

# Mincost Flow

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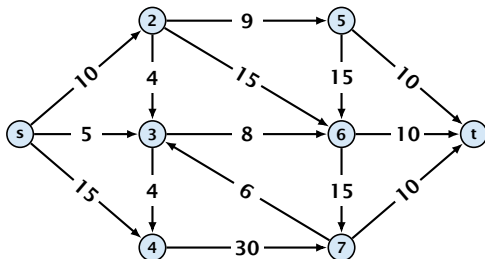
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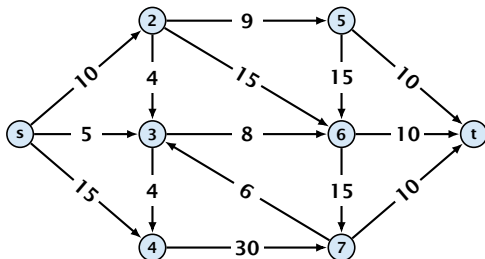
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(note that  $c(e)$  may be negative).
- ▶  $b : V \rightarrow \mathbb{R}, \sum_{v \in V} b(v) = 0$  is a **demand function**.

# Solve Maxflow Using Mincost Flow

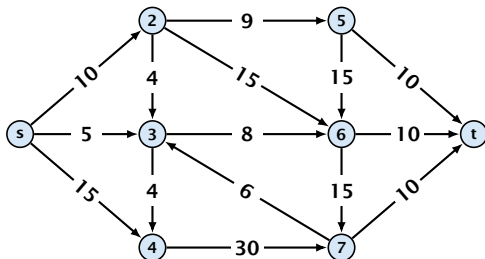


# Solve Maxflow Using Mincost Flow



- ▶ Given a flow network for a standard maxflow problem.

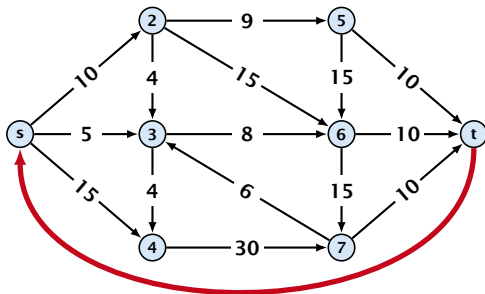
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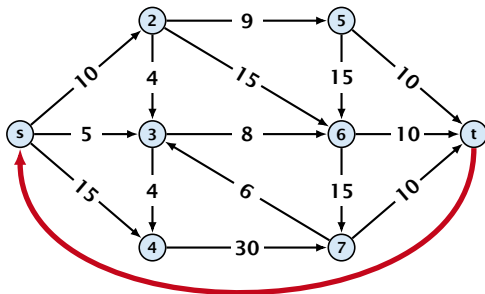


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- ▶ Add an edge from  $t$  to  $s$  with infinite capacity and cost  $-1$ .
- ▶ Then,  $\text{val}(f^*) = -\text{cost}(f_{\min})$ , where  $f^*$  is a maxflow, and  $f_{\min}$  is a mincost-flow.

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- ▶ Set edge-costs to zero, and keep the capacities.
- ▶ There exists a maxflow of value at least  $k$  if and only if the mincost-flow problem is feasible.

# Generalization

**Our model:**

$$\begin{aligned} \min \quad & \sum_e c(e) f(e) \\ \text{s.t.} \quad & \forall e \in E: 0 \leq f(e) \leq u(e) \\ & \forall v \in V: f(v) = b(v) \end{aligned}$$

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**A more general model?**

$$\begin{aligned} \min \quad & \sum_e c(e)f(e) \\ \text{s.t.} \quad & \forall e \in E: \ell(e) \leq f(e) \leq u(e) \\ & \forall v \in V: a(v) \leq f(v) \leq b(v) \end{aligned}$$

where  $a: V \rightarrow \mathbb{R}$ ,  $b: V \rightarrow \mathbb{R}$ ;  $\ell: E \rightarrow \mathbb{R} \cup \{-\infty\}$ ,  $u: E \rightarrow \mathbb{R} \cup \{\infty\}$   
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## Differences

- ▶ Flow along an edge  $e$  may have non-zero lower bound  $\ell(e)$ .
- ▶ Flow along  $e$  may have negative upper bound  $u(e)$ .
- ▶ The demand at a node  $v$  may have lower bound  $a(v)$  and upper bound  $b(v)$  instead of just lower bound = upper bound =  $b(v)$ .

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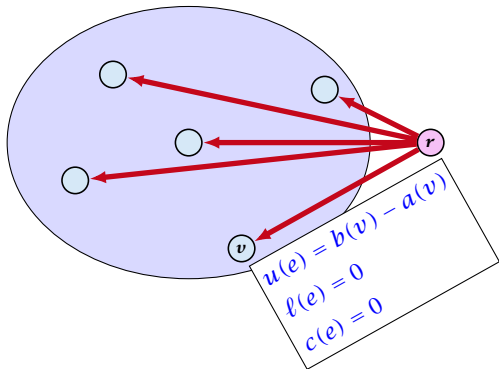
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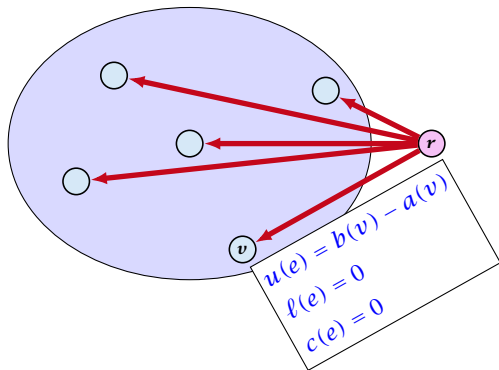


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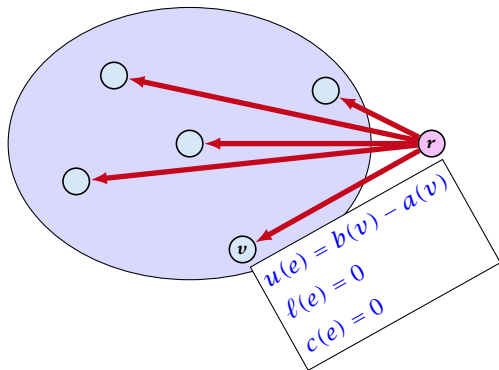
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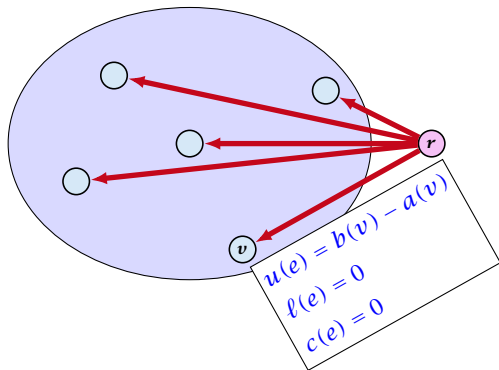
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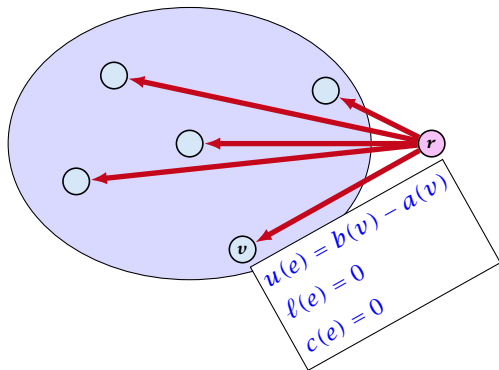
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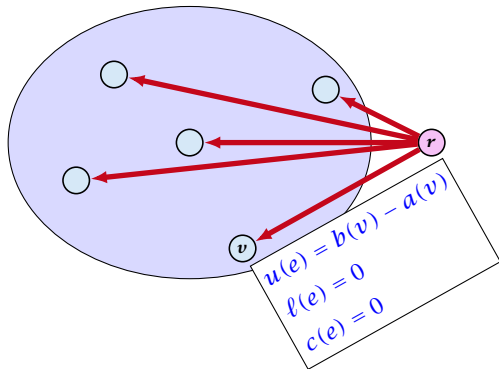
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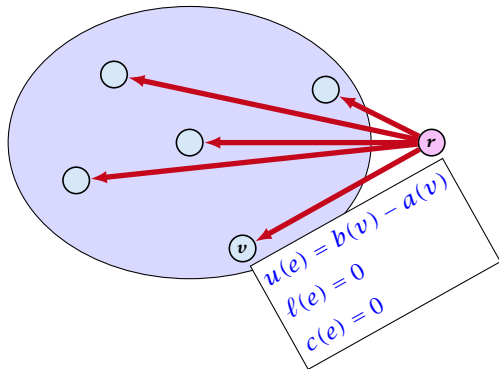
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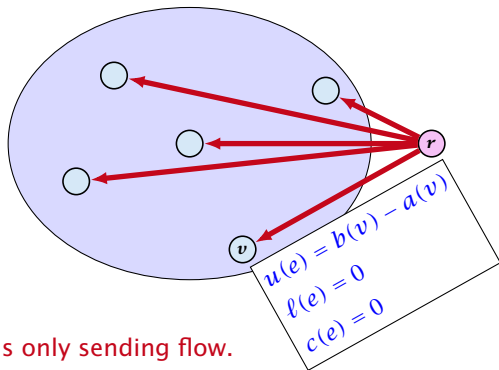
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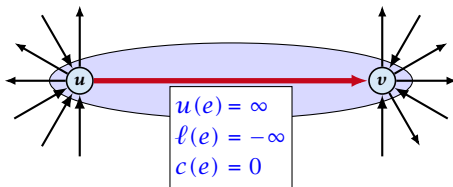
$-\sum_v b(v)$  is negative; hence  $r$  is only sending flow.



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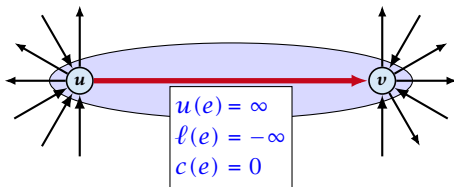
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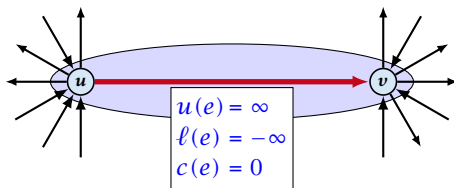


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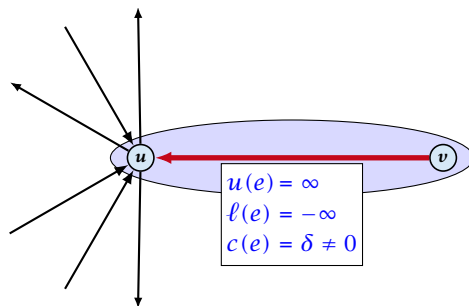


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If  $c(e) \neq 0$  we can transform the graph so that  $c(e) = 0$ .

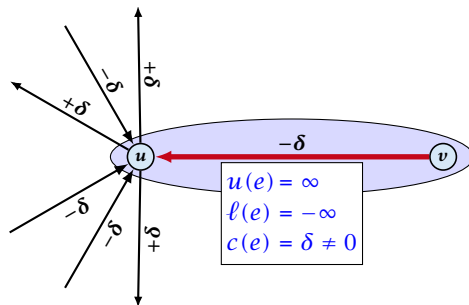
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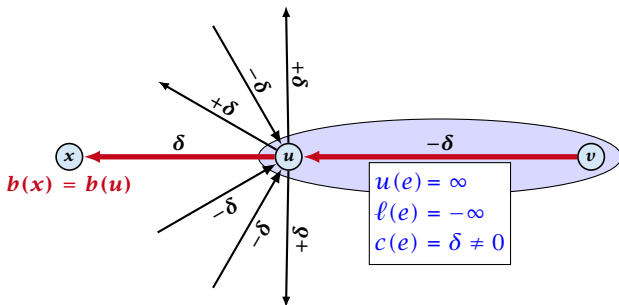
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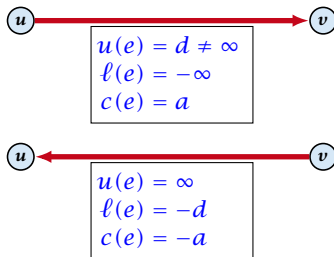


Additionally we set  $b(u) = 0$ .

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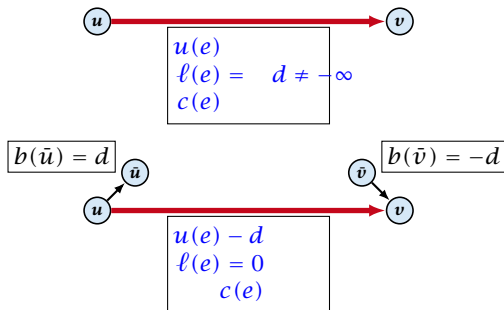


Replace the edge by an edge in opposite direction.

## Reduction IV

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We can assume that  $\ell(e) = 0$ :



The added edges have infinite capacity and cost  $c(e)/2$ .

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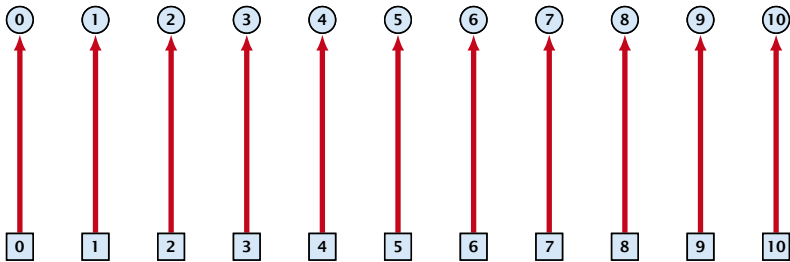
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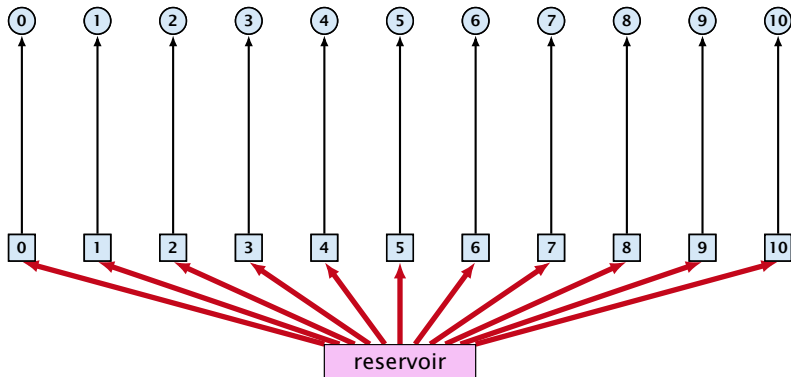
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- ▶ Minimize cost.





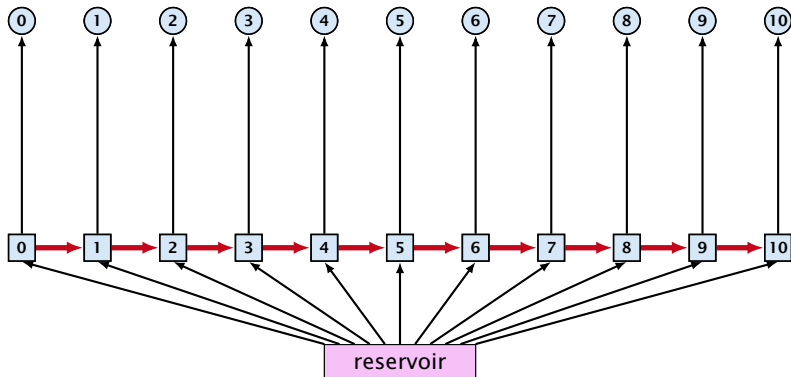
day edges:

upper bound:  $u(e_i) = \infty$ ;  
lower bound:  $\ell(e_i) = r_i$ ;  
cost:  $c(e) = 0$



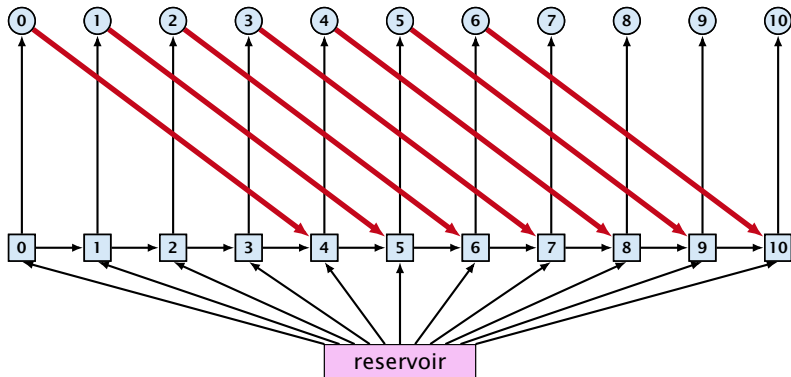
buy edges:

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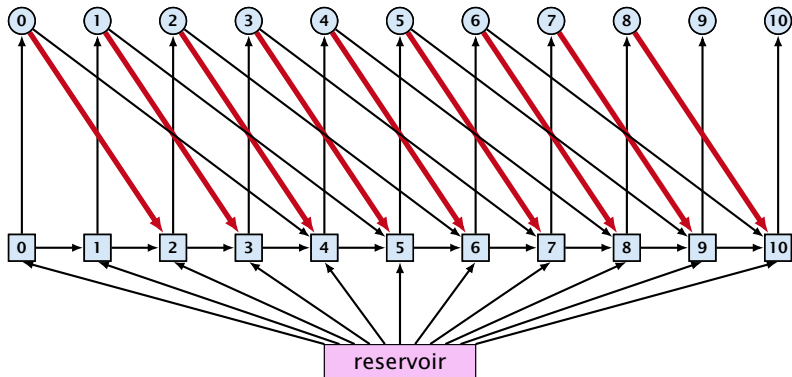
forward edges:

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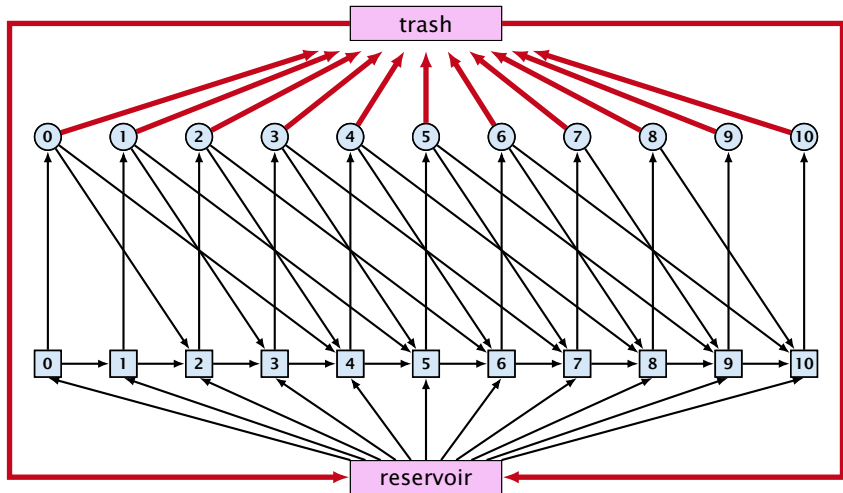
slow edges:

upper bound:  $u(e_i) = \infty$ ;  
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 cost:  $c(e) = s$



fast edges:

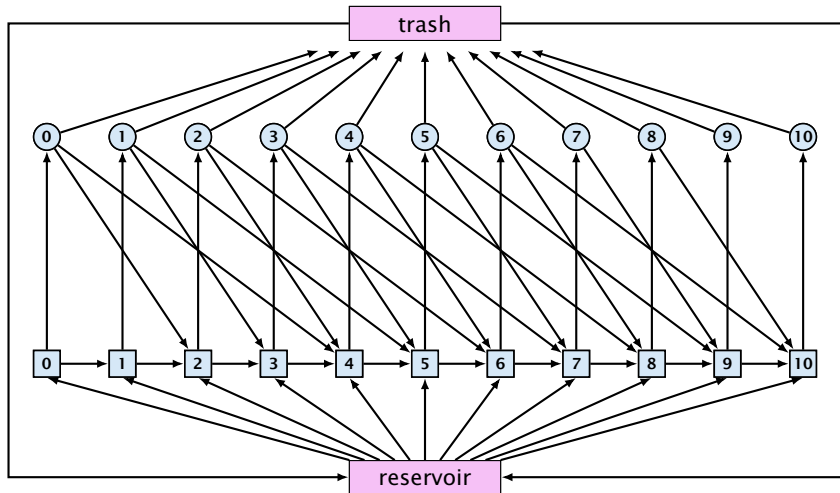
upper bound:  $u(e_i) = \infty$ ;  
 lower bound:  $\ell(e_i) = 0$ ;  
 cost:  $c(e) = f$



trash edges:

upper bound:  $u(e_i) = \infty$ ;  
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 cost:  $c(e) = 0$





# Residual Graph

## Version A:

The residual graph  $G'$  for a mincost flow is just a copy of the graph  $G$ .

If we send  $f(e)$  along an edge, the corresponding edge  $e'$  in the residual graph has its lower and upper bound changed to  $l(e') = l(e) - f(e)$  and  $u(e') = u(e) - f(e)$ .

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## Version B:

The residual graph for a mincost flow is exactly defined as the residual graph for standard flows, with the only exception that one needs to define a cost for the residual edge.

For a flow of  $z$  from  $u$  to  $v$  the residual edge  $(v, u)$  has capacity  $z$  and a cost of  $-c((u, v))$ .

# 14 Mincost Flow

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A circulation is **feasible** if it fulfills capacity constraints, i.e.,  $f(e) \leq u(e)$  for every edge of  $G$ .

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Clearly  $f^* - f$  is a circulation of negative cost. One can also easily see that it is feasible for the residual graph. (after sending  $-f$  in the residual graph (pushing all flow back) we arrive at the original graph; for this  $f^*$  is clearly feasible)

## 14 Mincost Flow

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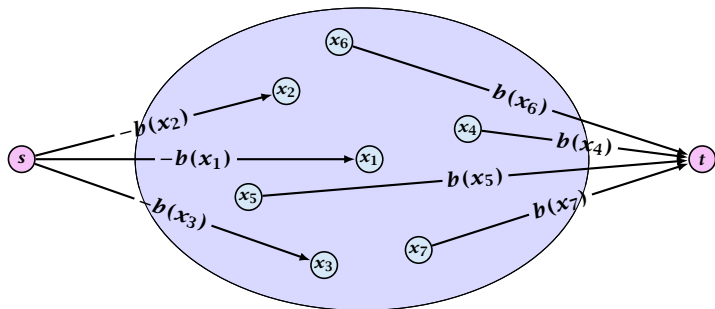
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- ▶ You still have a circulation with negative cost.
- ▶ Repeat.

## 14 Mincost Flow

### Algorithm 48 CycleCanceling( $G = (V, E), c, u, b$ )

- 1: establish a feasible flow  $f$  in  $G$
- 2: **while**  $G_f$  contains negative cycle **do**
- 3:     use Bellman-Ford to find a negative circuit  $Z$
- 4:      $\delta \leftarrow \min\{u_f(e) \mid e \in Z\}$
- 5:     augment  $\delta$  units along  $Z$  and update  $G_f$

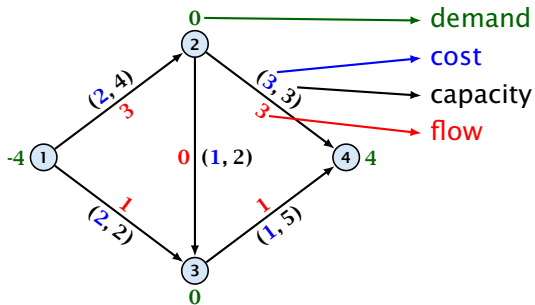
## How do we find the initial feasible flow?



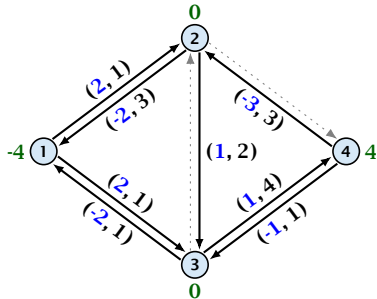
- ▶ Connect new node  $s$  to all nodes with negative  $b(v)$ -value.
- ▶ Connect nodes with positive  $b(v)$ -value to a new node  $t$ .
- ▶ There exist a feasible flow in the original graph iff in the resulting graph there exists an  $s$ - $t$  flow of value

$$\sum_{v:b(v)<0} (-b(v)) = \sum_{v:b(v)>0} b(v) .$$

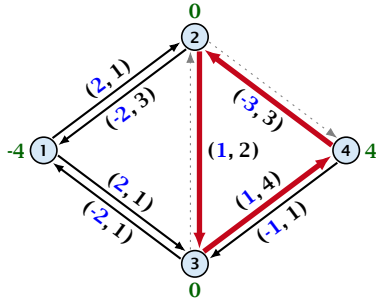
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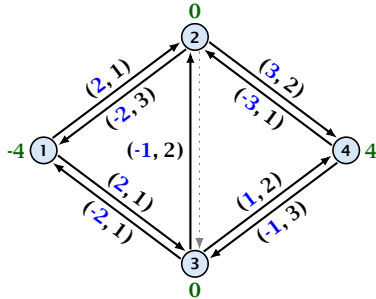
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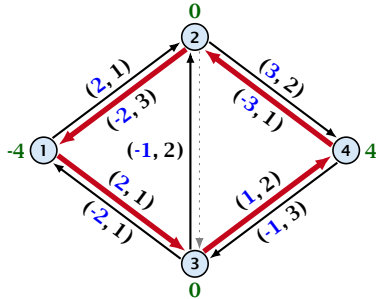
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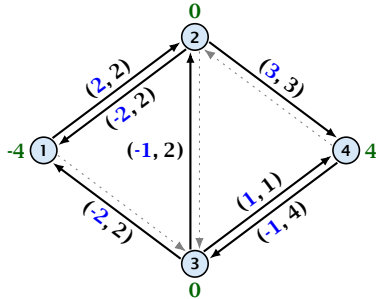


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## Lemma 8

*The improving cycle algorithm runs in time  $\mathcal{O}(nm^2CU)$ , for integer capacities and costs, when for all edges  $e$ ,  $|c(e)| \leq C$  and  $|u(e)| \leq U$ .*

- ▶ Running time of Bellman-Ford is  $\mathcal{O}(mn)$ .
- ▶ Pushing flow along the cycle can be done in time  $\mathcal{O}(n)$ .
- ▶ Each iteration decreases the total cost by at least 1.
- ▶ The true optimum cost must lie in the interval  $[-mCU, \dots, +mCU]$ .

Note that this lemma is weak since it does not allow for edges with infinite capacity.

# 14 Mincost Flow

A **general mincost flow problem** is of the following form:

$$\begin{aligned} \min \quad & \sum_e c(e)f(e) \\ \text{s.t.} \quad & \forall e \in E: \ell(e) \leq f(e) \leq u(e) \\ & \forall v \in V: a(v) \leq f(v) \leq b(v) \end{aligned}$$

where  $a: V \rightarrow \mathbb{R}$ ,  $b: V \rightarrow \mathbb{R}$ ;  $\ell: E \rightarrow \mathbb{R} \cup \{-\infty\}$ ,  $u: E \rightarrow \mathbb{R} \cup \{\infty\}$   
 $c: E \rightarrow \mathbb{R}$ ;

## Lemma 9 (without proof)

*A general mincost flow problem can be solved in polynomial time.*