinct keys. We have to show that the probability of a collision is only  $1/n$ .

►  $ax + b \not\equiv ay + b \pmod{p}$

# Universal Hashing

## Proof.

Let  $x, y \in U$  be two distinct keys. We have to show that the probability of a collision is only  $1/n$ .

►  $ax + b \not\equiv ay + b \pmod{p}$

If  $x \neq y$  then  $(x - y) \not\equiv 0 \pmod{p}$ .

# Universal Hashing

## Proof.

Let  $x, y \in U$  be two distinct keys. We have to show that the probability of a collision is only  $1/n$ .

►  $ax + b \not\equiv ay + b \pmod{p}$

If  $x \neq y$  then  $(x - y) \not\equiv 0 \pmod{p}$ .

Multiplying with  $a \not\equiv 0 \pmod{p}$  gives

$$a(x - y) \not\equiv 0 \pmod{p}$$

# Universal Hashing

## Proof.

Let  $x, y \in U$  be two distinct keys. We have to show that the probability of a collision is only  $1/n$ .

►  $ax + b \not\equiv ay + b \pmod{p}$

If  $x \neq y$  then  $(x - y) \not\equiv 0 \pmod{p}$ .

Multiplying with  $a \not\equiv 0 \pmod{p}$  gives

$$a(x - y) \not\equiv 0 \pmod{p}$$

where we use that  $\mathbb{Z}_p$  is a field (**Körper**) and, hence, has no zero divisors (**nullteilerfrei**).

## Universal Hashing

- ▶ The hash-function does not generate collisions before the  $(\text{mod } n)$ -operation. Furthermore, every choice  $(a, b)$  is mapped to a different pair  $(t_x, t_y)$  with  $t_x := ax + b$  and  $t_y := ay + b$ .



## Universal Hashing

- ▶ The hash-function does not generate collisions before the  $(\text{mod } n)$ -operation. Furthermore, every choice  $(a, b)$  is mapped to a different pair  $(t_x, t_y)$  with  $t_x := ax + b$  and  $t_y := ay + b$ .

This holds because we can compute  $a$  and  $b$  when given  $t_x$  and  $t_y$ :

## Universal Hashing

- ▶ The hash-function does not generate collisions before the  $(\text{mod } n)$ -operation. Furthermore, every choice  $(a, b)$  is mapped to a different pair  $(t_x, t_y)$  with  $t_x := ax + b$  and  $t_y := ay + b$ .

This holds because we can compute  $a$  and  $b$  when given  $t_x$  and  $t_y$ :

$$t_x \equiv ax + b \pmod{p}$$

$$t_y \equiv ay + b \pmod{p}$$

## Universal Hashing

- ▶ The hash-function does not generate collisions before the  $(\text{mod } n)$ -operation. Furthermore, every choice  $(a, b)$  is mapped to a different pair  $(t_x, t_y)$  with  $t_x := ax + b$  and  $t_y := ay + b$ .

This holds because we can compute  $a$  and  $b$  when given  $t_x$  and  $t_y$ :

$$t_x \equiv ax + b \pmod{p}$$

$$t_y \equiv ay + b \pmod{p}$$

$$t_x - t_y \equiv a(x - y) \pmod{p}$$

$$t_y \equiv ay + b \pmod{p}$$

## Universal Hashing

- ▶ The hash-function does not generate collisions before the  $(\text{mod } n)$ -operation. Furthermore, every choice  $(a, b)$  is mapped to a different pair  $(t_x, t_y)$  with  $t_x := ax + b$  and  $t_y := ay + b$ .

This holds because we can compute  $a$  and  $b$  when given  $t_x$  and  $t_y$ :

$$t_x \equiv ax + b \pmod{p}$$

$$t_y \equiv ay + b \pmod{p}$$

$$t_x - t_y \equiv a(x - y) \pmod{p}$$

$$t_y \equiv ay + b \pmod{p}$$

$$a \equiv (t_x - t_y)(x - y)^{-1} \pmod{p}$$

$$b \equiv t_y - ay \pmod{p}$$

# Universal Hashing

# Universal Hashing

There is a one-to-one correspondence between hash-functions (pairs  $(a, b)$ ,  $a \neq 0$ ) and pairs  $(t_x, t_y)$ ,  $t_x \neq t_y$ .

# Universal Hashing

There is a one-to-one correspondence between hash-functions (pairs  $(a, b)$ ,  $a \neq 0$ ) and pairs  $(t_x, t_y)$ ,  $t_x \neq t_y$ .

Therefore, we can view the first step (before the  $\text{mod } n$ -operation) as choosing a pair  $(t_x, t_y)$ ,  $t_x \neq t_y$  uniformly at random.

# Universal Hashing

There is a one-to-one correspondence between hash-functions (pairs  $(a, b)$ ,  $a \neq 0$ ) and pairs  $(t_x, t_y)$ ,  $t_x \neq t_y$ .

Therefore, we can view the first step (before the  $\text{mod } n$ -operation) as choosing a pair  $(t_x, t_y)$ ,  $t_x \neq t_y$  uniformly at random.

What happens when we do the  $\text{mod } n$  operation?



# Universal Hashing

There is a one-to-one correspondence between hash-functions (pairs  $(a, b)$ ,  $a \neq 0$ ) and pairs  $(t_x, t_y)$ ,  $t_x \neq t_y$ .

Therefore, we can view the first step (before the  $\text{mod } n$ -operation) as choosing a pair  $(t_x, t_y)$ ,  $t_x \neq t_y$  uniformly at random.

What happens when we do the  $\text{mod } n$  operation?

Fix a value  $t_x$ . There are  $p - 1$  possible values for choosing  $t_y$ .

# Universal Hashing

There is a one-to-one correspondence between hash-functions (pairs  $(a, b)$ ,  $a \neq 0$ ) and pairs  $(t_x, t_y)$ ,  $t_x \neq t_y$ .

Therefore, we can view the first step (before the  $\text{mod } n$ -operation) as choosing a pair  $(t_x, t_y)$ ,  $t_x \neq t_y$  uniformly at random.

What happens when we do the  $\text{mod } n$  operation?

Fix a value  $t_x$ . There are  $p - 1$  possible values for choosing  $t_y$ .

From the range  $0, \dots, p - 1$  the values  $t_x, t_x + n, t_x + 2n, \dots$  map to  $t_x$  after the modulo-operation. These are at most  $\lceil p/n \rceil$  values.

# Universal Hashing

# Universal Hashing

As  $t_y \neq t_x$  there are

$$\left[ \frac{p}{n} \right] - 1$$

# Universal Hashing

As  $t_y \neq t_x$  there are

$$\left\lceil \frac{p}{n} \right\rceil - 1 \leq \frac{p}{n} + \frac{n-1}{n} - 1$$

# Universal Hashing

As  $t_y \neq t_x$  there are

$$\left\lceil \frac{p}{n} \right\rceil - 1 \leq \frac{p}{n} + \frac{n-1}{n} - 1 \leq \frac{p-1}{n}$$

# Universal Hashing

As  $t_y \neq t_x$  there are

$$\left\lceil \frac{p}{n} \right\rceil - 1 \leq \frac{p}{n} + \frac{n-1}{n} - 1 \leq \frac{p-1}{n}$$

possibilities for choosing  $t_y$  such that the final hash-value creates a collision.

# Universal Hashing

As  $t_y \neq t_x$  there are

$$\left\lceil \frac{p}{n} \right\rceil - 1 \leq \frac{p}{n} + \frac{n-1}{n} - 1 \leq \frac{p-1}{n}$$

possibilities for choosing  $t_y$  such that the final hash-value creates a collision.

This happens with probability at most  $\frac{1}{n}$ .



# Universal Hashing

# Universal Hashing

It is also possible to show that  $\mathcal{H}$  is an (almost) pairwise independent class of hash-functions.

$$\Pr_{t_x \neq t_y \in \mathbb{Z}_p^2} \left[ \begin{array}{l} t_x \bmod n = h_1 \\ t_y \bmod n = h_2 \end{array} \right]$$

# Universal Hashing

It is also possible to show that  $\mathcal{H}$  is an (almost) pairwise independent class of hash-functions.

$$\frac{\left\lfloor \frac{p}{n} \right\rfloor^2}{p(p-1)} \leq \Pr_{t_x \neq t_y \in \mathbb{Z}_p^2} \left[ \begin{array}{l} t_x \bmod n = h_1 \\ t_y \bmod n = h_2 \end{array} \right] \leq \frac{\left\lceil \frac{p}{n} \right\rceil^2}{p(p-1)}$$

# Universal Hashing

It is also possible to show that  $\mathcal{H}$  is an (almost) pairwise independent class of hash-functions.

$$\frac{\lfloor \frac{p}{n} \rfloor^2}{p(p-1)} \leq \Pr_{t_x \neq t_y \in \mathbb{Z}_p^2} \left[ \begin{array}{l} t_x \bmod n = h_1 \\ t_y \bmod n = h_2 \end{array} \right] \leq \frac{\lceil \frac{p}{n} \rceil^2}{p(p-1)}$$

Note that the middle is the probability that  $h(x) = h_1$  and  $h(y) = h_2$ . The total number of choices for  $(t_x, t_y)$  is  $p(p-1)$ . The number of choices for  $t_x$  ( $t_y$ ) such that  $t_x \bmod n = h_1$  ( $t_y \bmod n = h_2$ ) lies between  $\lfloor \frac{p}{n} \rfloor$  and  $\lceil \frac{p}{n} \rceil$ .

# Universal Hashing

## Definition 10

Let  $d \in \mathbb{N}$ ;  $q \geq (d+1)n$  be a prime; and let  $\bar{a} \in \{0, \dots, q-1\}^{d+1}$ . Define for  $x \in \{0, \dots, q-1\}$

$$h_{\bar{a}}(x) := \left( \sum_{i=0}^d a_i x^i \bmod q \right) \bmod n .$$

Let  $\mathcal{H}_n^d := \{h_{\bar{a}} \mid \bar{a} \in \{0, \dots, q-1\}^{d+1}\}$ . The class  $\mathcal{H}_n^d$  is  $(e, d+1)$ -independent.

Note that in the previous case we had  $d = 1$  and chose  $a_d \neq 0$ .

# Universal Hashing

# Universal Hashing

For the coefficients  $\bar{a} \in \{0, \dots, q-1\}^{d+1}$  let  $f_{\bar{a}}$  denote the polynomial

$$f_{\bar{a}}(x) = \left( \sum_{i=0}^d a_i x^i \right) \bmod q$$

# Universal Hashing

For the coefficients  $\bar{a} \in \{0, \dots, q-1\}^{d+1}$  let  $f_{\bar{a}}$  denote the polynomial

$$f_{\bar{a}}(x) = \left( \sum_{i=0}^d a_i x^i \right) \bmod q$$

The polynomial is defined by  $d + 1$  distinct points.



# Universal Hashing

## Universal Hashing

Fix  $\ell \leq d + 1$ ; let  $x_1, \dots, x_\ell \in \{0, \dots, q - 1\}$  be keys, and let  $t_1, \dots, t_\ell$  denote the corresponding hash-function values.

## Universal Hashing

Fix  $\ell \leq d + 1$ ; let  $x_1, \dots, x_\ell \in \{0, \dots, q - 1\}$  be keys, and let  $t_1, \dots, t_\ell$  denote the corresponding hash-function values.

Let  $A^\ell = \{h_{\bar{a}} \in \mathcal{H} \mid h_{\bar{a}}(x_i) = t_i \text{ for all } i \in \{1, \dots, \ell\}\}$

# Universal Hashing

Fix  $\ell \leq d + 1$ ; let  $x_1, \dots, x_\ell \in \{0, \dots, q - 1\}$  be keys, and let  $t_1, \dots, t_\ell$  denote the corresponding hash-function values.

Let  $A^\ell = \{h_{\bar{a}} \in \mathcal{H} \mid h_{\bar{a}}(x_i) = t_i \text{ for all } i \in \{1, \dots, \ell\}\}$

Then

$$h_{\bar{a}} \in A^\ell \Leftrightarrow h_{\bar{a}} = f_{\bar{a}} \bmod n \text{ and}$$

$$f_{\bar{a}}(x_i) \in \underbrace{\{t_i + \alpha \cdot n \mid \alpha \in \{0, \dots, \lceil \frac{q}{n} \rceil - 1\}\}}_{=: B_i}$$

# Universal Hashing

Fix  $\ell \leq d + 1$ ; let  $x_1, \dots, x_\ell \in \{0, \dots, q - 1\}$  be keys, and let  $t_1, \dots, t_\ell$  denote the corresponding hash-function values.

Let  $A^\ell = \{h_{\bar{a}} \in \mathcal{H} \mid h_{\bar{a}}(x_i) = t_i \text{ for all } i \in \{1, \dots, \ell\}\}$

Then

$$h_{\bar{a}} \in A^\ell \Leftrightarrow h_{\bar{a}} = f_{\bar{a}} \bmod n \text{ and}$$

$$f_{\bar{a}}(x_i) \in \underbrace{\{t_i + \alpha \cdot n \mid \alpha \in \{0, \dots, \lceil \frac{q}{n} \rceil - 1\}\}}_{=: B_i}$$

In order to obtain the cardinality of  $A^\ell$  we choose our polynomial by fixing  $d + 1$  points.

# Universal Hashing

Fix  $\ell \leq d + 1$ ; let  $x_1, \dots, x_\ell \in \{0, \dots, q - 1\}$  be keys, and let  $t_1, \dots, t_\ell$  denote the corresponding hash-function values.

Let  $A^\ell = \{h_{\bar{a}} \in \mathcal{H} \mid h_{\bar{a}}(x_i) = t_i \text{ for all } i \in \{1, \dots, \ell\}\}$

Then

$$h_{\bar{a}} \in A^\ell \Leftrightarrow h_{\bar{a}} = f_{\bar{a}} \bmod n \text{ and}$$

$$f_{\bar{a}}(x_i) \in \underbrace{\{t_i + \alpha \cdot n \mid \alpha \in \{0, \dots, \lceil \frac{q}{n} \rceil - 1\}\}}_{=: B_i}$$

In order to obtain the cardinality of  $A^\ell$  we choose our polynomial by fixing  $d + 1$  points.

We first fix the values for inputs  $x_1, \dots, x_\ell$ .

# Universal Hashing

Fix  $\ell \leq d + 1$ ; let  $x_1, \dots, x_\ell \in \{0, \dots, q - 1\}$  be keys, and let  $t_1, \dots, t_\ell$  denote the corresponding hash-function values.

Let  $A^\ell = \{h_{\bar{a}} \in \mathcal{H} \mid h_{\bar{a}}(x_i) = t_i \text{ for all } i \in \{1, \dots, \ell\}\}$

Then

$$h_{\bar{a}} \in A^\ell \Leftrightarrow h_{\bar{a}} = f_{\bar{a}} \bmod n \text{ and}$$

$$f_{\bar{a}}(x_i) \in \underbrace{\{t_i + \alpha \cdot n \mid \alpha \in \{0, \dots, \lceil \frac{q}{n} \rceil - 1\}\}}_{=: B_i}$$

In order to obtain the cardinality of  $A^\ell$  we choose our polynomial by fixing  $d + 1$  points.

We first fix the values for inputs  $x_1, \dots, x_\ell$ .

We have

$$|B_1| \cdot \dots \cdot |B_\ell|$$

possibilities to do this (so that  $h_{\bar{a}}(x_i) = t_i$ ).

# Universal Hashing

Now, we choose  $d - \ell + 1$  other inputs and choose their value arbitrarily. We have  $q^{d-\ell+1}$  possibilities to do this.



# Universal Hashing

Now, we choose  $d - \ell + 1$  other inputs and choose their value arbitrarily. We have  $q^{d-\ell+1}$  possibilities to do this.

Therefore we have

$$|B_1| \cdot \dots \cdot |B_\ell| \cdot q^{d-\ell+1} \leq \left\lceil \frac{q}{n} \right\rceil^\ell \cdot q^{d-\ell+1}$$

possibilities to choose  $\bar{a}$  such that  $h_{\bar{a}} \in A_\ell$ .

# Universal Hashing

Therefore the probability of choosing  $h_{\bar{a}}$  from  $A_\ell$  is only

$$\frac{\left[\frac{q}{n}\right]^\ell \cdot q^{d-\ell+1}}{q^{d+1}}$$

# Universal Hashing

Therefore the probability of choosing  $h_{\bar{a}}$  from  $A_\ell$  is only

$$\frac{\lceil \frac{q}{n} \rceil^\ell \cdot q^{d-\ell+1}}{q^{d+1}} \leq \frac{(\frac{q+n}{n})^\ell}{q^\ell}$$

# Universal Hashing

Therefore the probability of choosing  $h_{\bar{a}}$  from  $A_\ell$  is only

$$\frac{\left[\frac{q}{n}\right]^\ell \cdot q^{d-\ell+1}}{q^{d+1}} \leq \frac{\left(\frac{q+n}{n}\right)^\ell}{q^\ell} \leq \left(\frac{q+n}{q}\right)^\ell \cdot \frac{1}{n^\ell}$$

# Universal Hashing

Therefore the probability of choosing  $h_{\bar{a}}$  from  $A_\ell$  is only

$$\begin{aligned} \frac{\lceil \frac{q}{n} \rceil^\ell \cdot q^{d-\ell+1}}{q^{d+1}} &\leq \frac{(\frac{q+n}{n})^\ell}{q^\ell} \leq \left(\frac{q+n}{q}\right)^\ell \cdot \frac{1}{n^\ell} \\ &\leq \left(1 + \frac{1}{\ell}\right)^\ell \cdot \frac{1}{n^\ell} \end{aligned}$$

# Universal Hashing

Therefore the probability of choosing  $h_{\bar{a}}$  from  $A_\ell$  is only

$$\begin{aligned} \frac{\lceil \frac{q}{n} \rceil^\ell \cdot q^{d-\ell+1}}{q^{d+1}} &\leq \frac{\left(\frac{q+n}{n}\right)^\ell}{q^\ell} \leq \left(\frac{q+n}{q}\right)^\ell \cdot \frac{1}{n^\ell} \\ &\leq \left(1 + \frac{1}{\ell}\right)^\ell \cdot \frac{1}{n^\ell} \leq \frac{e}{n^\ell} . \end{aligned}$$

# Universal Hashing

Therefore the probability of choosing  $h_{\bar{a}}$  from  $A_\ell$  is only

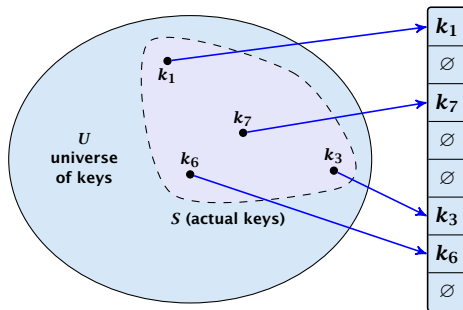
$$\begin{aligned}\frac{\lceil \frac{q}{n} \rceil^\ell \cdot q^{d-\ell+1}}{q^{d+1}} &\leq \frac{(\frac{q+n}{n})^\ell}{q^\ell} \leq \left(\frac{q+n}{q}\right)^\ell \cdot \frac{1}{n^\ell} \\ &\leq \left(1 + \frac{1}{\ell}\right)^\ell \cdot \frac{1}{n^\ell} \leq \frac{e}{n^\ell}.\end{aligned}$$

This shows that the  $\mathcal{H}$  is  $(e, d+1)$ -universal.

The last step followed from  $q \geq (d+1)n$ , and  $\ell \leq d+1$ .

# Perfect Hashing

Suppose that we **know** the set  $S$  of actual keys (no insert/no delete). Then we may want to design a **simple** hash-function that maps all these keys to different memory locations.





# Perfect Hashing

# Perfect Hashing

Let  $m = |S|$ . We could simply choose the hash-table size very large so that we don't get any collisions.

# Perfect Hashing

Let  $m = |S|$ . We could simply choose the hash-table size very large so that we don't get any collisions.

Using a universal hash-function the expected number of collisions is

$$E[\text{\#Collisions}] = \binom{m}{2} \cdot \frac{1}{n} .$$

# Perfect Hashing

Let  $m = |S|$ . We could simply choose the hash-table size very large so that we don't get any collisions.

Using a universal hash-function the expected number of collisions is

$$E[\#\text{Collisions}] = \binom{m}{2} \cdot \frac{1}{n} .$$

If we choose  $n = m^2$  the **expected number** of collisions is strictly less than  $\frac{1}{2}$ .

# Perfect Hashing

Let  $m = |S|$ . We could simply choose the hash-table size very large so that we don't get any collisions.

Using a universal hash-function the expected number of collisions is

$$E[\text{\#Collisions}] = \binom{m}{2} \cdot \frac{1}{n}.$$

If we choose  $n = m^2$  the **expected number** of collisions is strictly less than  $\frac{1}{2}$ .

Can we get an upper bound on the **probability of having collisions**?

# Perfect Hashing

Let  $m = |S|$ . We could simply choose the hash-table size very large so that we don't get any collisions.

Using a universal hash-function the expected number of collisions is

$$E[\#\text{Collisions}] = \binom{m}{2} \cdot \frac{1}{n}.$$

If we choose  $n = m^2$  the **expected number** of collisions is strictly less than  $\frac{1}{2}$ .

Can we get an upper bound on the **probability of having collisions**?

The probability of having **1** or more collisions can be at most  $\frac{1}{2}$  as otherwise the expectation would be larger than  $\frac{1}{2}$ .

# Perfect Hashing

# Perfect Hashing

We can find such a hash-function by a few trials.



# Perfect Hashing

We can find such a hash-function by a few trials.

However, a hash-table size of  $n = m^2$  is very very high.

# Perfect Hashing

We can find such a hash-function by a few trials.

However, a hash-table size of  $n = m^2$  is very very high.

We construct a two-level scheme. We first use a hash-function that maps elements from  $S$  to  $m$  buckets.

# Perfect Hashing

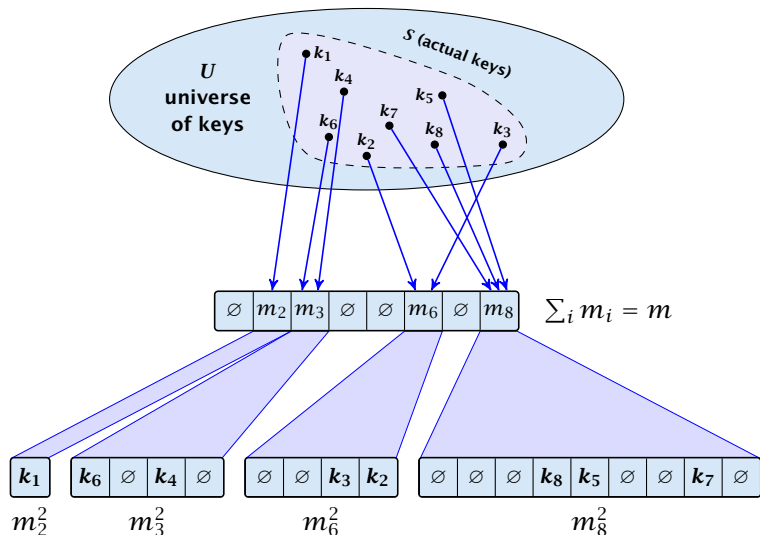
We can find such a hash-function by a few trials.

However, a hash-table size of  $n = m^2$  is very very high.

We construct a two-level scheme. We first use a hash-function that maps elements from  $S$  to  $m$  buckets.

Let  $m_j$  denote the number of items that are hashed to the  $j$ -th bucket. For each bucket we choose a second hash-function that maps the elements of the bucket into a table of size  $m_j^2$ . The second function can be chosen such that all elements are mapped to different locations.

# Perfect Hashing



# Perfect Hashing

# Perfect Hashing

The total memory that is required by all hash-tables is  $\mathcal{O}(\sum_j m_j^2)$ .  
Note that  $m_j$  is a random variable.

$$E \left[ \sum_j m_j^2 \right]$$

# Perfect Hashing

The total memory that is required by all hash-tables is  $\mathcal{O}(\sum_j m_j^2)$ .  
Note that  $m_j$  is a random variable.

$$\mathbb{E} \left[ \sum_j m_j^2 \right] = \mathbb{E} \left[ 2 \sum_j \binom{m_j}{2} + \sum_j m_j \right]$$

# Perfect Hashing

The total memory that is required by all hash-tables is  $\mathcal{O}(\sum_j m_j^2)$ .  
Note that  $m_j$  is a random variable.

$$\begin{aligned} \mathbb{E} \left[ \sum_j m_j^2 \right] &= \mathbb{E} \left[ 2 \sum_j \binom{m_j}{2} + \sum_j m_j \right] \\ &= 2 \mathbb{E} \left[ \sum_j \binom{m_j}{2} \right] + \mathbb{E} \left[ \sum_j m_j \right] \end{aligned}$$



# Perfect Hashing

The total memory that is required by all hash-tables is  $\mathcal{O}(\sum_j m_j^2)$ . Note that  $m_j$  is a random variable.

$$\begin{aligned} \mathbb{E} \left[ \sum_j m_j^2 \right] &= \mathbb{E} \left[ 2 \sum_j \binom{m_j}{2} + \sum_j m_j \right] \\ &= 2 \mathbb{E} \left[ \sum_j \binom{m_j}{2} \right] + \mathbb{E} \left[ \sum_j m_j \right] \end{aligned}$$

The first expectation is simply the expected number of collisions, for the first level. Since we use universal hashing we have

# Perfect Hashing

The total memory that is required by all hash-tables is  $\mathcal{O}(\sum_j m_j^2)$ . Note that  $m_j$  is a random variable.

$$\begin{aligned} \mathbb{E} \left[ \sum_j m_j^2 \right] &= \mathbb{E} \left[ 2 \sum_j \binom{m_j}{2} + \sum_j m_j \right] \\ &= 2 \mathbb{E} \left[ \sum_j \binom{m_j}{2} \right] + \mathbb{E} \left[ \sum_j m_j \right] \end{aligned}$$

The first expectation is simply the expected number of collisions, for the first level. Since we use universal hashing we have

$$= 2 \binom{m}{2} \frac{1}{m} + m = 2m - 1 .$$

# Perfect Hashing

We need only  $\mathcal{O}(m)$  time to construct a hash-function  $h$  with  $\sum_j m_j^2 = \mathcal{O}(4m)$ , because with probability at least  $1/2$  a random function from a universal family will have this property.

Then we construct a hash-table  $h_j$  for every bucket. This takes expected time  $\mathcal{O}(m_j)$  for every bucket. A random function  $h_j$  is collision-free with probability at least  $1/2$ . We need  $\mathcal{O}(m_j)$  to test this.

We only need that the hash-functions are chosen from a universal family!!!

# Cuckoo Hashing

# Cuckoo Hashing

## Goal:

Try to generate a hash-table with constant worst-case search time in a dynamic scenario.

# Cuckoo Hashing

## Goal:

Try to generate a hash-table with constant worst-case search time in a dynamic scenario.

- ▶ Two hash-tables  $T_1[0, \dots, n - 1]$  and  $T_2[0, \dots, n - 1]$ , with hash-functions  $h_1$ , and  $h_2$ .

# Cuckoo Hashing

## Goal:

Try to generate a hash-table with constant worst-case search time in a dynamic scenario.

- ▶ Two hash-tables  $T_1[0, \dots, n - 1]$  and  $T_2[0, \dots, n - 1]$ , with hash-functions  $h_1$ , and  $h_2$ .
- ▶ An object  $x$  is either stored at location  $T_1[h_1(x)]$  or  $T_2[h_2(x)]$ .

# Cuckoo Hashing

## Goal:

Try to generate a hash-table with constant worst-case search time in a dynamic scenario.

- ▶ Two hash-tables  $T_1[0, \dots, n - 1]$  and  $T_2[0, \dots, n - 1]$ , with hash-functions  $h_1$ , and  $h_2$ .
- ▶ An object  $x$  is either stored at location  $T_1[h_1(x)]$  or  $T_2[h_2(x)]$ .
- ▶ A search clearly takes constant time if the above constraint is met.



# Cuckoo Hashing

Insert:



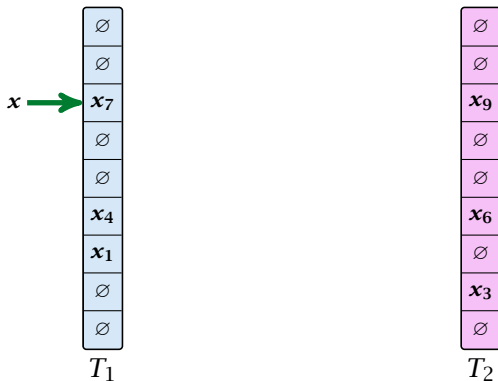
$T_1$



$T_2$

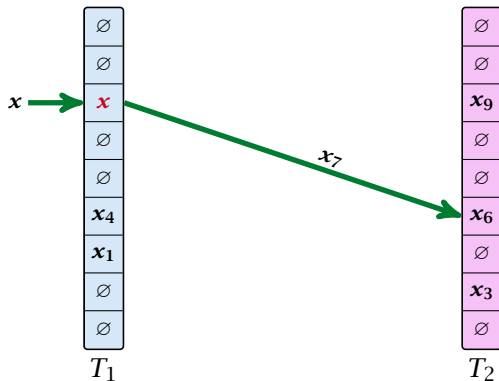
# Cuckoo Hashing

Insert:



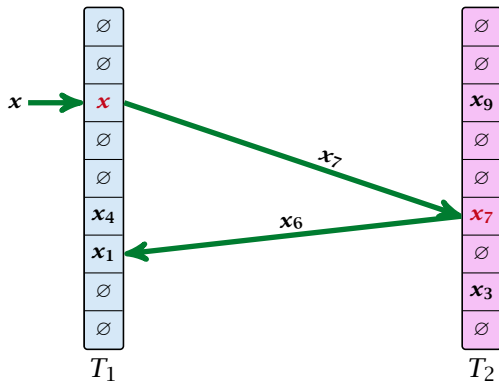
# Cuckoo Hashing

Insert:



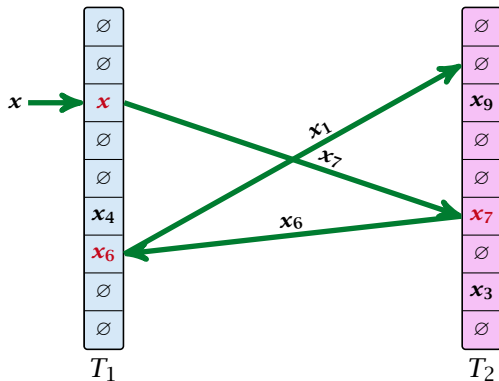
# Cuckoo Hashing

Insert:



# Cuckoo Hashing

Insert:



## Algorithm 38 Cuckoo-Insert( $x$ )

```
1: if  $T_1[h_1(x)] = x \vee T_2[h_2(x)] = x$  then return  
2: steps  $\leftarrow 1$   
3: while steps  $\leq$  maxsteps do  
4:     exchange  $x$  and  $T_1[h_1(x)]$   
5:     if  $x = \text{null}$  then return  
6:     exchange  $x$  and  $T_2[h_2(x)]$   
7:     if  $x = \text{null}$  then return  
8:     steps  $\leftarrow$  steps + 1  
9: rehash() // change hash-functions; rehash everything  
10: Cuckoo-Insert( $x$ )
```

# Cuckoo Hashing

- ▶ We call one iteration through the while-loop a **step** of the algorithm.

# Cuckoo Hashing

- ▶ We call one iteration through the while-loop a **step** of the algorithm.
- ▶ We call a sequence of iterations through the while-loop without the termination condition becoming true a **phase** of the algorithm.



# Cuckoo Hashing

- ▶ We call one iteration through the while-loop a **step** of the algorithm.
- ▶ We call a sequence of iterations through the while-loop without the termination condition becoming true a **phase** of the algorithm.
- ▶ We say a phase is **successful** if it is not terminated by the **maxstep**-condition, but the while loop is left because  $x = \text{null}$ .

# Cuckoo Hashing

**What is the expected time for an insert-operation?**

## What is the expected time for an insert-operation?

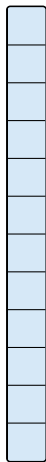
We first analyze the probability that we end-up in an infinite loop (that is then terminated after **maxsteps** steps).

## What is the expected time for an insert-operation?

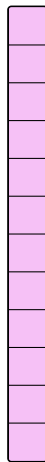
We first analyze the probability that we end-up in an infinite loop (that is then terminated after  $\text{maxsteps}$  steps).

Formally what is the probability to enter an infinite loop that touches  $s$  different keys?

# Cuckoo Hashing: Insert

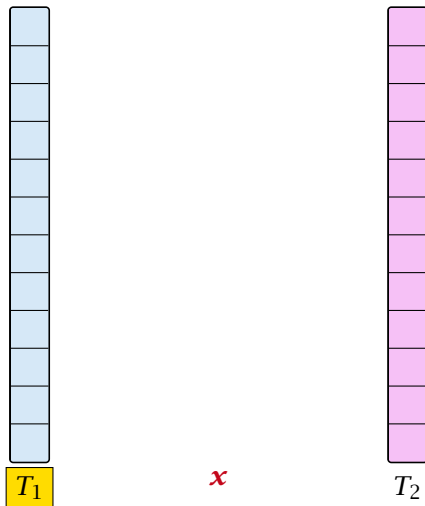


$T_1$

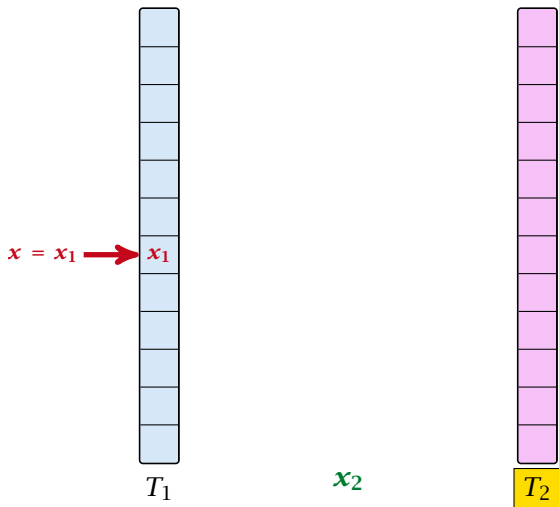


$T_2$

# Cuckoo Hashing: Insert

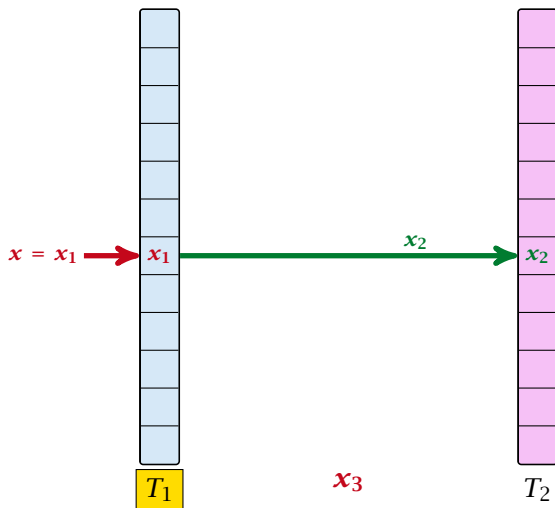


# Cuckoo Hashing: Insert

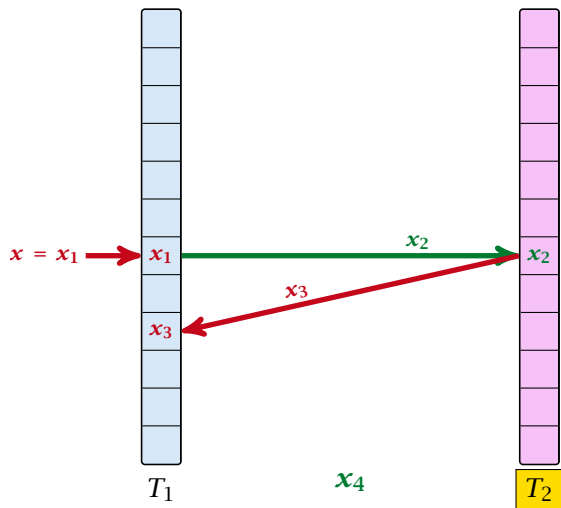




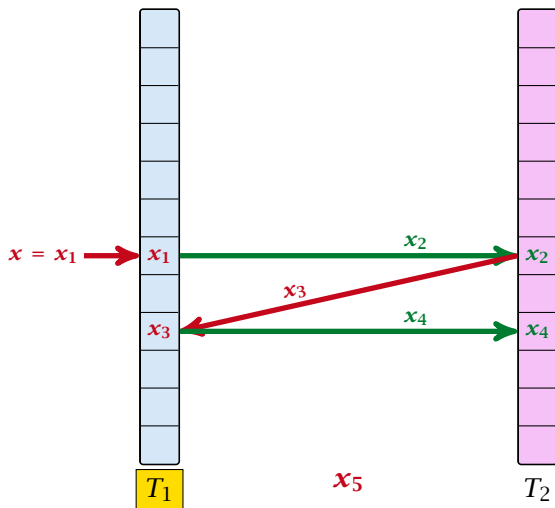
# Cuckoo Hashing: Insert



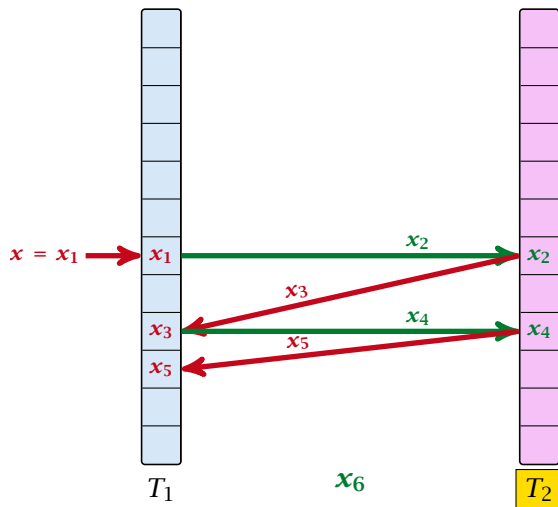
# Cuckoo Hashing: Insert



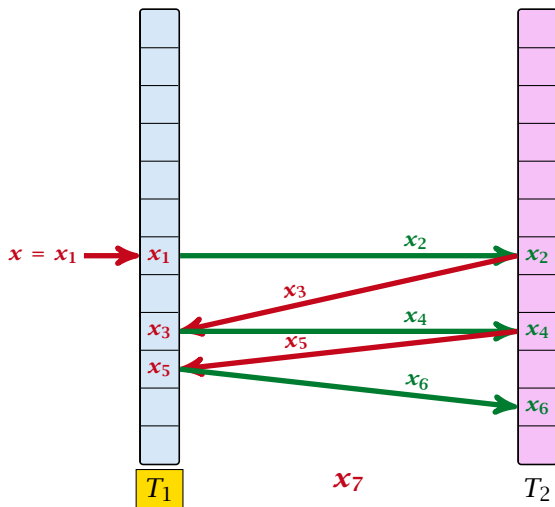
# Cuckoo Hashing: Insert



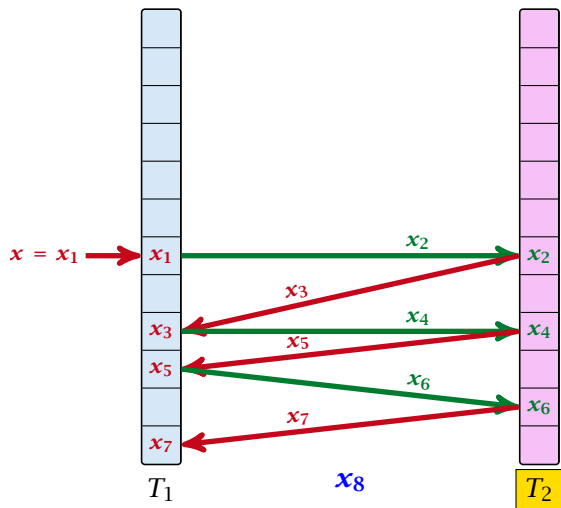
# Cuckoo Hashing: Insert



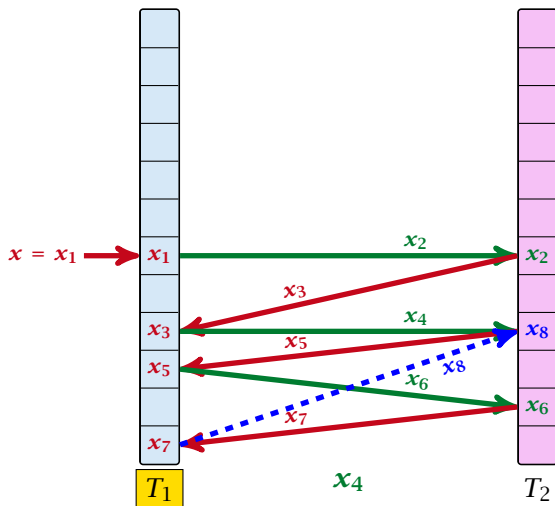
# Cuckoo Hashing: Insert



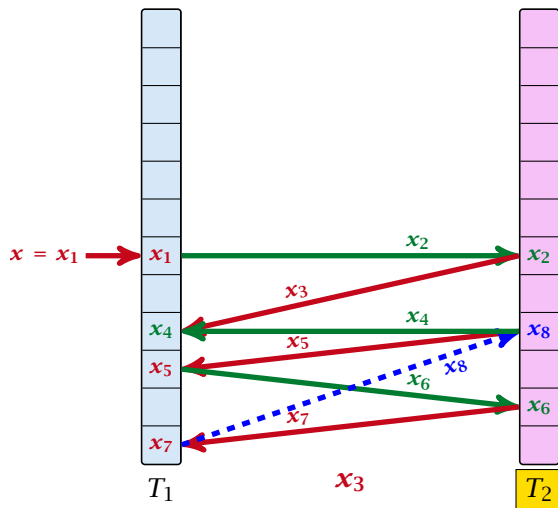
# Cuckoo Hashing: Insert



# Cuckoo Hashing: Insert

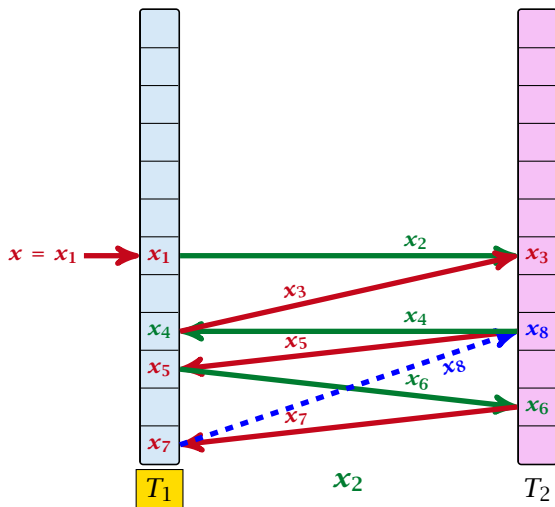


# Cuckoo Hashing: Insert

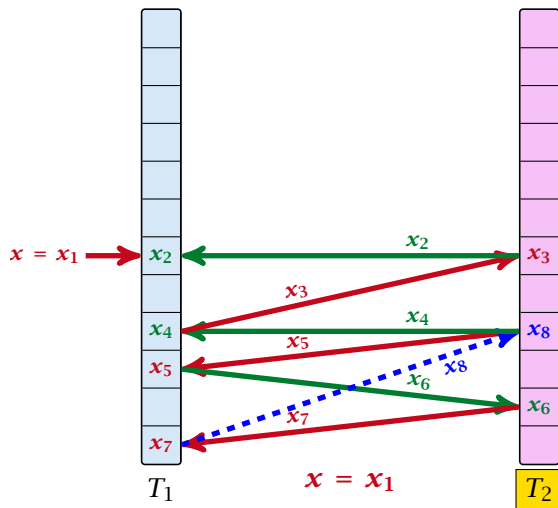




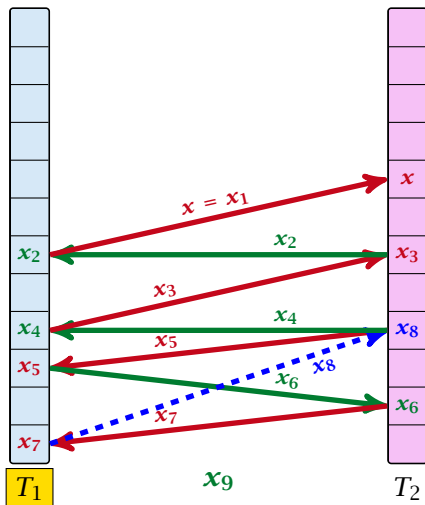
# Cuckoo Hashing: Insert



# Cuckoo Hashing: Insert



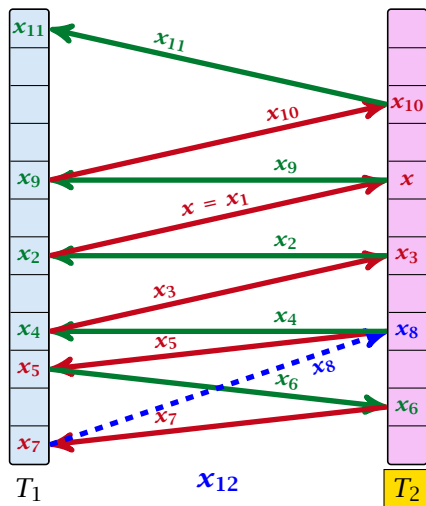
# Cuckoo Hashing: Insert



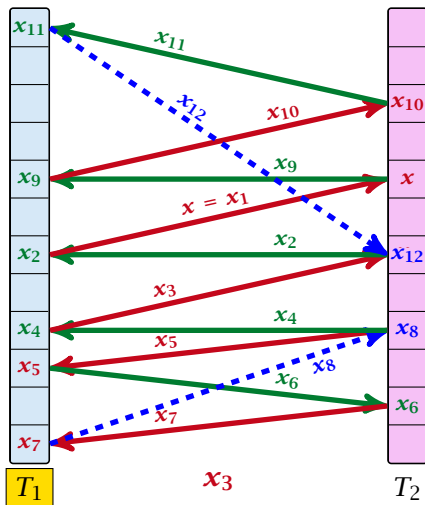




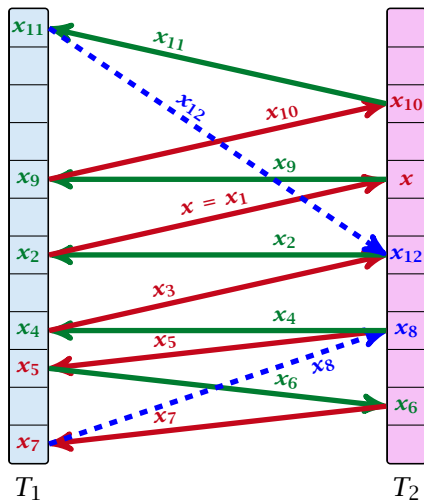
# Cuckoo Hashing: Insert



# Cuckoo Hashing: Insert

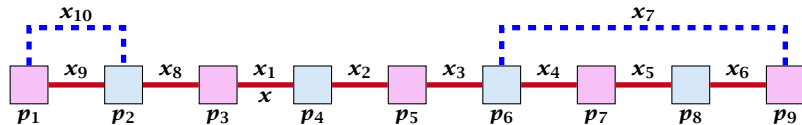


# Cuckoo Hashing: Insert



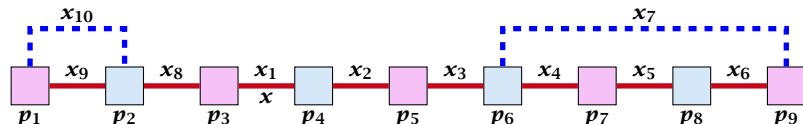


# Cuckoo Hashing



A cycle-structure of size  $s$  is defined by

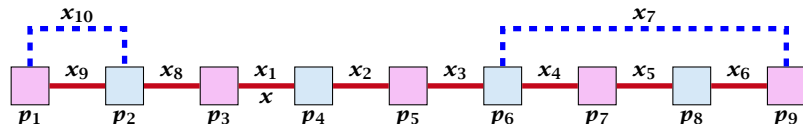
# Cuckoo Hashing



A cycle-structure of size  $s$  is defined by

- ▶  $s - 1$  different cells (alternating btw. cells from  $T_1$  and  $T_2$ ).

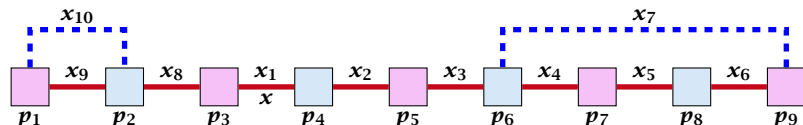
# Cuckoo Hashing



A cycle-structure of size  $s$  is defined by

- ▶  $s - 1$  different cells (alternating btw. cells from  $T_1$  and  $T_2$ ).
- ▶  $s$  distinct keys  $x = x_1, x_2, \dots, x_s$ , linking the cells.

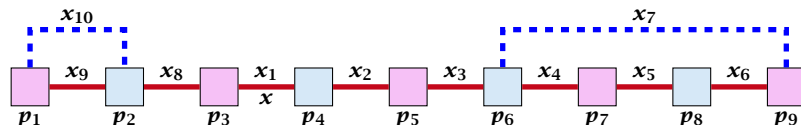
# Cuckoo Hashing



A **cycle-structure of size  $s$**  is defined by

- ▶  $s - 1$  different cells (alternating btw. cells from  $T_1$  and  $T_2$ ).
- ▶  $s$  distinct keys  $x = x_1, x_2, \dots, x_s$ , linking the cells.
- ▶ The leftmost cell is “linked forward” to some cell on the right.

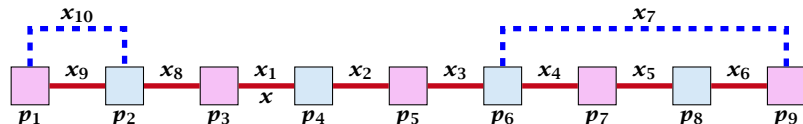
# Cuckoo Hashing



A **cycle-structure of size  $s$**  is defined by

- ▶  $s - 1$  different cells (alternating btw. cells from  $T_1$  and  $T_2$ ).
- ▶  $s$  distinct keys  $x = x_1, x_2, \dots, x_s$ , linking the cells.
- ▶ The leftmost cell is “linked forward” to some cell on the right.
- ▶ The rightmost cell is “linked backward” to a cell on the left.

# Cuckoo Hashing



A **cycle-structure of size  $s$**  is defined by

- ▶  $s - 1$  different cells (alternating btw. cells from  $T_1$  and  $T_2$ ).
- ▶  $s$  distinct keys  $x = x_1, x_2, \dots, x_s$ , linking the cells.
- ▶ The leftmost cell is “linked forward” to some cell on the right.
- ▶ The rightmost cell is “linked backward” to a cell on the left.
- ▶ One link represents key  $x$ ; this is where the counting starts.

# Cuckoo Hashing

A cycle-structure is **active** if for every key  $x_\ell$  (linking a cell  $p_i$  from  $T_1$  and a cell  $p_j$  from  $T_2$ ) we have

$$h_1(x_\ell) = p_i \quad \text{and} \quad h_2(x_\ell) = p_j$$

# Cuckoo Hashing

A cycle-structure is **active** if for every key  $x_\ell$  (linking a cell  $p_i$  from  $T_1$  and a cell  $p_j$  from  $T_2$ ) we have

$$h_1(x_\ell) = p_i \quad \text{and} \quad h_2(x_\ell) = p_j$$

## Observation:

If during a phase the insert-procedure runs into a cycle there must exist an active cycle structure of size  $s \geq 3$ .



# Cuckoo Hashing

What is the probability that all keys in a cycle-structure of size  $s$  correctly map into their  $T_1$ -cell?

# Cuckoo Hashing

What is the probability that all keys in a cycle-structure of size  $s$  correctly map into their  $T_1$ -cell?

This probability is at most  $\frac{\mu}{n^s}$  since  $h_1$  is a  $(\mu, s)$ -independent hash-function.

# Cuckoo Hashing

What is the probability that all keys in a cycle-structure of size  $s$  correctly map into their  $T_1$ -cell?

This probability is at most  $\frac{\mu}{n^s}$  since  $h_1$  is a  $(\mu, s)$ -independent hash-function.

What is the probability that all keys in the cycle-structure of size  $s$  correctly map into their  $T_2$ -cell?

# Cuckoo Hashing

What is the probability that all keys in a cycle-structure of size  $s$  correctly map into their  $T_1$ -cell?

This probability is at most  $\frac{\mu}{n^s}$  since  $h_1$  is a  $(\mu, s)$ -independent hash-function.

What is the probability that all keys in the cycle-structure of size  $s$  correctly map into their  $T_2$ -cell?

This probability is at most  $\frac{\mu}{n^s}$  since  $h_2$  is a  $(\mu, s)$ -independent hash-function.

# Cuckoo Hashing

What is the probability that all keys in a cycle-structure of size  $s$  correctly map into their  $T_1$ -cell?

This probability is at most  $\frac{\mu}{n^s}$  since  $h_1$  is a  $(\mu, s)$ -independent hash-function.

What is the probability that all keys in the cycle-structure of size  $s$  correctly map into their  $T_2$ -cell?

This probability is at most  $\frac{\mu}{n^s}$  since  $h_2$  is a  $(\mu, s)$ -independent hash-function.

These events are independent.

# Cuckoo Hashing

The probability that a given cycle-structure of size  $s$  is active is at most  $\frac{\mu^2}{n^{2s}}$ .

# Cuckoo Hashing

The probability that a given cycle-structure of size  $s$  is active is at most  $\frac{\mu^2}{n^{2s}}$ .

What is the probability that **there exists** an active cycle structure of size  $s$ ?

# Cuckoo Hashing

The number of cycle-structures of size  $s$  is at most

$$s^3 \cdot n^{s-1} \cdot m^{s-1} .$$



# Cuckoo Hashing

The number of cycle-structures of size  $s$  is at most

$$s^3 \cdot n^{s-1} \cdot m^{s-1} .$$

- ▶ There are at most  $s^2$  possibilities where to attach the forward and backward links.

# Cuckoo Hashing

The number of cycle-structures of size  $s$  is at most

$$s^3 \cdot n^{s-1} \cdot m^{s-1} .$$

- ▶ There are at most  $s^2$  possibilities where to attach the forward and backward links.
- ▶ There are at most  $s$  possibilities to choose where to place key  $x$ .

# Cuckoo Hashing

The number of cycle-structures of size  $s$  is at most

$$s^3 \cdot n^{s-1} \cdot m^{s-1} .$$

- ▶ There are at most  $s^2$  possibilities where to attach the forward and backward links.
- ▶ There are at most  $s$  possibilities to choose where to place key  $x$ .
- ▶ There are  $m^{s-1}$  possibilities to choose the keys apart from  $x$ .

# Cuckoo Hashing

The number of cycle-structures of size  $s$  is at most

$$s^3 \cdot n^{s-1} \cdot m^{s-1} .$$

- ▶ There are at most  $s^2$  possibilities where to attach the forward and backward links.
- ▶ There are at most  $s$  possibilities to choose where to place key  $x$ .
- ▶ There are  $m^{s-1}$  possibilities to choose the keys apart from  $x$ .
- ▶ There are  $n^{s-1}$  possibilities to choose the cells.

# Cuckoo Hashing

The probability that there exists an active cycle-structure is therefore at most

$$\sum_{s=3}^{\infty} s^3 \cdot n^{s-1} \cdot m^{s-1} \cdot \frac{\mu^2}{n^{2s}}$$

# Cuckoo Hashing

The probability that there exists an active cycle-structure is therefore at most

$$\sum_{s=3}^{\infty} s^3 \cdot n^{s-1} \cdot m^{s-1} \cdot \frac{\mu^2}{n^{2s}} = \frac{\mu^2}{nm} \sum_{s=3}^{\infty} s^3 \left(\frac{m}{n}\right)^s$$

# Cuckoo Hashing

The probability that there exists an active cycle-structure is therefore at most

$$\begin{aligned} \sum_{s=3}^{\infty} s^3 \cdot n^{s-1} \cdot m^{s-1} \cdot \frac{\mu^2}{n^{2s}} &= \frac{\mu^2}{nm} \sum_{s=3}^{\infty} s^3 \left(\frac{m}{n}\right)^s \\ &\leq \frac{\mu^2}{m^2} \sum_{s=3}^{\infty} s^3 \left(\frac{1}{1+\epsilon}\right)^s \end{aligned}$$

# Cuckoo Hashing

The probability that there exists an active cycle-structure is therefore at most

$$\begin{aligned} \sum_{s=3}^{\infty} s^3 \cdot n^{s-1} \cdot m^{s-1} \cdot \frac{\mu^2}{n^{2s}} &= \frac{\mu^2}{nm} \sum_{s=3}^{\infty} s^3 \left(\frac{m}{n}\right)^s \\ &\leq \frac{\mu^2}{m^2} \sum_{s=3}^{\infty} s^3 \left(\frac{1}{1+\epsilon}\right)^s \leq \mathcal{O}\left(\frac{1}{m^2}\right). \end{aligned}$$



# Cuckoo Hashing

The probability that there exists an active cycle-structure is therefore at most

$$\begin{aligned} \sum_{s=3}^{\infty} s^3 \cdot n^{s-1} \cdot m^{s-1} \cdot \frac{\mu^2}{n^{2s}} &= \frac{\mu^2}{nm} \sum_{s=3}^{\infty} s^3 \left(\frac{m}{n}\right)^s \\ &\leq \frac{\mu^2}{m^2} \sum_{s=3}^{\infty} s^3 \left(\frac{1}{1+\epsilon}\right)^s \leq \mathcal{O}\left(\frac{1}{m^2}\right). \end{aligned}$$

Here we used the fact that  $(1 + \epsilon)m \leq n$ .

# Cuckoo Hashing

The probability that there exists an active cycle-structure is therefore at most

$$\begin{aligned} \sum_{s=3}^{\infty} s^3 \cdot n^{s-1} \cdot m^{s-1} \cdot \frac{\mu^2}{n^{2s}} &= \frac{\mu^2}{nm} \sum_{s=3}^{\infty} s^3 \left(\frac{m}{n}\right)^s \\ &\leq \frac{\mu^2}{m^2} \sum_{s=3}^{\infty} s^3 \left(\frac{1}{1+\epsilon}\right)^s \leq \mathcal{O}\left(\frac{1}{m^2}\right). \end{aligned}$$

Here we used the fact that  $(1 + \epsilon)m \leq n$ .

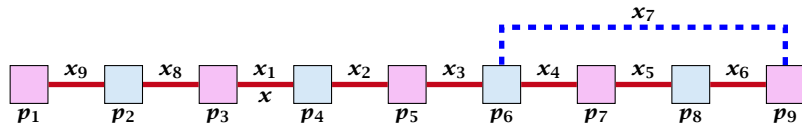
Hence,

$$\Pr[\text{cycle}] = \mathcal{O}\left(\frac{1}{m^2}\right).$$

# Cuckoo Hashing

Now, we analyze the probability that a phase is not successful without running into a closed cycle.

# Cuckoo Hashing



Sequence of visited keys:

$x = x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_3, x_2, x_1 = x, x_8, x_9, \dots$

# Cuckoo Hashing

Consider the sequence of not necessarily distinct keys starting with  $x$  in the order that they are visited during the phase.

# Cuckoo Hashing

Consider the sequence of not necessarily distinct keys starting with  $x$  in the order that they are visited during the phase.

## Lemma 11

*If the sequence is of length  $p$  then there exists a sub-sequence of at least  $\frac{p+2}{3}$  keys starting with  $x$  of *distinct* keys.*

# Cuckoo Hashing

## Proof.

Let  $i$  be the number of keys (including  $x$ ) that we see before the first repeated key. Let  $j$  denote the total number of distinct keys.

The sequence is of the form:

$$x = x_1 \rightarrow x_2 \rightarrow \dots \rightarrow x_i \rightarrow x_r \rightarrow x_{r-1} \rightarrow \dots \rightarrow x_1 \rightarrow x_{i+1} \rightarrow \dots \rightarrow x_j$$

As  $r \leq i - 1$  the length  $p$  of the sequence is

$$p = i + r + (j - i) \leq i + j - 1 .$$

# Cuckoo Hashing

## Proof.

Let  $i$  be the number of keys (including  $x$ ) that we see before the first repeated key. Let  $j$  denote the total number of distinct keys.

The sequence is of the form:

$$x = x_1 \rightarrow x_2 \rightarrow \dots \rightarrow x_i \rightarrow x_r \rightarrow x_{r-1} \rightarrow \dots \rightarrow x_1 \rightarrow x_{i+1} \rightarrow \dots \rightarrow x_j$$

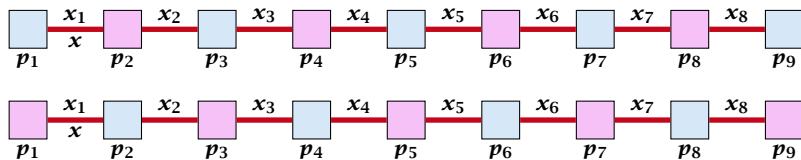
As  $r \leq i - 1$  the length  $p$  of the sequence is

$$p = i + r + (j - i) \leq i + j - 1 .$$

Either sub-sequence  $x_1 \rightarrow x_2 \rightarrow \dots \rightarrow x_i$  or sub-sequence  $x_1 \rightarrow x_{i+1} \rightarrow \dots \rightarrow x_j$  has at least  $\frac{p+2}{3}$  elements. □

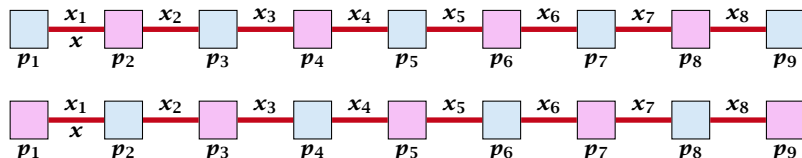


# Cuckoo Hashing



A path-structure of size  $s$  is defined by

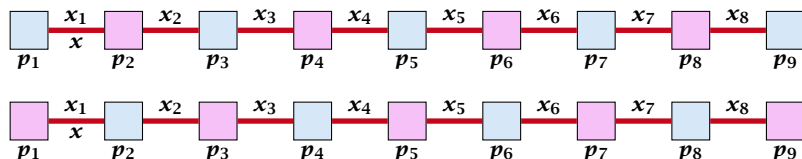
# Cuckoo Hashing



A path-structure of size  $s$  is defined by

- ▶  $s + 1$  different cells (alternating btw. cells from  $T_1$  and  $T_2$ ).

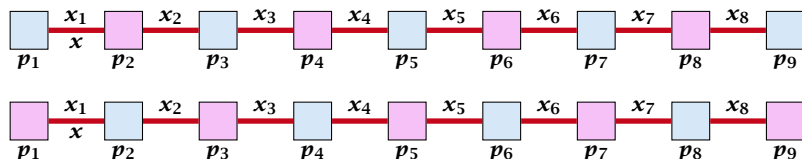
# Cuckoo Hashing



A path-structure of size  $s$  is defined by

- ▶  $s + 1$  different cells (alternating btw. cells from  $T_1$  and  $T_2$ ).
- ▶  $s$  distinct keys  $x = x_1, x_2, \dots, x_s$ , linking the cells.

# Cuckoo Hashing



A path-structure of size  $s$  is defined by

- ▶  $s + 1$  different cells (alternating btw. cells from  $T_1$  and  $T_2$ ).
- ▶  $s$  distinct keys  $x = x_1, x_2, \dots, x_s$ , linking the cells.
- ▶ The leftmost cell is either from  $T_1$  or  $T_2$ .

# Cuckoo Hashing

A path-structure is **active** if for every key  $x_\ell$  (linking a cell  $p_i$  from  $T_1$  and a cell  $p_j$  from  $T_2$ ) we have

$$h_1(x_\ell) = p_i \quad \text{and} \quad h_2(x_\ell) = p_j$$

## Observation:

If a phase takes at least  $t$  steps without running into a cycle there must exist an active path-structure of size  $(2t + 2)/3$ .

# Cuckoo Hashing

The probability that a given path-structure of size  $s$  is active is at most  $\frac{\mu^2}{n^{2s}}$ .

# Cuckoo Hashing

The probability that a given path-structure of size  $s$  is active is at most  $\frac{\mu^2}{n^{2s}}$ .

The probability that there exists an active path-structure of size  $s$  is at most

$$2 \cdot n^{s+1} \cdot m^{s-1} \cdot \frac{\mu^2}{n^{2s}}$$

# Cuckoo Hashing

The probability that a given path-structure of size  $s$  is active is at most  $\frac{\mu^2}{n^{2s}}$ .

The probability that there exists an active path-structure of size  $s$  is at most

$$2 \cdot n^{s+1} \cdot m^{s-1} \cdot \frac{\mu^2}{n^{2s}} \\ \leq 2\mu^2 \left(\frac{m}{n}\right)^{s-1}$$



# Cuckoo Hashing

The probability that a given path-structure of size  $s$  is active is at most  $\frac{\mu^2}{n^{2s}}$ .

The probability that there exists an active path-structure of size  $s$  is at most

$$2 \cdot n^{s+1} \cdot m^{s-1} \cdot \frac{\mu^2}{n^{2s}} \\ \leq 2\mu^2 \left(\frac{m}{n}\right)^{s-1} \leq 2\mu^2 \left(\frac{1}{1+\epsilon}\right)^{s-1}$$

# Cuckoo Hashing

The probability that a given path-structure of size  $s$  is active is at most  $\frac{\mu^2}{n^{2s}}$ .

The probability that there exists an active path-structure of size  $s$  is at most

$$2 \cdot n^{s+1} \cdot m^{s-1} \cdot \frac{\mu^2}{n^{2s}} \\ \leq 2\mu^2 \left(\frac{m}{n}\right)^{s-1} \leq 2\mu^2 \left(\frac{1}{1+\epsilon}\right)^{s-1}$$

Plugging in  $s = (2t + 2)/3$  gives

# Cuckoo Hashing

The probability that a given path-structure of size  $s$  is active is at most  $\frac{\mu^2}{n^{2s}}$ .

The probability that there exists an active path-structure of size  $s$  is at most

$$2 \cdot n^{s+1} \cdot m^{s-1} \cdot \frac{\mu^2}{n^{2s}} \\ \leq 2\mu^2 \left(\frac{m}{n}\right)^{s-1} \leq 2\mu^2 \left(\frac{1}{1+\epsilon}\right)^{s-1}$$

Plugging in  $s = (2t + 2)/3$  gives

$$\leq 2\mu^2 \left(\frac{1}{1+\epsilon}\right)^{(2t+2)/3-1}$$

# Cuckoo Hashing

The probability that a given path-structure of size  $s$  is active is at most  $\frac{\mu^2}{n^{2s}}$ .

The probability that there exists an active path-structure of size  $s$  is at most

$$2 \cdot n^{s+1} \cdot m^{s-1} \cdot \frac{\mu^2}{n^{2s}} \\ \leq 2\mu^2 \left(\frac{m}{n}\right)^{s-1} \leq 2\mu^2 \left(\frac{1}{1+\epsilon}\right)^{s-1}$$

Plugging in  $s = (2t + 2)/3$  gives

$$\leq 2\mu^2 \left(\frac{1}{1+\epsilon}\right)^{(2t+2)/3-1} = 2\mu^2 \left(\frac{1}{1+\epsilon}\right)^{(2t-1)/3}.$$

# Cuckoo Hashing

We choose  $\text{maxsteps} \geq 3\ell/2 + 1/2$ .

# Cuckoo Hashing

We choose  $\text{maxsteps} \geq 3\ell/2 + 1/2$ . Then the probability that a phase terminates unsuccessfully without running into a cycle is at most

$$\Pr[\text{unsuccessful} \mid \text{no cycle}]$$

# Cuckoo Hashing

We choose  $\text{maxsteps} \geq 3\ell/2 + 1/2$ . Then the probability that a phase terminates unsuccessfully without running into a cycle is at most

$$\begin{aligned} & \Pr[\text{unsuccessful} \mid \text{no cycle}] \\ & \leq \Pr[\exists \text{ active path-structure of size at least } \frac{2\text{maxsteps}+2}{3}] \end{aligned}$$

# Cuckoo Hashing

We choose  $\text{maxsteps} \geq 3\ell/2 + 1/2$ . Then the probability that a phase terminates unsuccessfully without running into a cycle is at most

$$\begin{aligned} & \Pr[\text{unsuccessful} \mid \text{no cycle}] \\ & \leq \Pr[\exists \text{ active path-structure of size at least } \frac{2\text{maxsteps}+2}{3}] \\ & \leq \Pr[\exists \text{ active path-structure of size at least } \ell + 1] \end{aligned}$$



# Cuckoo Hashing

We choose  $\text{maxsteps} \geq 3\ell/2 + 1/2$ . Then the probability that a phase terminates unsuccessfully without running into a cycle is at most

$$\begin{aligned} & \Pr[\text{unsuccessful} \mid \text{no cycle}] \\ & \leq \Pr[\exists \text{ active path-structure of size at least } \frac{2\text{maxsteps}+2}{3}] \\ & \leq \Pr[\exists \text{ active path-structure of size at least } \ell + 1] \\ & \leq \Pr[\exists \text{ active path-structure of size exactly } \ell + 1] \end{aligned}$$

# Cuckoo Hashing

We choose  $\text{maxsteps} \geq 3\ell/2 + 1/2$ . Then the probability that a phase terminates unsuccessfully without running into a cycle is at most

$$\begin{aligned} & \Pr[\text{unsuccessful} \mid \text{no cycle}] \\ & \leq \Pr[\exists \text{ active path-structure of size at least } \frac{2\text{maxsteps}+2}{3}] \\ & \leq \Pr[\exists \text{ active path-structure of size at least } \ell + 1] \\ & \leq \Pr[\exists \text{ active path-structure of size exactly } \ell + 1] \\ & \leq 2\mu^2 \left( \frac{1}{1 + \epsilon} \right)^\ell \end{aligned}$$

# Cuckoo Hashing

We choose  $\text{maxsteps} \geq 3\ell/2 + 1/2$ . Then the probability that a phase terminates unsuccessfully without running into a cycle is at most

$$\begin{aligned} & \Pr[\text{unsuccessful} \mid \text{no cycle}] \\ & \leq \Pr[\exists \text{ active path-structure of size at least } \frac{2\text{maxsteps}+2}{3}] \\ & \leq \Pr[\exists \text{ active path-structure of size at least } \ell + 1] \\ & \leq \Pr[\exists \text{ active path-structure of size exactly } \ell + 1] \\ & \leq 2\mu^2 \left( \frac{1}{1+\epsilon} \right)^\ell \leq \frac{1}{m^2} \end{aligned}$$

# Cuckoo Hashing

We choose  $\text{maxsteps} \geq 3\ell/2 + 1/2$ . Then the probability that a phase terminates unsuccessfully without running into a cycle is at most

$$\begin{aligned} & \Pr[\text{unsuccessful} \mid \text{no cycle}] \\ & \leq \Pr[\exists \text{ active path-structure of size at least } \frac{2\text{maxsteps}+2}{3}] \\ & \leq \Pr[\exists \text{ active path-structure of size at least } \ell + 1] \\ & \leq \Pr[\exists \text{ active path-structure of size exactly } \ell + 1] \\ & \leq 2\mu^2 \left( \frac{1}{1+\epsilon} \right)^\ell \leq \frac{1}{m^2} \end{aligned}$$

by choosing  $\ell \geq \log \left( \frac{1}{2\mu^2 m^2} \right) / \log \left( \frac{1}{1+\epsilon} \right) = \log (2\mu^2 m^2) / \log (1 + \epsilon)$

# Cuckoo Hashing

We choose  $\text{maxsteps} \geq 3\ell/2 + 1/2$ . Then the probability that a phase terminates unsuccessfully without running into a cycle is at most

$$\begin{aligned} & \Pr[\text{unsuccessful} \mid \text{no cycle}] \\ & \leq \Pr[\exists \text{ active path-structure of size at least } \frac{2\text{maxsteps}+2}{3}] \\ & \leq \Pr[\exists \text{ active path-structure of size at least } \ell + 1] \\ & \leq \Pr[\exists \text{ active path-structure of size exactly } \ell + 1] \\ & \leq 2\mu^2 \left(\frac{1}{1+\epsilon}\right)^\ell \leq \frac{1}{m^2} \end{aligned}$$

by choosing  $\ell \geq \log\left(\frac{1}{2\mu^2 m^2}\right) / \log\left(\frac{1}{1+\epsilon}\right) = \log(2\mu^2 m^2) / \log(1+\epsilon)$

This gives  $\text{maxsteps} = \Theta(\log m)$ .

# Cuckoo Hashing

So far we estimated

$$\Pr[\text{cycle}] \leq \mathcal{O}\left(\frac{1}{m^2}\right)$$

and

$$\Pr[\text{unsuccessful} \mid \text{no cycle}] \leq \mathcal{O}\left(\frac{1}{m^2}\right)$$

# Cuckoo Hashing

So far we estimated

$$\Pr[\text{cycle}] \leq \mathcal{O}\left(\frac{1}{m^2}\right)$$

and

$$\Pr[\text{unsuccessful} \mid \text{no cycle}] \leq \mathcal{O}\left(\frac{1}{m^2}\right)$$

Observe that

$$\Pr[\text{successful}] = \Pr[\text{no cycle}] - \Pr[\text{unsuccessful} \mid \text{no cycle}]$$

# Cuckoo Hashing

So far we estimated

$$\Pr[\text{cycle}] \leq \mathcal{O}\left(\frac{1}{m^2}\right)$$

and

$$\Pr[\text{unsuccessful} \mid \text{no cycle}] \leq \mathcal{O}\left(\frac{1}{m^2}\right)$$

Observe that

$$\begin{aligned}\Pr[\text{successful}] &= \Pr[\text{no cycle}] - \Pr[\text{unsuccessful} \mid \text{no cycle}] \\ &\geq c \cdot \Pr[\text{no cycle}]\end{aligned}$$



# Cuckoo Hashing

So far we estimated

$$\Pr[\text{cycle}] \leq \mathcal{O}\left(\frac{1}{m^2}\right)$$

and

$$\Pr[\text{unsuccessful} \mid \text{no cycle}] \leq \mathcal{O}\left(\frac{1}{m^2}\right)$$

Observe that

$$\begin{aligned}\Pr[\text{successful}] &= \Pr[\text{no cycle}] - \Pr[\text{unsuccessful} \mid \text{no cycle}] \\ &\geq c \cdot \Pr[\text{no cycle}]\end{aligned}$$

for a suitable constant  $c > 0$ .

## Cuckoo Hashing

The expected number of complete steps in the **successful phase** of an insert operation is:

# Cuckoo Hashing

The expected number of complete steps in the **successful phase** of an insert operation is:

$$E[\text{number of steps} \mid \text{phase successful}]$$

# Cuckoo Hashing

The expected number of complete steps in the **successful phase** of an insert operation is:

$$\begin{aligned} & E[\text{number of steps} \mid \text{phase successful}] \\ &= \sum_{t \geq 1} \Pr[\text{search takes at least } t \text{ steps} \mid \text{phase successful}] \end{aligned}$$

# Cuckoo Hashing

The expected number of complete steps in the **successful phase** of an insert operation is:

$$\begin{aligned} & E[\text{number of steps} \mid \text{phase successful}] \\ &= \sum_{t \geq 1} \Pr[\text{search takes at least } t \text{ steps} \mid \text{phase successful}] \end{aligned}$$

We have

$$\Pr[\text{search at least } t \text{ steps} \mid \text{successful}]$$

# Cuckoo Hashing

The expected number of complete steps in the **successful phase** of an insert operation is:

$$\begin{aligned} & E[\text{number of steps} \mid \text{phase successful}] \\ &= \sum_{t \geq 1} \Pr[\text{search takes at least } t \text{ steps} \mid \text{phase successful}] \end{aligned}$$

We have

$$\begin{aligned} & \Pr[\text{search at least } t \text{ steps} \mid \text{successful}] \\ &= \Pr[\text{search at least } t \text{ steps} \wedge \text{successful}] / \Pr[\text{successful}] \end{aligned}$$

# Cuckoo Hashing

The expected number of complete steps in the **successful phase** of an insert operation is:

$$\begin{aligned} & E[\text{number of steps} \mid \text{phase successful}] \\ &= \sum_{t \geq 1} \Pr[\text{search takes at least } t \text{ steps} \mid \text{phase successful}] \end{aligned}$$

We have

$$\begin{aligned} & \Pr[\text{search at least } t \text{ steps} \mid \text{successful}] \\ &= \Pr[\text{search at least } t \text{ steps} \wedge \text{successful}] / \Pr[\text{successful}] \\ &\leq \frac{1}{c} \Pr[\text{search at least } t \text{ steps} \wedge \text{successful}] / \Pr[\text{no cycle}] \end{aligned}$$

# Cuckoo Hashing

The expected number of complete steps in the **successful phase** of an insert operation is:

$$\begin{aligned} & E[\text{number of steps} \mid \text{phase successful}] \\ &= \sum_{t \geq 1} \Pr[\text{search takes at least } t \text{ steps} \mid \text{phase successful}] \end{aligned}$$

We have

$$\begin{aligned} & \Pr[\text{search at least } t \text{ steps} \mid \text{successful}] \\ &= \Pr[\text{search at least } t \text{ steps} \wedge \text{successful}] / \Pr[\text{successful}] \\ &\leq \frac{1}{c} \Pr[\text{search at least } t \text{ steps} \wedge \text{successful}] / \Pr[\text{no cycle}] \\ &\leq \frac{1}{c} \Pr[\text{search at least } t \text{ steps} \wedge \text{no cycle}] / \Pr[\text{no cycle}] \end{aligned}$$



# Cuckoo Hashing

The expected number of complete steps in the **successful phase** of an insert operation is:

$$\begin{aligned} & E[\text{number of steps} \mid \text{phase successful}] \\ &= \sum_{t \geq 1} \Pr[\text{search takes at least } t \text{ steps} \mid \text{phase successful}] \end{aligned}$$

We have

$$\begin{aligned} & \Pr[\text{search at least } t \text{ steps} \mid \text{successful}] \\ &= \Pr[\text{search at least } t \text{ steps} \wedge \text{successful}] / \Pr[\text{successful}] \\ &\leq \frac{1}{c} \Pr[\text{search at least } t \text{ steps} \wedge \text{successful}] / \Pr[\text{no cycle}] \\ &\leq \frac{1}{c} \Pr[\text{search at least } t \text{ steps} \wedge \text{no cycle}] / \Pr[\text{no cycle}] \\ &= \frac{1}{c} \Pr[\text{search at least } t \text{ steps} \mid \text{no cycle}] . \end{aligned}$$

# Cuckoo Hashing

Hence,

$E[\text{number of steps} \mid \text{phase successful}]$

# Cuckoo Hashing

Hence,

$E[\text{number of steps} \mid \text{phase successful}]$

$$\leq \frac{1}{c} \sum_{t \geq 1} \Pr[\text{search at least } t \text{ steps} \mid \text{no cycle}]$$

# Cuckoo Hashing

Hence,

$E[\text{number of steps} \mid \text{phase successful}]$

$$\leq \frac{1}{c} \sum_{t \geq 1} \Pr[\text{search at least } t \text{ steps} \mid \text{no cycle}]$$

$$\leq \frac{1}{c} \sum_{t \geq 1} 2\mu^2 \left( \frac{1}{1 + \epsilon} \right)^{(2t-1)/3}$$

# Cuckoo Hashing

Hence,

$E[\text{number of steps} \mid \text{phase successful}]$

$$\leq \frac{1}{c} \sum_{t \geq 1} \Pr[\text{search at least } t \text{ steps} \mid \text{no cycle}]$$

$$\leq \frac{1}{c} \sum_{t \geq 1} 2\mu^2 \left( \frac{1}{1 + \epsilon} \right)^{(2t-1)/3} = \frac{1}{c} \sum_{t \geq 0} 2\mu^2 \left( \frac{1}{1 + \epsilon} \right)^{(2(t+1)-1)/3}$$

# Cuckoo Hashing

Hence,

$E[\text{number of steps} \mid \text{phase successful}]$

$$\begin{aligned} &\leq \frac{1}{c} \sum_{t \geq 1} \Pr[\text{search at least } t \text{ steps} \mid \text{no cycle}] \\ &\leq \frac{1}{c} \sum_{t \geq 1} 2\mu^2 \left(\frac{1}{1+\epsilon}\right)^{(2t-1)/3} = \frac{1}{c} \sum_{t \geq 0} 2\mu^2 \left(\frac{1}{1+\epsilon}\right)^{(2(t+1)-1)/3} \\ &= \frac{2\mu^2}{c(1+\epsilon)^{1/3}} \sum_{t \geq 0} \left(\frac{1}{(1+\epsilon)^{2/3}}\right)^t \end{aligned}$$

# Cuckoo Hashing

Hence,

$E[\text{number of steps} \mid \text{phase successful}]$

$$\begin{aligned} &\leq \frac{1}{c} \sum_{t \geq 1} \Pr[\text{search at least } t \text{ steps} \mid \text{no cycle}] \\ &\leq \frac{1}{c} \sum_{t \geq 1} 2\mu^2 \left(\frac{1}{1+\epsilon}\right)^{(2t-1)/3} = \frac{1}{c} \sum_{t \geq 0} 2\mu^2 \left(\frac{1}{1+\epsilon}\right)^{(2(t+1)-1)/3} \\ &= \frac{2\mu^2}{c(1+\epsilon)^{1/3}} \sum_{t \geq 0} \left(\frac{1}{(1+\epsilon)^{2/3}}\right)^t = \mathcal{O}(1) . \end{aligned}$$

# Cuckoo Hashing

Hence,

$E[\text{number of steps} \mid \text{phase successful}]$

$$\begin{aligned} &\leq \frac{1}{c} \sum_{t \geq 1} \Pr[\text{search at least } t \text{ steps} \mid \text{no cycle}] \\ &\leq \frac{1}{c} \sum_{t \geq 1} 2\mu^2 \left(\frac{1}{1+\epsilon}\right)^{(2t-1)/3} = \frac{1}{c} \sum_{t \geq 0} 2\mu^2 \left(\frac{1}{1+\epsilon}\right)^{(2(t+1)-1)/3} \\ &= \frac{2\mu^2}{c(1+\epsilon)^{1/3}} \sum_{t \geq 0} \left(\frac{1}{(1+\epsilon)^{2/3}}\right)^t = \mathcal{O}(1) . \end{aligned}$$

This means the expected cost for a successful phase is constant (even after accounting for the cost of the incomplete step that finishes the phase).



# Cuckoo Hashing

# Cuckoo Hashing

A phase that is not successful induces cost for doing a complete rehash (this dominates the cost for the steps in the phase).

# Cuckoo Hashing

A phase that is not successful induces cost for doing a complete rehash (this dominates the cost for the steps in the phase).

The probability that a phase is not successful is  $q = \mathcal{O}(1/m^2)$  (probability  $\mathcal{O}(1/m^2)$  of running into a cycle and probability  $\mathcal{O}(1/m^2)$  of reaching `maxsteps` without running into a cycle).

# Cuckoo Hashing

A phase that is not successful induces cost for doing a complete rehash (this dominates the cost for the steps in the phase).

The probability that a phase is not successful is  $q = \mathcal{O}(1/m^2)$  (probability  $\mathcal{O}(1/m^2)$  of running into a cycle and probability  $\mathcal{O}(1/m^2)$  of reaching `maxsteps` without running into a cycle).

A rehash try requires  $m$  insertions and takes expected constant time per insertion. It fails with probability  $p := \mathcal{O}(1/m)$ .

# Cuckoo Hashing

A phase that is not successful induces cost for doing a complete rehash (this dominates the cost for the steps in the phase).

The probability that a phase is not successful is  $q = \mathcal{O}(1/m^2)$  (probability  $\mathcal{O}(1/m^2)$  of running into a cycle and probability  $\mathcal{O}(1/m^2)$  of reaching `maxsteps` without running into a cycle).

A rehash try requires  $m$  insertions and takes expected constant time per insertion. It fails with probability  $p := \mathcal{O}(1/m)$ .

The expected number of unsuccessful rehashes is

$$\sum_{i \geq 1} p^i = \frac{1}{1-p} - 1 = \frac{p}{1-p} = \mathcal{O}(p).$$

# Cuckoo Hashing

A phase that is not successful induces cost for doing a complete rehash (this dominates the cost for the steps in the phase).

The probability that a phase is not successful is  $q = \mathcal{O}(1/m^2)$  (probability  $\mathcal{O}(1/m^2)$  of running into a cycle and probability  $\mathcal{O}(1/m^2)$  of reaching `maxsteps` without running into a cycle).

A rehash try requires  $m$  insertions and takes expected constant time per insertion. It fails with probability  $p := \mathcal{O}(1/m)$ .

The expected number of unsuccessful rehashes is

$$\sum_{i \geq 1} p^i = \frac{1}{1-p} - 1 = \frac{p}{1-p} = \mathcal{O}(p).$$

Therefore the expected cost for re-hashes is  $\mathcal{O}(m) \cdot \mathcal{O}(p) = \mathcal{O}(1)$ .

## Formal Proof

Let  $Y_i$  denote the event that the  $i$ -th rehash occurs and does not lead to a valid configuration (i.e., one of the  $m + 1$  insertions fails):

## Formal Proof

Let  $Y_i$  denote the event that the  $i$ -th rehash occurs and does not lead to a valid configuration (i.e., one of the  $m + 1$  insertions fails):

$$\Pr[Y_i | Z_i] \leq (m + 1) \cdot \mathcal{O}(1/m^2) \leq \mathcal{O}(1/m) =: p .$$



## Formal Proof

Let  $Y_i$  denote the event that the  $i$ -th rehash occurs and does not lead to a valid configuration (i.e., one of the  $m + 1$  insertions fails):

$$\Pr[Y_i | Z_i] \leq (m + 1) \cdot \mathcal{O}(1/m^2) \leq \mathcal{O}(1/m) =: p .$$

Let  $Z_i$  denote the event that the  $i$ -th rehash occurs:

## Formal Proof

Let  $Y_i$  denote the event that the  $i$ -th rehash occurs and does not lead to a valid configuration (i.e., one of the  $m + 1$  insertions fails):

$$\Pr[Y_i | Z_i] \leq (m + 1) \cdot \mathcal{O}(1/m^2) \leq \mathcal{O}(1/m) =: p .$$

Let  $Z_i$  denote the event that the  $i$ -th rehash occurs:

$$\Pr[Z_i] \leq \Pr[\wedge_{j=0}^{i-1} Y_j] \leq p^i$$

## Formal Proof

Let  $Y_i$  denote the event that the  $i$ -th rehash occurs and does not lead to a valid configuration (i.e., one of the  $m + 1$  insertions fails):

$$\Pr[Y_i | Z_i] \leq (m + 1) \cdot \mathcal{O}(1/m^2) \leq \mathcal{O}(1/m) =: p .$$

Let  $Z_i$  denote the event that the  $i$ -th rehash occurs:

$$\Pr[Z_i] \leq \Pr[\wedge_{j=0}^{i-1} Y_j] \leq p^i$$

Let  $X_i^s$ ,  $s \in \{1, \dots, m + 1\}$  denote the cost for inserting the  $s$ -th element during the  $i$ -th rehash (assuming  $i$ -th rehash occurs):

$$E[X_i^s]$$

## Formal Proof

Let  $Y_i$  denote the event that the  $i$ -th rehash occurs and does not lead to a valid configuration (i.e., one of the  $m + 1$  insertions fails):

$$\Pr[Y_i | Z_i] \leq (m + 1) \cdot \mathcal{O}(1/m^2) \leq \mathcal{O}(1/m) =: p .$$

Let  $Z_i$  denote the event that the  $i$ -th rehash occurs:

$$\Pr[Z_i] \leq \Pr[\wedge_{j=0}^{i-1} Y_j] \leq p^i$$

Let  $X_i^s$ ,  $s \in \{1, \dots, m + 1\}$  denote the cost for inserting the  $s$ -th element during the  $i$ -th rehash (assuming  $i$ -th rehash occurs):

$$\begin{aligned} E[X_i^s] &= E[\text{steps} \mid \text{phase successful}] \cdot \Pr[\text{phase successful}] \\ &\quad + \text{maxsteps} \cdot \Pr[\text{not successful}] \end{aligned}$$

## Formal Proof

Let  $Y_i$  denote the event that the  $i$ -th rehash occurs and does not lead to a valid configuration (i.e., one of the  $m + 1$  insertions fails):

$$\Pr[Y_i | Z_i] \leq (m + 1) \cdot \mathcal{O}(1/m^2) \leq \mathcal{O}(1/m) =: p .$$

Let  $Z_i$  denote the event that the  $i$ -th rehash occurs:

$$\Pr[Z_i] \leq \Pr[\wedge_{j=0}^{i-1} Y_j] \leq p^i$$

Let  $X_i^s$ ,  $s \in \{1, \dots, m + 1\}$  denote the cost for inserting the  $s$ -th element during the  $i$ -th rehash (assuming  $i$ -th rehash occurs):

$$\begin{aligned} \mathbb{E}[X_i^s] &= \mathbb{E}[\text{steps} \mid \text{phase successful}] \cdot \Pr[\text{phase successful}] \\ &\quad + \text{maxsteps} \cdot \Pr[\text{not successful}] = \mathcal{O}(1) . \end{aligned}$$

The expected cost for all rehashes is

$$E \left[ \sum_i \sum_s Z_i X_i^s \right]$$

The expected cost for all rehashes is

$$E \left[ \sum_i \sum_s Z_i X_i^s \right]$$

Note that  $Z_i$  is independent of  $X_j^s$ ,  $j \geq i$  (however, it is not independent of  $X_j^s$ ,  $j < i$ ). Hence,

$$E \left[ \sum_i \sum_s Z_i X_i^s \right] = \sum_i \sum_s E[Z_i] \cdot E[X_i^s]$$

The expected cost for all rehashes is

$$E \left[ \sum_i \sum_s Z_i X_i^s \right]$$

Note that  $Z_i$  is independent of  $X_j^s$ ,  $j \geq i$  (however, it is not independent of  $X_j^s$ ,  $j < i$ ). Hence,

$$\begin{aligned} E \left[ \sum_i \sum_s Z_i X_i^s \right] &= \sum_i \sum_s E[Z_i] \cdot E[X_i^s] \\ &\leq \mathcal{O}(m) \cdot \sum_i p^i \end{aligned}$$



The expected cost for all rehashes is

$$E \left[ \sum_i \sum_s Z_i X_i^s \right]$$

Note that  $Z_i$  is independent of  $X_j^s$ ,  $j \geq i$  (however, it is not independent of  $X_j^s$ ,  $j < i$ ). Hence,

$$\begin{aligned} E \left[ \sum_i \sum_s Z_i X_i^s \right] &= \sum_i \sum_s E[Z_i] \cdot E[X_i^s] \\ &\leq \mathcal{O}(m) \cdot \sum_i p^i \\ &\leq \mathcal{O}(m) \cdot \frac{p}{1-p} \end{aligned}$$

The expected cost for all rehashes is

$$\mathbb{E} \left[ \sum_i \sum_s Z_i X_i^s \right]$$

Note that  $Z_i$  is independent of  $X_j^s$ ,  $j \geq i$  (however, it is not independent of  $X_j^s$ ,  $j < i$ ). Hence,

$$\begin{aligned} \mathbb{E} \left[ \sum_i \sum_s Z_i X_i^s \right] &= \sum_i \sum_s \mathbb{E}[Z_i] \cdot \mathbb{E}[X_i^s] \\ &\leq \mathcal{O}(m) \cdot \sum_i p^i \\ &\leq \mathcal{O}(m) \cdot \frac{p}{1-p} \\ &= \mathcal{O}(1) . \end{aligned}$$

**What kind of hash-functions do we need?**

## What kind of hash-functions do we need?

Since `maxsteps` is  $\Theta(\log m)$  the largest size of a path-structure or cycle-structure contains just  $\Theta(\log m)$  different keys.

## What kind of hash-functions do we need?

Since  $\text{maxsteps}$  is  $\Theta(\log m)$  the largest size of a path-structure or cycle-structure contains just  $\Theta(\log m)$  different keys.

Therefore, it is sufficient to have  $(\mu, \Theta(\log m))$ -independent hash-functions.

# Cuckoo Hashing

How do we make sure that  $n \geq (1 + \epsilon)m$ ?

- ▶ Let  $\alpha := 1/(1 + \epsilon)$ .

# Cuckoo Hashing

How do we make sure that  $n \geq (1 + \epsilon)m$ ?

- ▶ Let  $\alpha := 1/(1 + \epsilon)$ .
- ▶ Keep track of the number of elements in the table. When  $m \geq \alpha n$  we double  $n$  and do a complete re-hash (table-expand).

# Cuckoo Hashing

How do we make sure that  $n \geq (1 + \epsilon)m$ ?

- ▶ Let  $\alpha := 1/(1 + \epsilon)$ .
- ▶ Keep track of the number of elements in the table. When  $m \geq \alpha n$  we double  $n$  and do a complete re-hash (**table-expand**).
- ▶ Whenever  $m$  drops below  $\alpha n/4$  we divide  $n$  by 2 and do a rehash (**table-shrink**).



# Cuckoo Hashing

How do we make sure that  $n \geq (1 + \epsilon)m$ ?

- ▶ Let  $\alpha := 1/(1 + \epsilon)$ .
- ▶ Keep track of the number of elements in the table. When  $m \geq \alpha n$  we double  $n$  and do a complete re-hash (**table-expand**).
- ▶ Whenever  $m$  drops below  $\alpha n/4$  we divide  $n$  by 2 and do a rehash (**table-shrink**).
- ▶ Note that right after a change in table-size we have  $m = \alpha n/2$ . In order for a table-expand to occur at least  $\alpha n/2$  insertions are required. Similar, for a table-shrink at least  $\alpha n/4$  deletions must occur.

# Cuckoo Hashing

How do we make sure that  $n \geq (1 + \epsilon)m$ ?

- ▶ Let  $\alpha := 1/(1 + \epsilon)$ .
- ▶ Keep track of the number of elements in the table. When  $m \geq \alpha n$  we double  $n$  and do a complete re-hash (**table-expand**).
- ▶ Whenever  $m$  drops below  $\alpha n/4$  we divide  $n$  by 2 and do a rehash (**table-shrink**).
- ▶ Note that right after a change in table-size we have  $m = \alpha n/2$ . In order for a table-expand to occur at least  $\alpha n/2$  insertions are required. Similar, for a table-shrink at least  $\alpha n/4$  deletions must occur.
- ▶ Therefore we can amortize the rehash cost after a change in table-size against the cost for insertions and deletions.

# Cuckoo Hashing

## Lemma 12

*Cuckoo Hashing has an expected constant insert-time and a worst-case constant search-time.*

## Lemma 12

*Cuckoo Hashing has an expected constant insert-time and a worst-case constant search-time.*

Note that the above lemma only holds if the fill-factor (number of keys/total number of hash-table slots) is at most  $\frac{1}{2(1+\epsilon)}$ .