4 Modelling Issues

What do you measure?

- Memory requirement
- Running time
- Number of comparisons
- Number of multiplications
- Number of hard-disc accesses
- Program size
- Power consumption
- ...

4 Modelling Issues

How do you measure?

- Implementing and testing on representative inputs
  - How do you choose your inputs?
  - May be very time-consuming.
  - Very reliable results if done correctly.
  - Results only hold for a specific machine and for a specific set of inputs.
- Theoretical analysis in a specific model of computation.
  - Gives asymptotic bounds like “this algorithm always runs in time $\mathcal{O}(n^2)$”.
  - Typically focuses on the worst case.
  - Can give lower bounds like “any comparison-based sorting algorithm needs at least $\Omega(n \log n)$ comparisons in the worst case”.

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Input length

The theoretical bounds are usually given by a function $f : \mathbb{N} \to \mathbb{N}$ that maps the input length to the running time (or storage space, comparisons, multiplications, program size etc.).

The input length may e.g. be

- the size of the input (number of bits)
- the number of arguments

Example 1

Suppose n numbers from the interval $\{1, \dots, N\}$ have to be sorted. In this case we usually say that the input length is n instead of e.g. $n \log N$, which would be the number of bits required to encode the input.

Model of Computation

How to measure performance

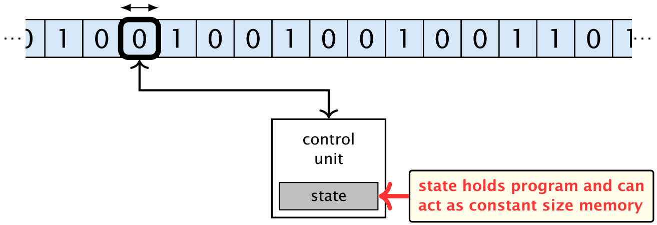
1. Calculate running time and storage space etc. on a simplified, idealized model of computation, e.g. Random Access Machine (RAM), Turing Machine (TM), ...
2. Calculate number of certain basic operations: comparisons, multiplications, harddisc accesses, ...

Version 2. is often easier, but focusing on one type of operation makes it more difficult to obtain meaningful results.

Turing Machine

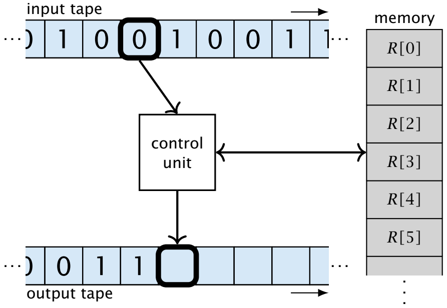
- Very simple model of computation.
- Only the “current” memory location can be altered.
- Very good model for discussing computabiliy, or polynomial vs. exponential time.
- Some simple problems like recognizing whether input is of the form xx, where x is a string, have quadratic lower bound.

$\Longrightarrow$ **Not a good model for developing efficient algorithms.**



Random Access Machine (RAM)

- Input tape and output tape (sequences of zeros and ones; unbounded length).
- Memory unit: infinite but countable number of registers $R[0], R[1], R[2], \ldots$.
- Registers hold integers.
- Indirect addressing.



Note that in the picture on the right the tapes are one-directional, and that a READ- or WRITE-operation always advances its tape.

Random Access Machine (RAM)

Operations

- input operations (input tape $\rightarrow R[i]$)
  - `READ` i
- output operations ($R[i] \rightarrow$ output tape)
  - `WRITE` i
- register-register transfers
  - $R[j]$ `:=` $R[i]$
  - $R[j]$ `:=` 4
- **indirect** addressing
  - $R[j]$ `:=` $R[R[i]]$
    loads the content of the $R[i]$-th register into the j-th register
  - $R[R[i]] := R[j]$
    loads the content of the j-th into the $R[i]$-th register

Random Access Machine (RAM)

Operations

- branching (including loops) based on comparisons
  - `jump` x
    jumps to position x in the program;
    sets instruction counter to x;
    reads the next operation to perform from register $R[x]$
  - `jumpz` x $R[i]$
    jump to x if $R[i] = 0$
    if not the instruction counter is increased by 1;
  - `jumpi` i
    jump to $R[i]$ (indirect jump);
- arithmetic instructions: $+$, $-$, $\times$, $/$
  - $R[i]$ `:=` $R[j]$ `+` $R[k]$`;`
    $R[i]$ `:= -`$R[k]$`;`

The jump-directives are very close to the jump-instructions contained in the assembler language of real machines.

Model of Computation

- uniform cost model
  Every operation takes time 1.
- logarithmic cost model
  The cost depends on the content of memory cells:
  - The time for a step is equal to the largest operand involved;
  - The storage space of a register is equal to the length (in bits) of the largest value ever stored in it.

Bounded word RAM model: cost is uniform but the largest value stored in a register may not exceed 2^w, where usually $w = \log_2 n$.

The latter model is quite realistic as the word-size of a standard computer that handles a problem of size n must be at least $\log_2 n$ as otherwise the computer could either not store the problem instance or not address all its memory.

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Example 2

Algorithm 1 RepeatedSquaring(n)

```
1: $r \leftarrow 2$;
2: for $i = 1 \rightarrow n$ do
3:     $r \leftarrow r^2$
4: return $r$
```

- running time (for Line 3):
  - uniform model: n steps
  - logarithmic model:
    $2 + 3 + 5 + \cdots + (1 + 2^n) = 2^{n+1} - 1 + n = \Theta(2^n)$
- space requirement:
  - uniform model: $\mathcal{O}(1)$
  - logarithmic model: $\mathcal{O}(2^n)$

There are different types of complexity bounds:

- best-case complexity:

$$C_{\text{bc}}(n) := \min\{C(x) \mid |x| = n\}$$

  Usually easy to analyze, but not very meaningful.

- worst-case complexity:

$$C_{\text{wc}}(n) := \max\{C(x) \mid |x| = n\}$$

  Usually moderately easy to analyze; sometimes too pessimistic.

- average case complexity:

$$C_{\text{avg}}(n) := \frac{1}{|I_n|} \sum_{|x|=n} C(x)$$

  more general: probability measure μ

$$C_{\text{avg}}(n) := \sum_{x \in I_n} \mu(x) \cdot C(x)$$

μ is a probability distribution over inputs of length n.

$C(x)$	cost of instance x
$\lvert x\rvert$	input length of instance x
I_n	set of instances of length n

There are different types of complexity bounds:

- amortized complexity:
  The average cost of data structure operations over a worst case sequence of operations.
- randomized complexity:
  The algorithm may use random bits. Expected running time (over all possible choices of random bits) for a fixed input x. Then take the worst-case over all x with $|x| = n$.

$C(x)$	cost of instance x
$\lvert x\rvert$	input length of instance x
I_n	set of instances of length n

μ is a probability distribution over inputs of length n.

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Bibliography

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[CLRS90] Thomas H. Cormen, Charles E. Leiserson, Ron L. Rivest, Clifford Stein:
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Chapter 2.1 and 2.2 of [MS08] and Chapter 2 of [CLRS90] are relevant for this section.