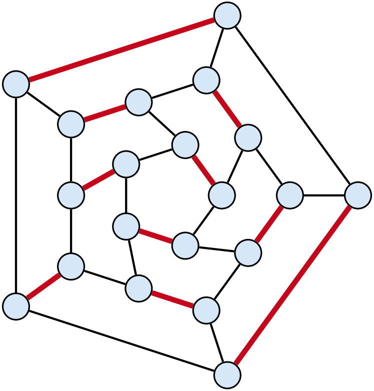
Part V

Matchings

Matching

- Input: undirected graph $G = (V, E)$.
- $M \subseteq E$ is a matching if each node appears in at most one edge in M.
- Maximum Matching: find a matching of maximum cardinality



16 Bipartite Matching via Flows

Which flow algorithm to use?

- Generic augmenting path: $\mathcal{O}(m\,\mathrm{val}(f^*)) = \mathcal{O}(mn)$.
- Capacity scaling: $\mathcal{O}(m^2 \log C) = \mathcal{O}(m^2)$.
- Shortest augmenting path: $\mathcal{O}(mn^2)$.

For unit capacity simple graphs shortest augmenting path can be implemented in time $\mathcal{O}(m\sqrt{n})$.

17 Augmenting Paths for Matchings

Definitions.

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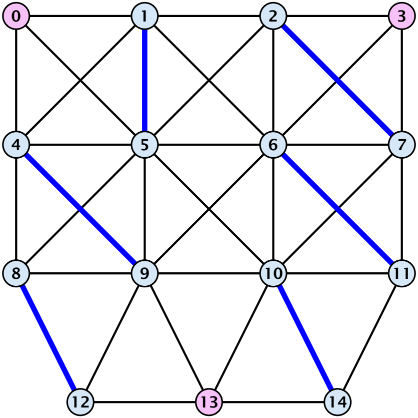
Theorem 6

A matching M is a maximum matching if and only if there is no augmenting path w. r. t. M.

Augmenting Paths in Action



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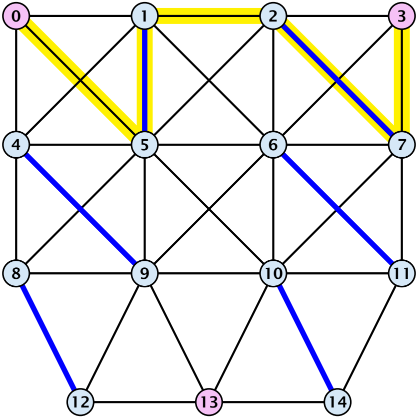

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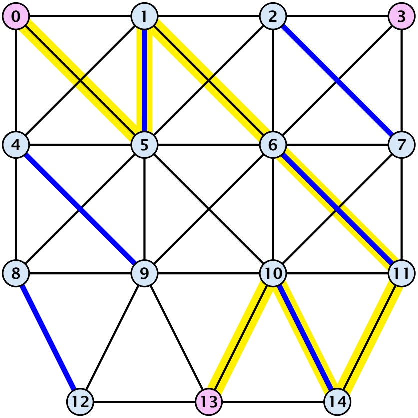

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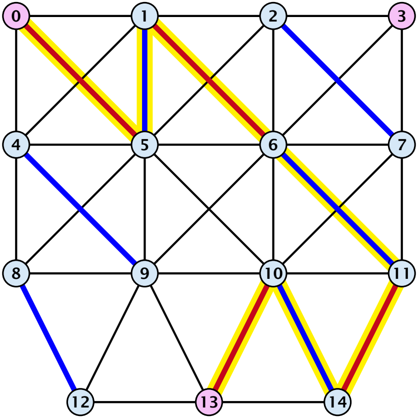

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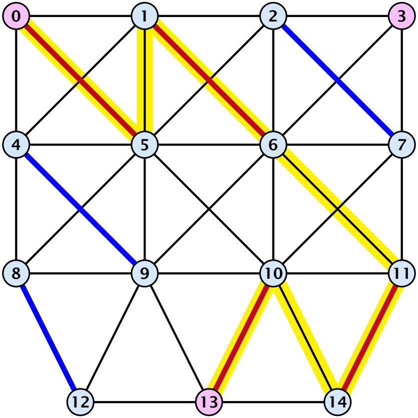

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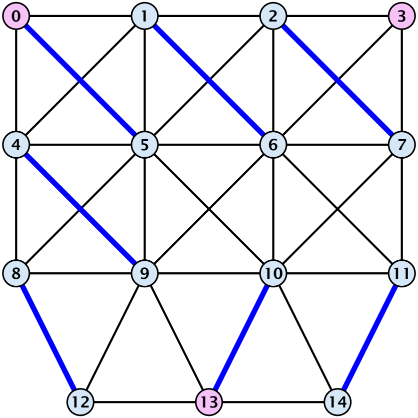

Proof.

- $\Rightarrow$ If M is maximum there is no augmenting path P, because we could switch matching and non-matching edges along P. This gives matching $M' = M \oplus P$ with larger cardinality.

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  Each vertex can be incident to at most two edges (one from M and one from M'). Hence, the connected components are alternating cycles or alternating path.

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  As $|M'| > |M|$ there is one connected component that is a path P for which both endpoints are incident to edges from M'. P is an alternating path.

17 Augmenting Paths for Matchings

Algorithmic idea:

As long as you find an augmenting path augment your matching using this path. When you arrive at a matching for which no augmenting path exists you have a maximum matching.

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As long as you find an augmenting path augment your matching using this path. When you arrive at a matching for which no augmenting path exists you have a maximum matching.

Theorem 7
Let G be a graph, M a matching in G, and let u be a free vertex w.r.t. M. Further let P denote an augmenting path w.r.t. M and let $M' = M \oplus P$ denote the matching resulting from augmenting M with P. If there was no augmenting path starting at u in M then there is no augmenting path starting at u in M'.

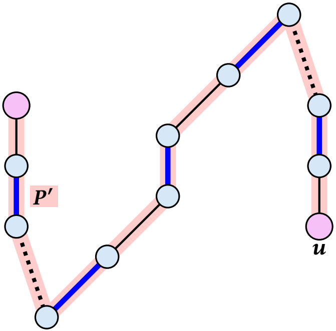
The above theorem allows for an easier implementation of an augmenting path algorithm. Once we checked for augmenting paths starting from u we don't have to check for such paths in future rounds.

17 Augmenting Paths for Matchings

Proof

Proof

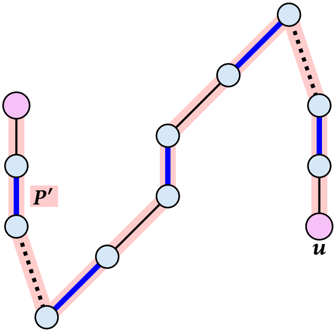
- Assume there is an augmenting path P' w.r.t. M' starting at u.



17 Augmenting Paths for Matchings

Proof

- Assume there is an augmenting path P' w.r.t. M' starting at u.
- If P' and P are node-disjoint, P' is also augmenting path w.r.t. M ($\lightning$).



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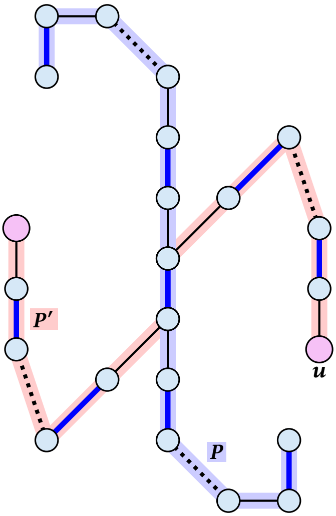

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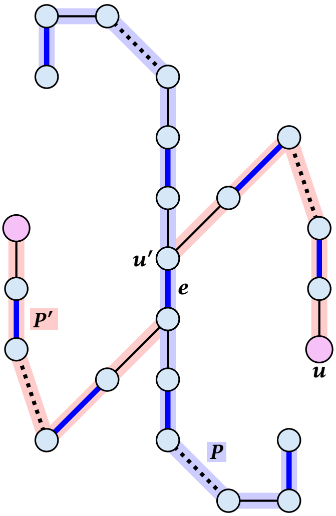
- Let u' be the first node on P' that is in P, and let e be the matching edge from M' incident to u'.



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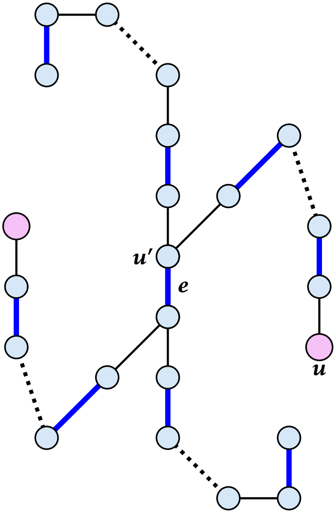

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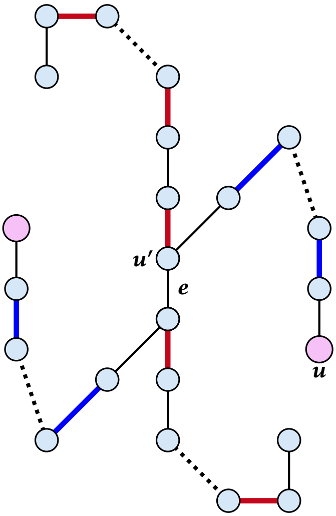

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- Let u' be the first node on P' that is in P, and let e be the matching edge from M' incident to u'.
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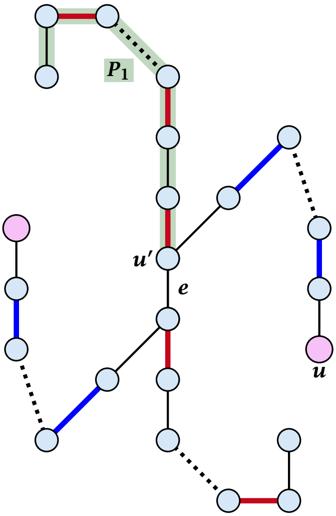

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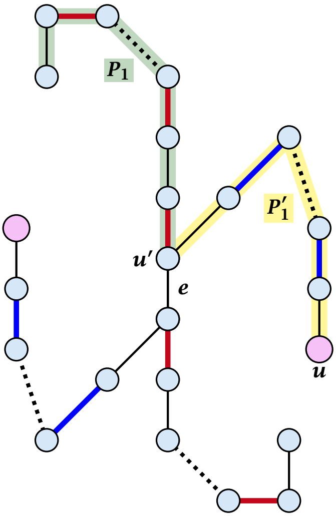

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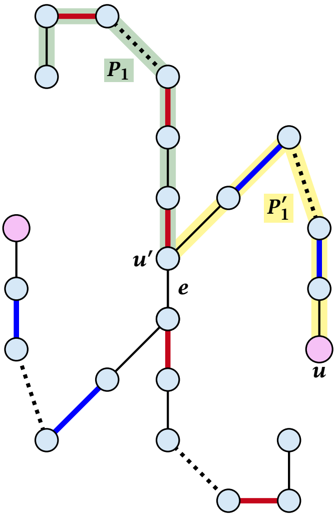
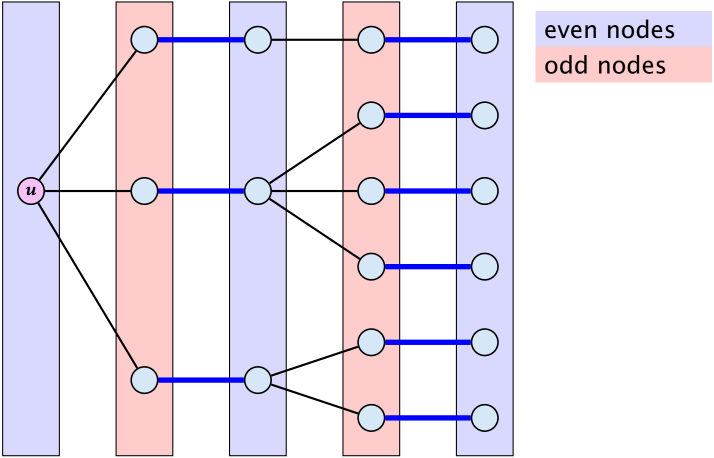
- Let u' be the first node on P' that is in P, and let e be the matching edge from M' incident to u'.
- u' splits P into two parts one of which does not contain e. Call this part P_1. Denote the sub-path of P' from u to u' with P_1'.
- $P_1 \circ P_1'$ is augmenting path in M ($\lightning$).



How to find an augmenting path?

Construct an alternating tree.



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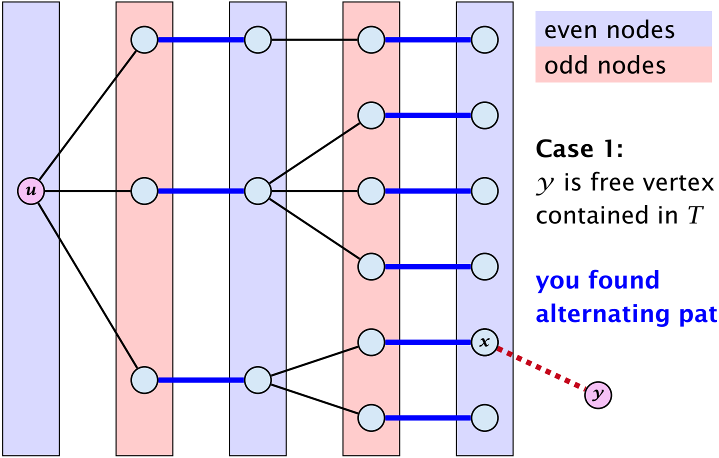

Case 1:
y is free vertex not contained in T

you found alternating path

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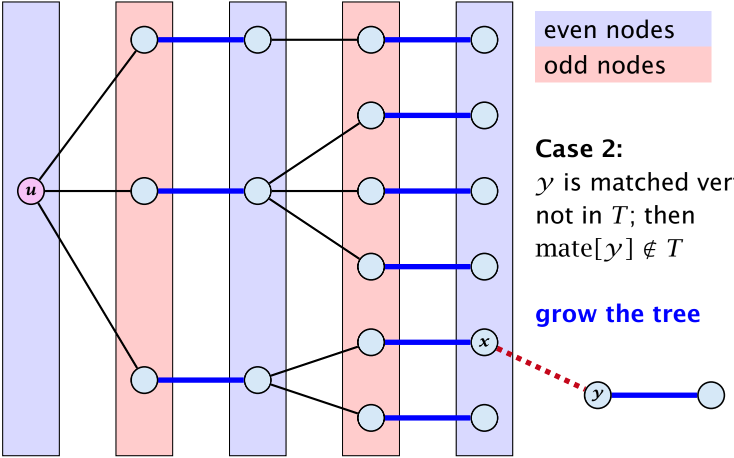
Construct an alternating tree.

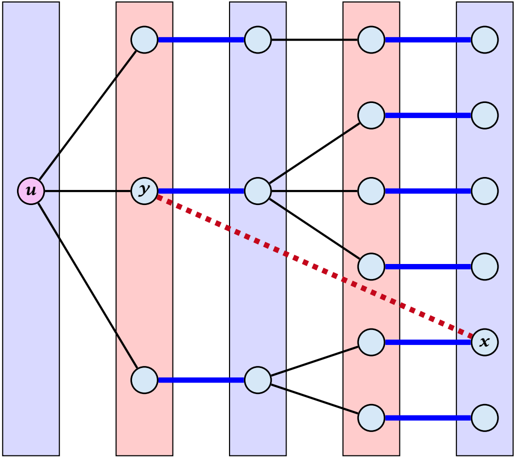


Case 2:
y is matched vertex not in T; then $\text{mate}[y] \notin T$

grow the tree

How to find an augmenting path?

Construct an alternating tree.



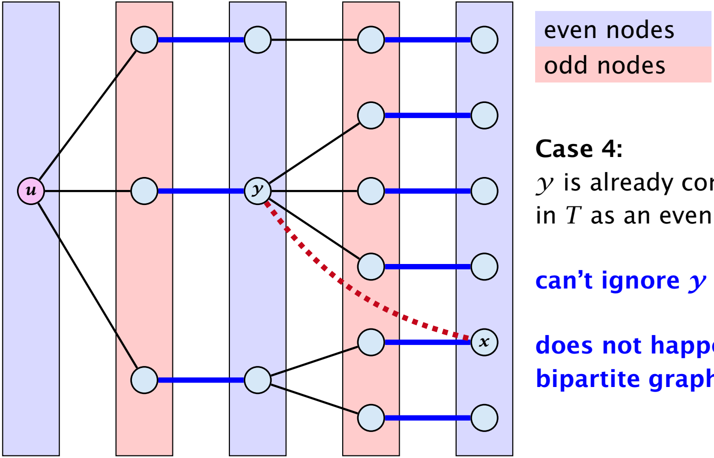
even nodes
odd nodes

Case 3:
y is already contained in T as an odd vertex

ignore successor y

How to find an augmenting path?

Construct an alternating tree.



Case 4:
y is already contained in T as an even vertex

can't ignore y

does not happen in bipartite graphs

Algorithm 49 BiMatch(G, $match$)

```
for x ∈ V do mate[x] ← 0;
r ← 0; free ← n;
while free ≥ 1 and r < n  do
    r ← r + 1
    if mate[r] = 0  then
        for i = 1 to n do parent[i′] ← 0
        Q ← ∅; Q.append(r); aug ← false;
        while aug = false and Q ≠ ∅  do
            x ← Q.dequeue();
            for y ∈ A_x do
                if mate[y] = 0  then
                    augm(mate, parent, y);
                    aug ← true;
                    free ← free − 1;
                else
                    if parent[y] = 0  then
                        parent[y] ← x;
                        Q.enqueue(mate[y]);
```

graph $G = (S \cup S', E)$

$$S = \{1, \ldots, n\}$$

$$S' = \{1', \ldots, n'\}$$

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```

start with an empty matching

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free: number of unmatched nodes in S

r: root of current tree

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```

as long as there are unmatched nodes and we did not yet try to grow from all nodes we continue

Algorithm 49 BiMatch(G, $match$)

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r is the new node that we grow from.

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                    if parent[y] = 0 then
                        parent[y] ← x;
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```

If r is free start tree construction

Algorithm 49 BiMatch(G, $match$)

for $x \in V$ **do** $mate[x] \leftarrow 0$;
$r \leftarrow 0$; $free \leftarrow n$;
while $free \geq 1$ **and** $r < n$ **do**
    $r \leftarrow r + 1$
    if $mate[r] = 0$ **then**
        for $i = 1$ **to** n **do** $parent[i'] \leftarrow 0$
        $Q \leftarrow \emptyset$; $Q.\text{append}(r)$; $aug \leftarrow \text{false}$;
        while $aug = \text{false}$ **and** $Q \neq \emptyset$ **do**
            $x \leftarrow Q.\text{dequeue}()$;
            for $y \in A_x$ **do**
                if $mate[y] = 0$ **then**
                    augm($mate, parent, y$);
                    $aug \leftarrow \text{true}$;
                    $free \leftarrow free - 1$;
                else
                    if $parent[y] = 0$ **then**
                        $parent[y] \leftarrow x$;
                        $Q.\text{enqueue}(mate[y])$;

Initialize an empty tree. Note that only nodes i' have parent pointers.

Algorithm 49 BiMatch$(G, match)$

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                    if parent[y] = 0 then
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Q is a queue (BFS!!!).

aug is a Boolean that stores whether we already found an augmenting path.

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as long as we did not augment and there are still unexamined leaves continue...

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        $Q \leftarrow \emptyset$; $Q.\mathrm{append}(r)$; $aug \leftarrow \mathrm{false}$;
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                if $mate[y] = 0$ **then**
                    $\mathrm{augm}(mate, parent, y)$;
                    $aug \leftarrow \mathrm{true}$;
                    $free \leftarrow free - 1$;
                else
                    if $parent[y] = 0$ **then**
                        $parent[y] \leftarrow x$;
                        $Q.\mathrm{enqueue}(mate[y])$;

take next unexamined leaf

Algorithm 49 BiMatch(G, $match$)

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```

if x has unmatched neighbour we found an augmenting path (note that $y \neq r$ because we are in a bipartite graph)

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do an augmentation...

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setting aug = true ensures that the tree construction will not continue

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            for y ∈ A_x do
                if mate[y] = 0  then
                    augm(mate, parent, y);
                    aug ← true;
                    free ← free − 1;
                else
                    if parent[y] = 0  then
                        parent[y] ← x;
                        Q.enqueue(mate[y]);
```

reduce number of free nodes

Algorithm 49 BiMatch(G, $match$)

```
for x ∈ V do mate[x] ← 0;
r ← 0; free ← n;
while free ≥ 1 and r < n  do
    r ← r + 1
    if mate[r] = 0  then
        for i = 1 to n do parent[i'] ← 0
        Q ← ∅; Q.append(r); aug ← false;
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if y is not in the tree yet

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...put it into the tree

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add its buddy to the set of unexamined leaves

18 Weighted Bipartite Matching

Weighted Bipartite Matching/Assignment

- Input: undirected, bipartite graph $G = L \cup R, E$.
- an edge $e = (\ell, r)$ has weight $w_e \geq 0$
- find a matching of maximum weight, where the weight of a matching is the sum of the weights of its edges

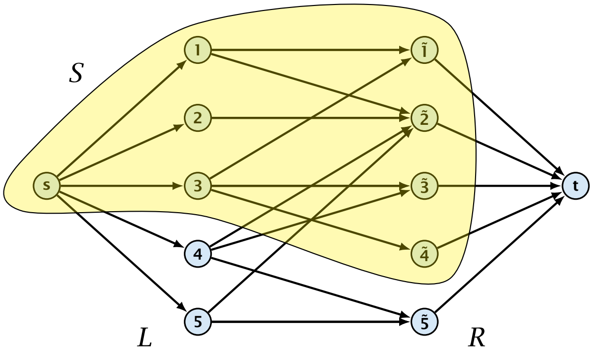
Simplifying Assumptions (wlog [why?]):

- assume that $|L| = |R| = n$
- assume that there is an edge between every pair of nodes $(\ell, r) \in V \times V$
- can assume goal is to construct maximum weight perfect matching

Weighted Bipartite Matching

Theorem 8 (Halls Theorem)
A bipartite graph $G = (L \cup R, E)$ has a perfect matching if and only if for all sets $S \subseteq L$, $|\Gamma(S)| \geq |S|$, where $\Gamma(S)$ denotes the set of nodes in R that have a neighbour in S.

18 Weighted Bipartite Matching



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  - The size of the cut is $|L| - |L_S| + |R_S|$.
  - Using the fact that $|\Gamma(L_S)| \geq L_S$ gives that this is at least $|L|$.

Algorithm Outline

Idea:

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- Try to compute a perfect matching in the subgraph $H(\vec{x})$. If you are successful you found an optimal matching.

Algorithm Outline

Reason:

- The weight of your matching M^* is

$$\sum_{(u,v)\in M^*} w_{(u,v)} = \sum_{(u,v)\in M^*} (x_u + x_v) = \sum_v x_v \enspace .$$

- Any other perfect matching M (in G, not necessarily in $H(\vec{x})$) has

$$\sum_{(u,v)\in M} w_{(u,v)} \le \sum_{(u,v)\in M} (x_u + x_v) = \sum_v x_v \enspace .$$

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What if you don't find a perfect matching?

Then, Halls theorem guarantees you that there is a set $S \subseteq L$, with $|\Gamma(S)| < |S|$, where Γ denotes the neighbourhood w.r.t. the subgraph $H(\vec{x})$.

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- the total weight assigned to nodes decreases
- the weight function still dominates the edge-weights

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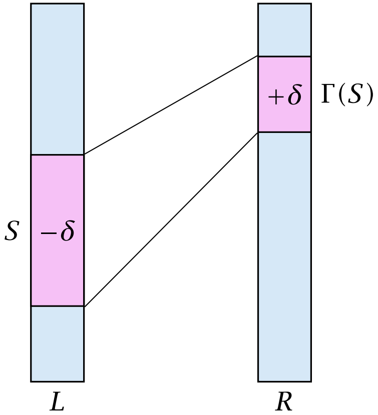
Idea: reweight such that:

- the total weight assigned to nodes decreases
- the weight function still dominates the edge-weights

If we can do this we have an algorithm that terminates with an optimal solution (we analyze the running time later).

Changing Node Weights

Increase node-weights in $\Gamma(S)$ by $+\delta$, and decrease the node-weights in S by $-\delta$.



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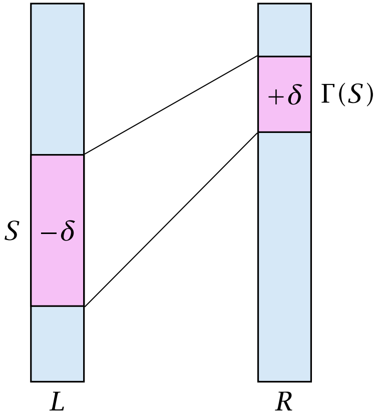

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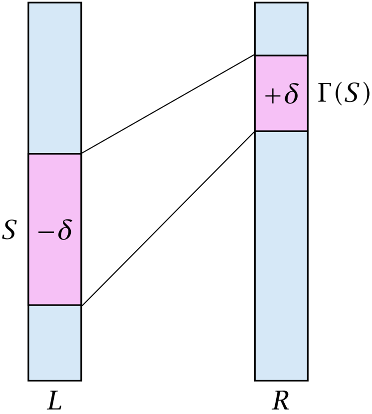

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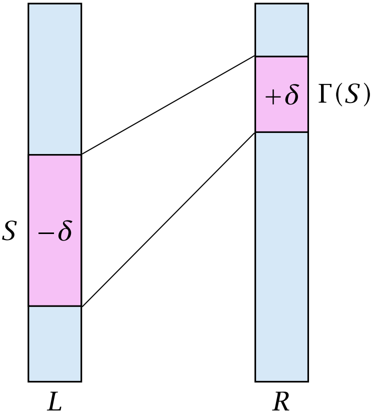
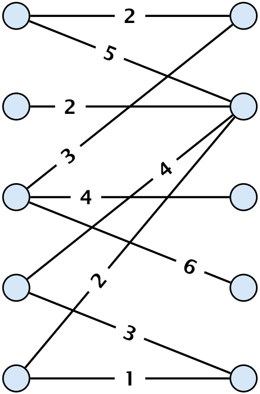
- Total node-weight decreases.
- Only edges from S to $R-\Gamma(S)$ decrease in their weight.
- Since, none of these edges is tight (otw. the edge would be contained in $H(\vec{x})$, and hence would go between S and $\Gamma(S)$) we can do this decrement for small enough $\delta>0$ until a new edge gets tight.



Weighted Bipartite Matching

Edges not drawn have weight 0.



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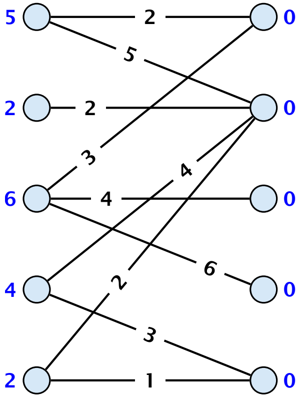

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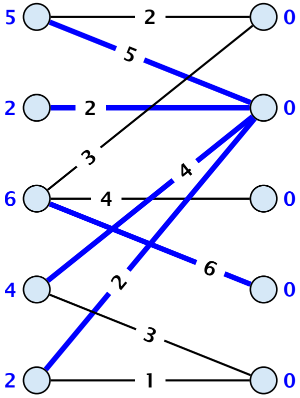

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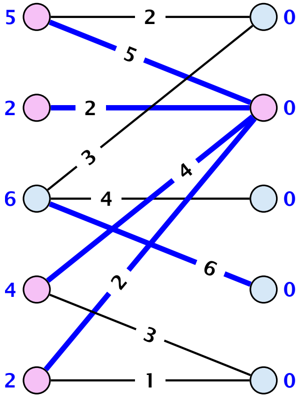

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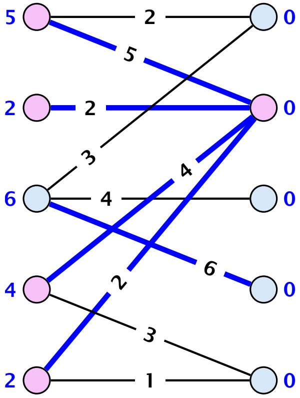

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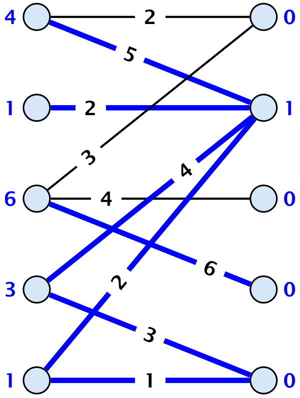

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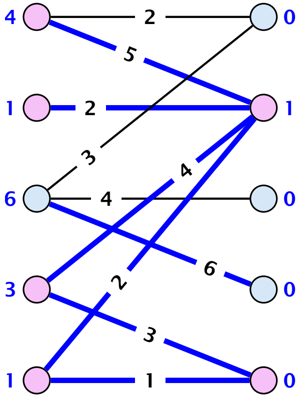

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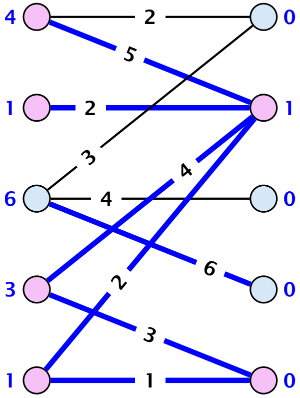

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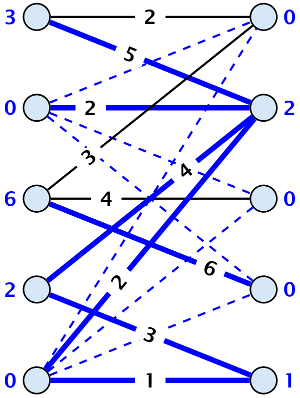

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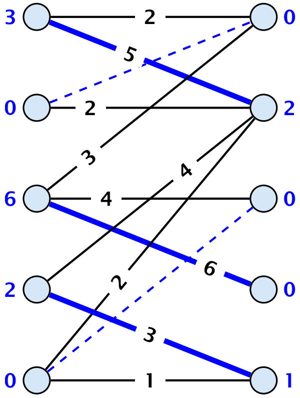

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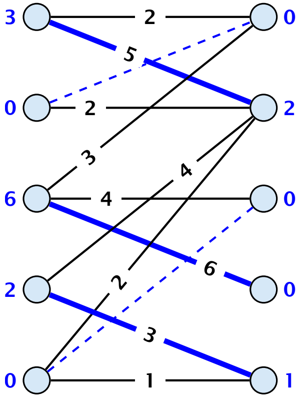

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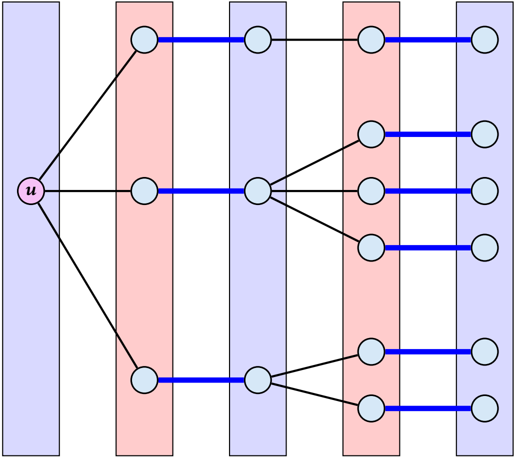
- Hence, reweighting does not decrease the size of a maximum matching in the tight sub-graph.

Analysis

- We will show that after at most n reweighting steps the size of the maximum matching can be increased by finding an augmenting path.
- This gives a polynomial running time.

How to find an augmenting path?

Construct an alternating tree.



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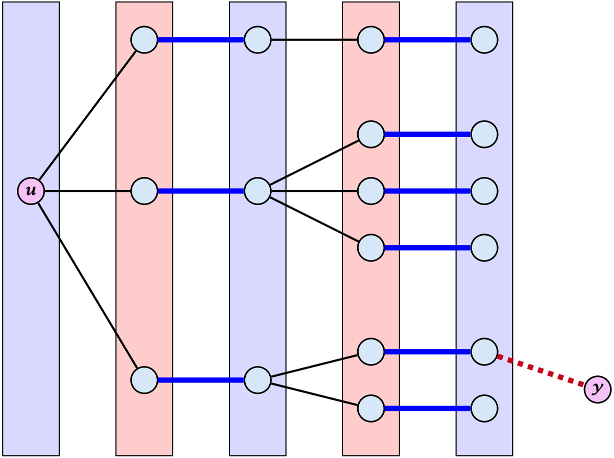

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- All odd vertices are matched to even vertices. Furthermore, the even vertices additionally contain the free vertex u. Hence, $|V_{\text{odd}}| = |\Gamma(V_{\text{even}})| < |V_{\text{even}}|$, and all odd vertices are saturated in the current matching.

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- A more careful implementation of the algorithm obtains a running time of $\mathcal{O}(n^3)$.

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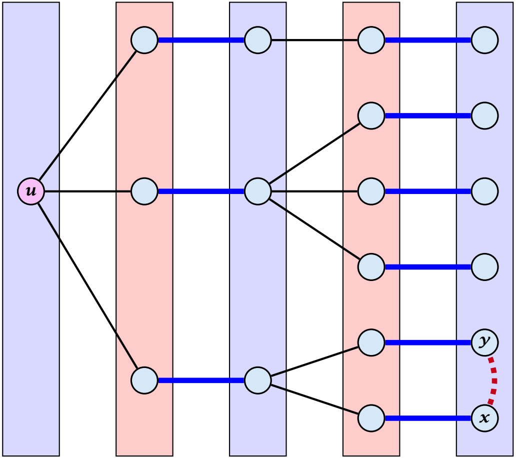

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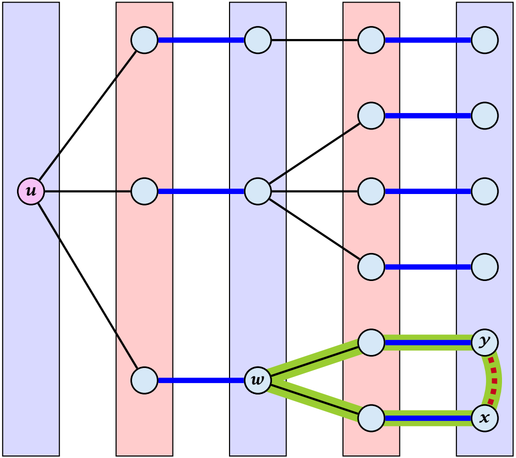

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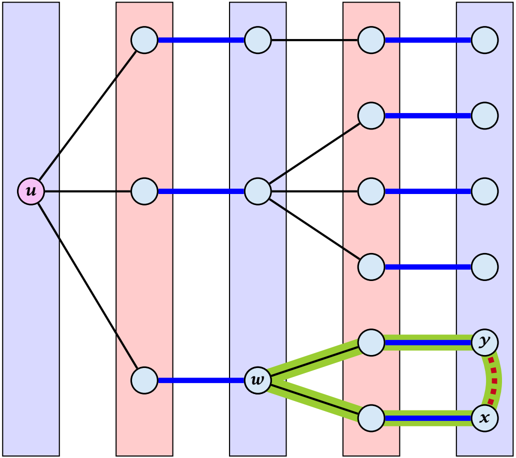

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The cycle $w \leftrightarrow y - x \leftrightarrow w$ is called a blossom.
w is called the base of the blossom (even node!!!).
The path u-w is called the stem of the blossom.

Flowers and Blossoms

Definition 9

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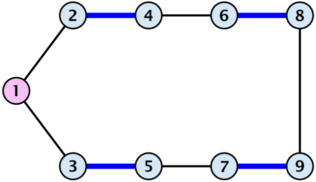
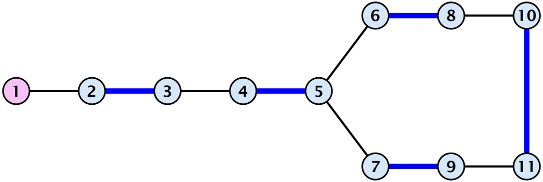
- A blossom is an odd length alternating cycle that starts and terminates at the terminal node w of a stem and has no other node in common with the stem. w is called the base of the blossom.

Flowers and Blossoms





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Properties:

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2. A blossom spans $2k + 1$ nodes and contains k matched edges for some integer $k \geq 1$. The matched edges match all nodes of the blossom except the base.

Flowers and Blossoms

Properties:

1. A stem spans $2\ell + 1$ nodes and contains ℓ matched edges for some integer $\ell \geq 0$.
2. A blossom spans $2k + 1$ nodes and contains k matched edges for some integer $k \geq 1$. The matched edges match all nodes of the blossom except the base.
3. The base of a blossom is an even node (if the stem is part of an alternating tree starting at r).

Flowers and Blossoms

Properties:

4. Every node x in the blossom (except its base) is reachable from the root (or from the base of the blossom) through two distinct alternating paths; one with even and one with odd length.

Flowers and Blossoms

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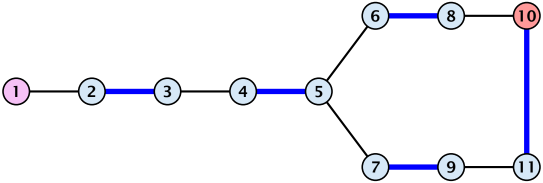
4. Every node x in the blossom (except its base) is reachable from the root (or from the base of the blossom) through two distinct alternating paths; one with even and one with odd length.
5. The even alternating path to x terminates with a matched edge and the odd path with an unmatched edge.

Flowers and Blossoms



Shrinking Blossoms

When during the alternating tree construction we discover a blossom B we replace the graph G by $G' = G/B$, which is obtained from G by contracting the blossom B.

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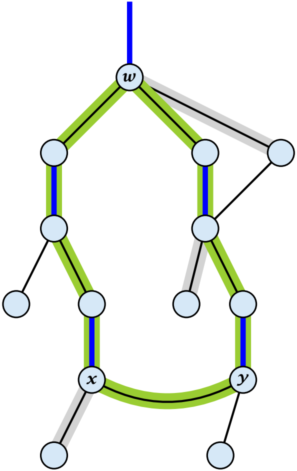
Shrinking Blossoms

When during the alternating tree construction we discover a blossom B we replace the graph G by $G' = G/B$, which is obtained from G by contracting the blossom B.

- Delete all vertices in B (and its incident edges) from G.
- Add a new (pseudo-)vertex b. The new vertex b is connected to all vertices in $V \setminus B$ that had at least one edge to a vertex from B.

Shrinking Blossoms

- Edges of T that connect a node u not in B to a node in B become tree edges in T' connecting u to b.
- Matching edges (there is at most one) that connect a node u not in B to a node in B become matching edges in M'.
- Nodes that are connected in G to at least one node in B become connected to b in G'.



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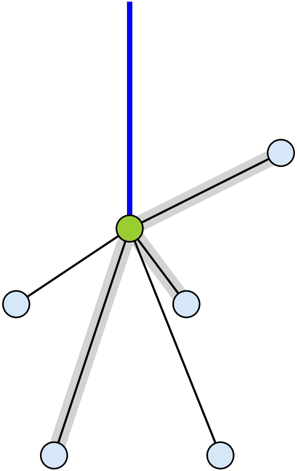
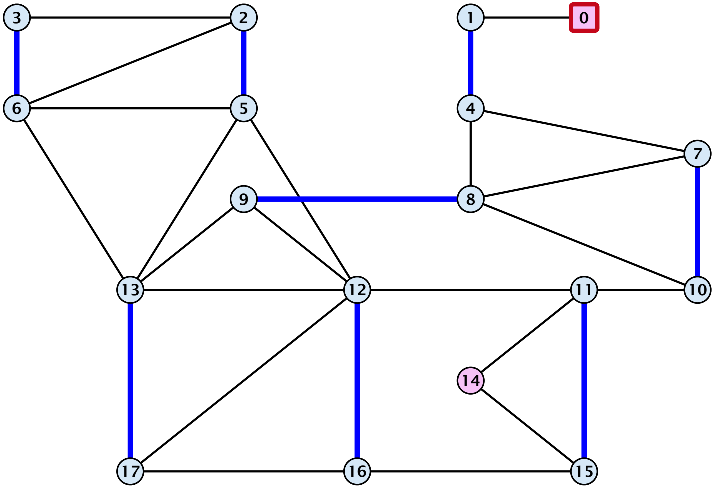


Example: Blossom Algorithm



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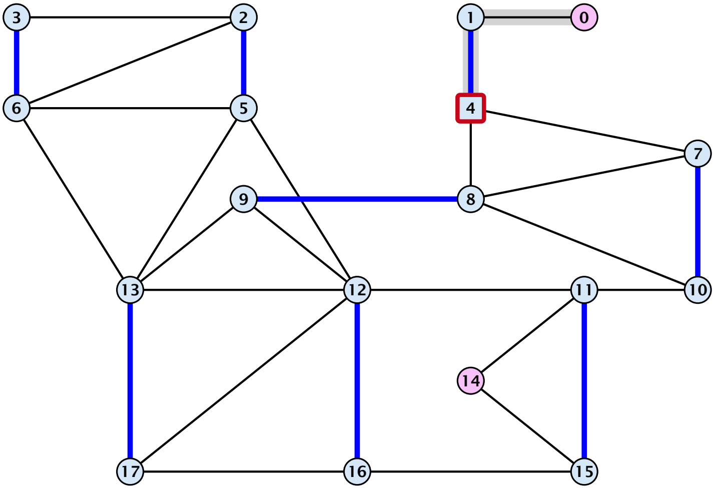

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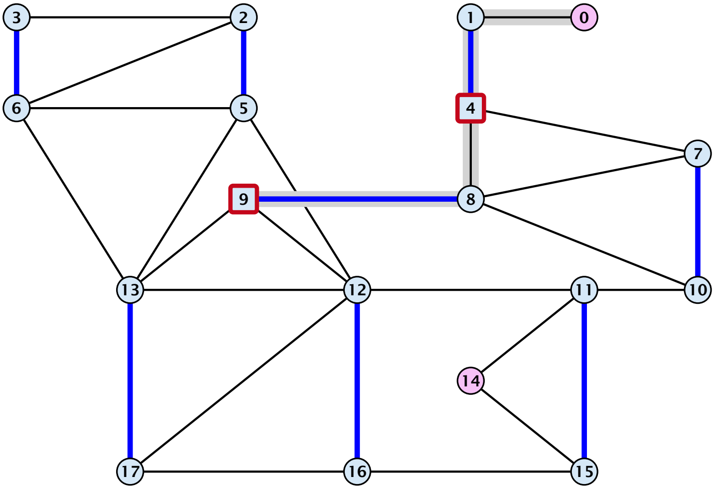

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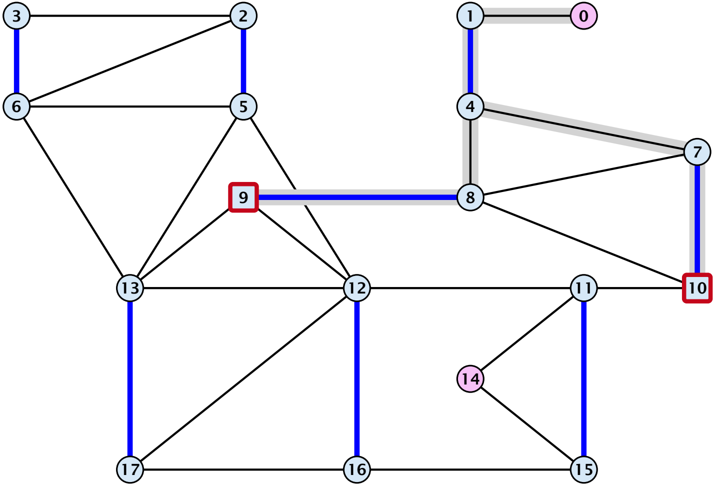

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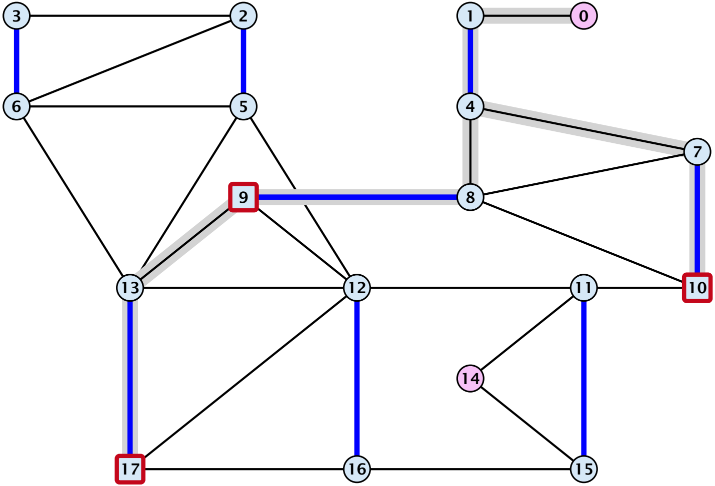

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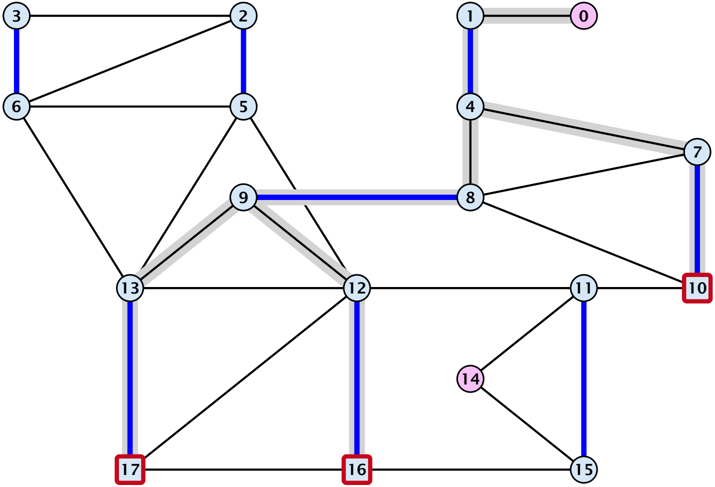

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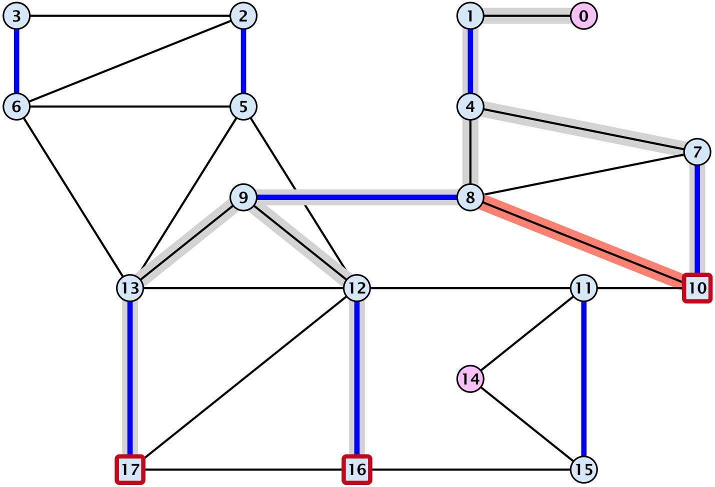

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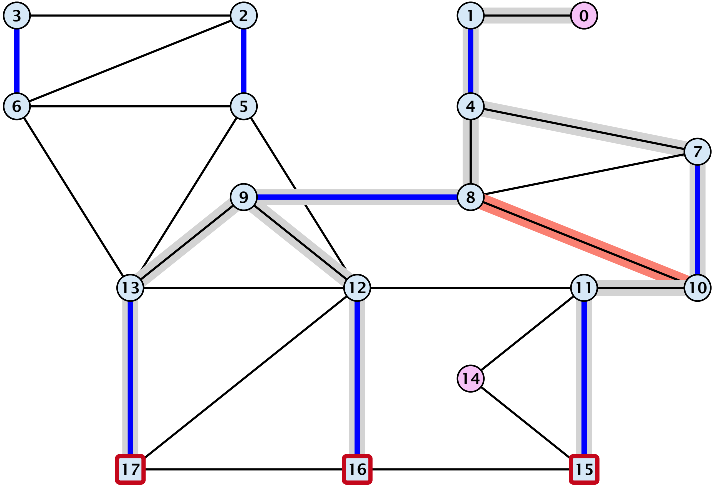

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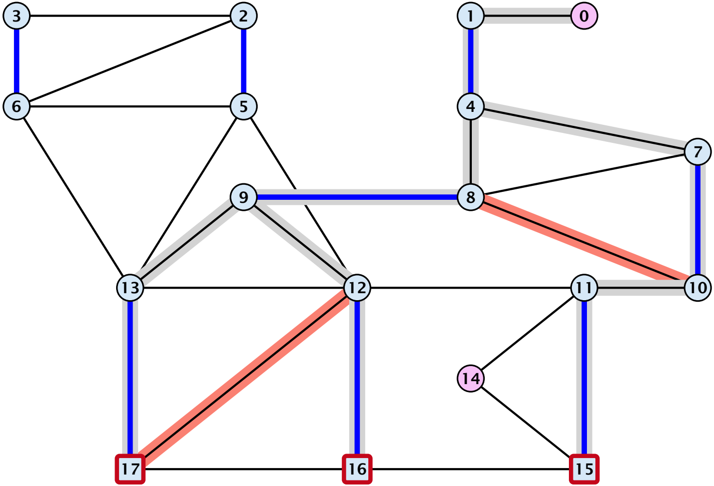

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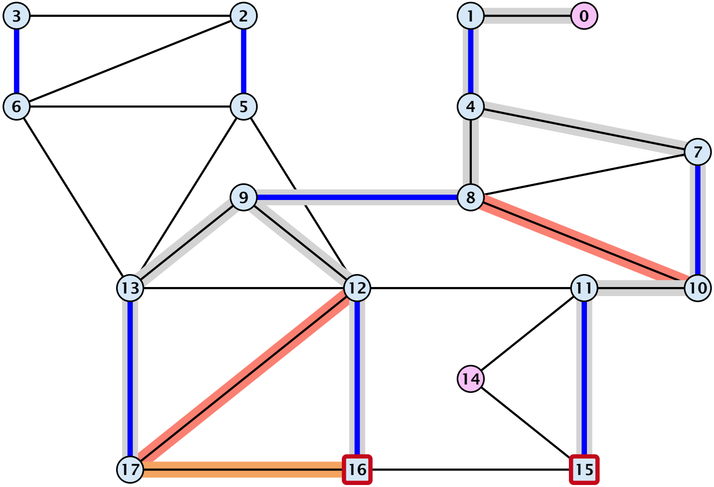

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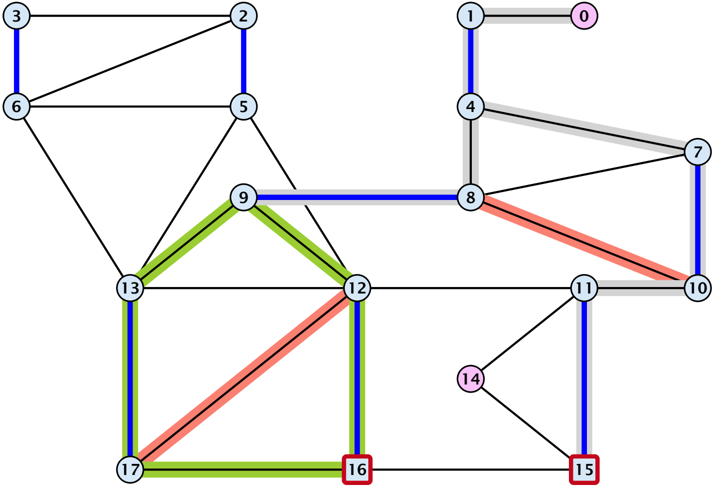

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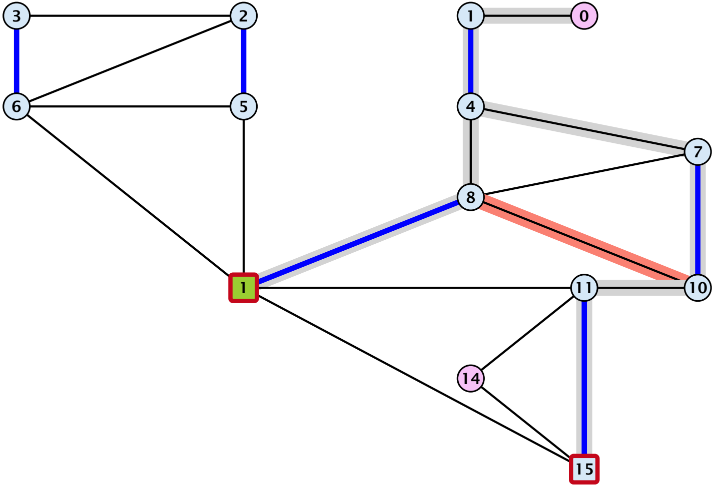

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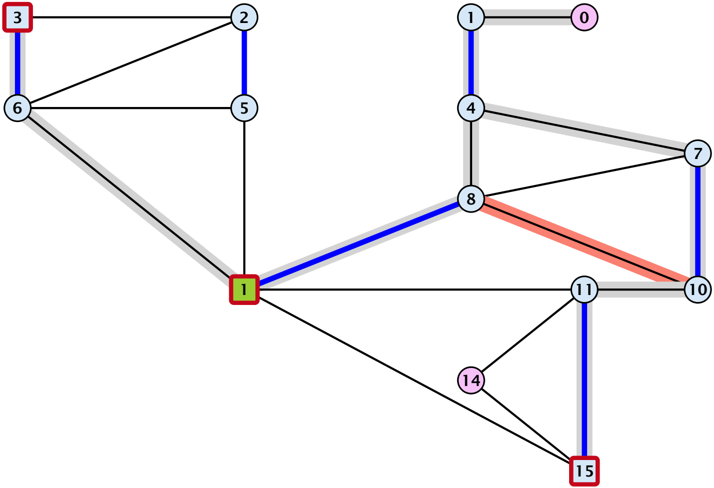

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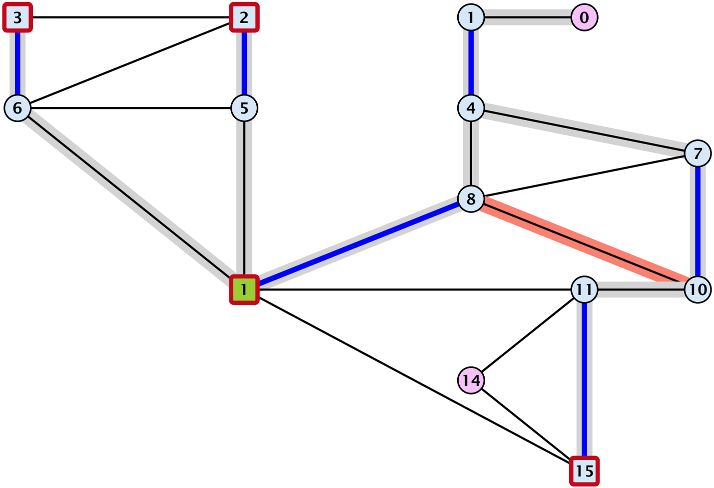

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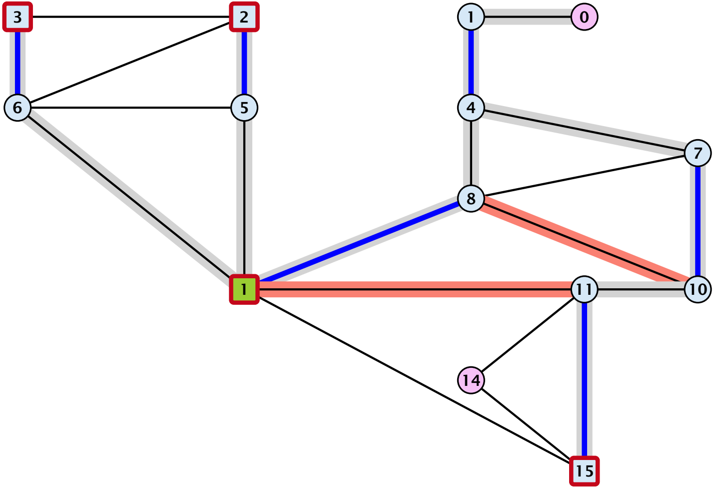

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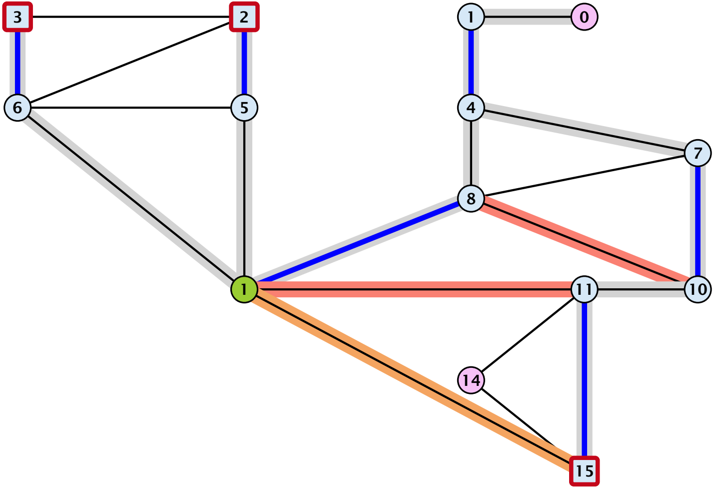

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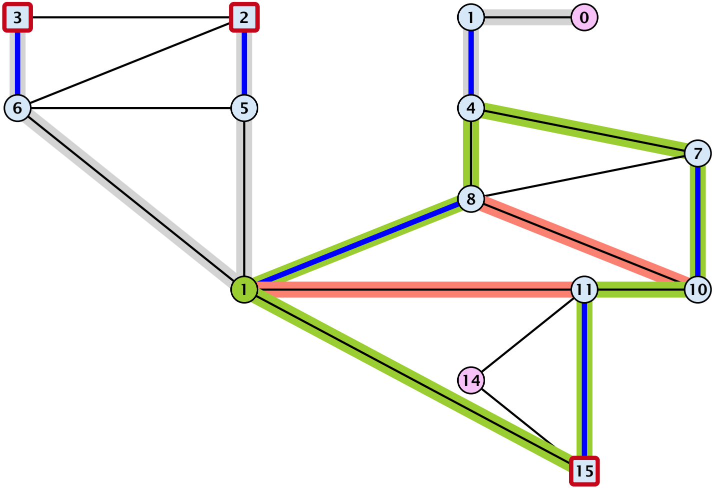

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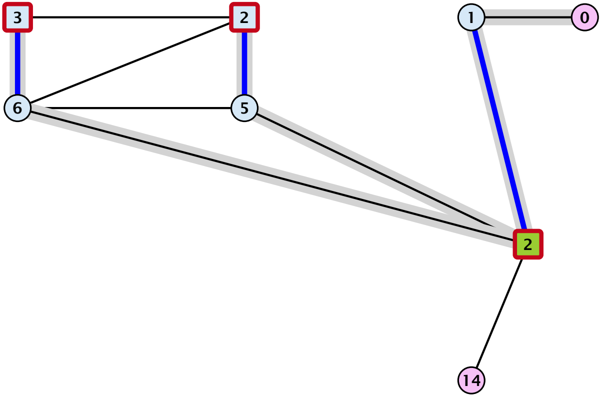

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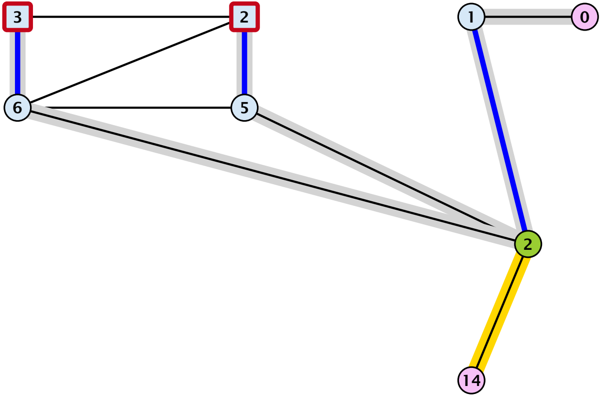

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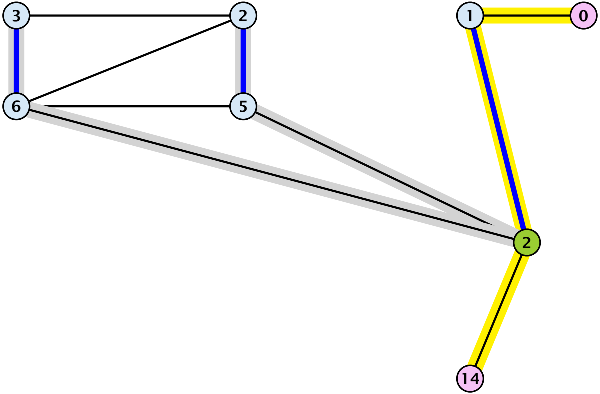

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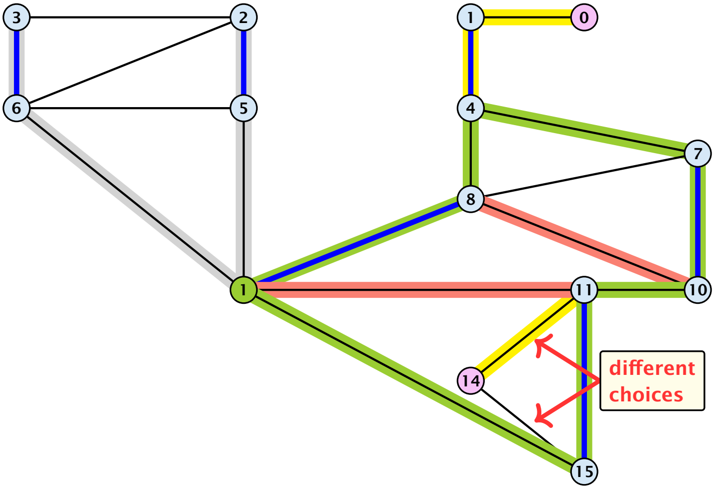

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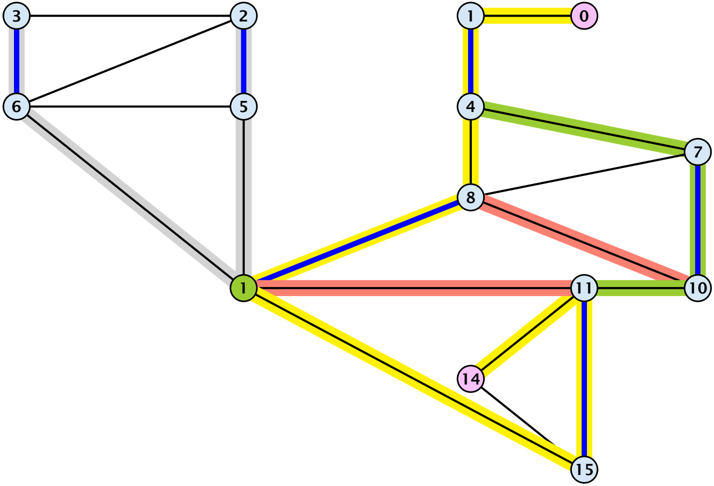

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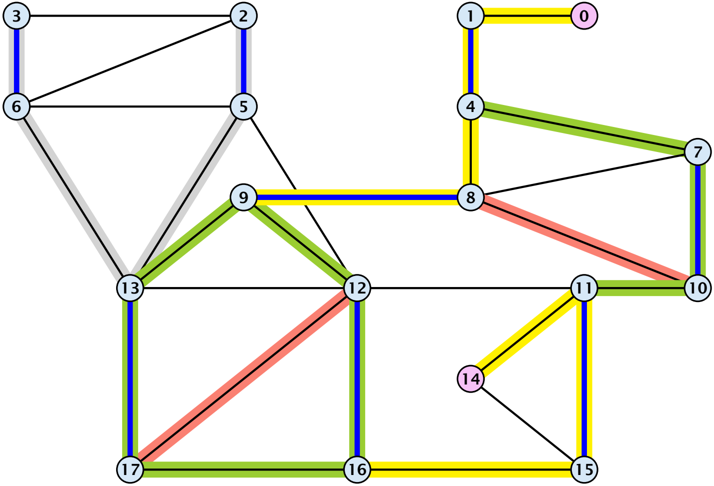

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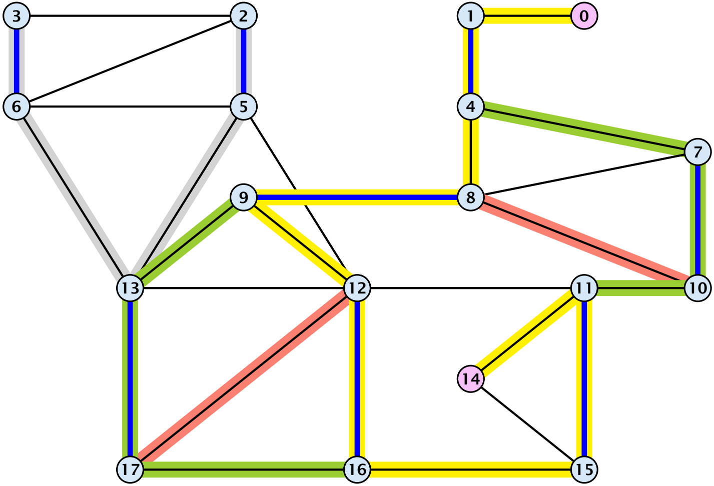

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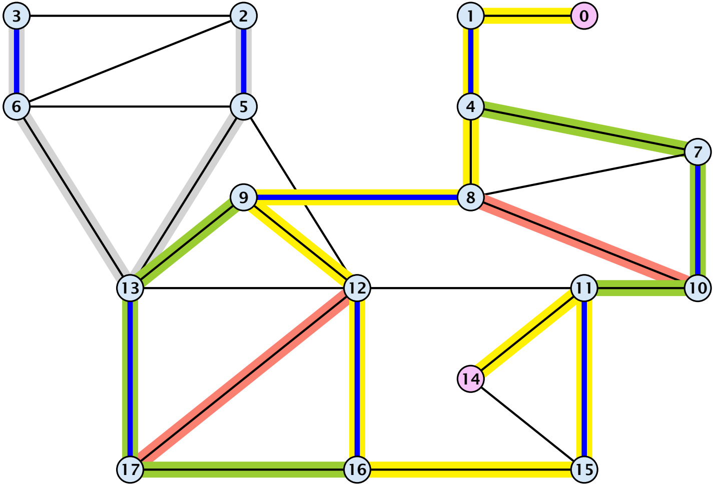

Correctness

Assume that in G we have a flower w.r.t. matching M. Let r be the root, B the blossom, and w the base. Let graph $G' = G/B$ with pseudonode b. Let M' be the matching in the contracted graph.

Correctness

Assume that in G we have a flower w.r.t. matching M. Let r be the root, B the blossom, and w the base. Let graph $G' = G/B$ with pseudonode b. Let M' be the matching in the contracted graph.

Lemma 10

If G' contains an augmenting path P' starting at r (or the pseudo-node containing r) w.r.t. the matching M' then G contains an augmenting path starting at r w.r.t. matching M.

Correctness

Proof.

If P' does not contain b it is also an augmenting path in G.

Correctness

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Case 1: non-empty stem

- Next suppose that the stem is non-empty.

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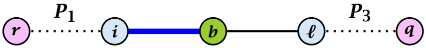

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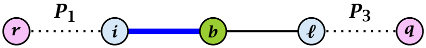

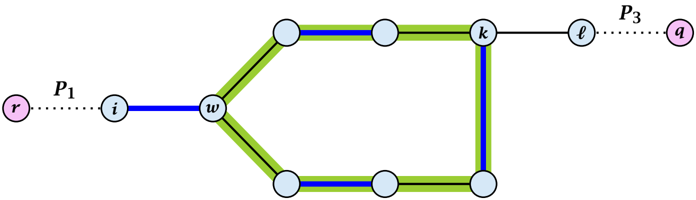

Correctness

- After the expansion ℓ must be incident to some node in the blossom. Let this node be k.
- If $k \neq w$ there is an alternating path P_2 from w to k that ends in a matching edge.
- $P_1 \circ (i,w) \circ P_2 \circ (k,\ell) \circ P_3$ is an alternating path.
- If $k = w$ then $P_1 \circ (i,w) \circ (w,\ell) \circ P_3$ is an alternating path.

Correctness

Proof.

Case 2: empty stem

- If the stem is empty then after expanding the blossom, $w = r$.

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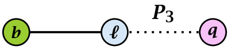

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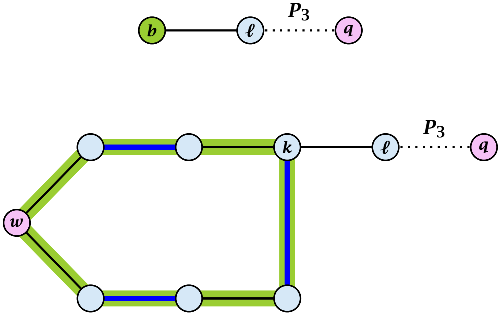

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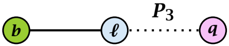

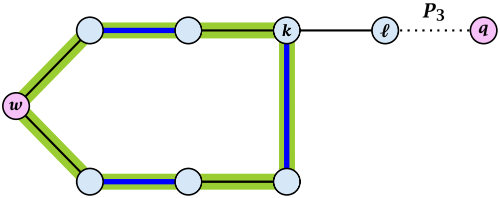

- The path $r \circ P_2 \circ (k, \ell) \circ P_3$ is an alternating path.

Correctness

Lemma 11
If G contains an augmenting path P from r to q w.r.t. matching M then G' contains an augmenting path from r (or the pseudo-node containing r) to q w.r.t. M'.

Correctness

Proof.

- If P does not contain a node from B there is nothing to prove.

Correctness

Proof.

- If P does not contain a node from B there is nothing to prove.
- We can assume that r and q are the only free nodes in G.

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Case 1: empty stem

Let i be the last node on the path P that is part of the blossom.

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Let i be the last node on the path P that is part of the blossom.

P is of the form $P_1 \circ (i,j) \circ P_2$, for some node j and (i,j) is unmatched.

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Case 1: empty stem

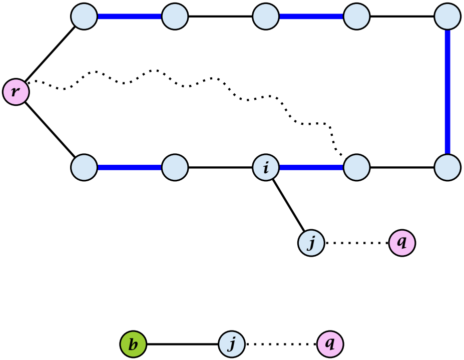
Let i be the last node on the path P that is part of the blossom.

P is of the form $P_1 \circ (i, j) \circ P_2$, for some node j and (i, j) is unmatched.

$(b, j) \circ P_2$ is an augmenting path in the contracted network.

Correctness

Illustration for Case 1:



Correctness

Case 2: non-empty stem

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Let P_3 be alternating path from r to w; this exists because r and w are root and base of a blossom. Define $M_+ = M \oplus P_3$.

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In M_+, r is matched and w is unmatched.

G must contain an augmenting path w.r.t. matching M_+, since M and M_+ have same cardinality.

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In M_+, r is matched and w is unmatched.

G must contain an augmenting path w.r.t. matching M_+, since M and M_+ have same cardinality.

This path must go between w and q as these are the only unmatched vertices w.r.t. M_+.

Correctness

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For M'_+ the blossom has an empty stem. Case 1 applies.

Correctness

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For M'_+ the blossom has an empty stem. Case 1 applies.

G' has an augmenting path w.r.t. M'_+. It must also have an augmenting path w.r.t. M', as both matchings have the same cardinality.

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Case 2: non-empty stem

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This path must go between w and q as these are the only unmatched vertices w.r.t. M_+.

For M'_+ the blossom has an empty stem. Case 1 applies.

G' has an augmenting path w.r.t. M'_+. It must also have an augmenting path w.r.t. M', as both matchings have the same cardinality.

This path must go between r and q.

Algorithm 50 search(r, *found*)

set $\bar{A}(i) \leftarrow A(i)$ for all nodes i
found $\leftarrow$ false
unlabel all nodes;
give an even label to r and initialize *list* $\leftarrow \{r\}$
while *list* $\neq \emptyset$ **do**
    delete a node i from *list*
    examine(i, *found*)
    if *found* $=$ true **then return**

Search for an augmenting path starting at r.

Algorithm 50 search(r, *found*)

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give an even label to r and initialize *list* $\leftarrow \{r\}$
while *list* $\neq \emptyset$ **do**
    delete a node i from *list*
    examine(i, *found*)
    if *found* $=$ true **then return**

$A(i)$ contains neighbours of node i.

We create a copy $\bar{A}(i)$ so that we later can shrink blossoms.

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    if *found* $=$ true **then return**

found is just a Boolean that allows to abort the search process...

Algorithm 50 search(r, *found*)

```
set Ā(i) ← A(i) for all nodes i
found ← false
unlabel all nodes;
give an even label to r and initialize list ← {r}
while list ≠ ∅ do
    delete a node i from list
    examine(i, found)
    if found = true then return
```

In the beginning no node is in the tree.

Algorithm 50 search(r, *found*)

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    if *found* $=$ true **then return**

Put the root in the tree.

list could also be a set or a stack.

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while *list* $\neq \emptyset$ **do**
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As long as there are nodes with unexamined neighbours...

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```
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while list ≠ ∅ do
    delete a node i from list
    examine(i, found)
    if found = true then return
```

...examine the next one

Algorithm 50 search(r, *found*)

set $\bar{A}(i) \leftarrow A(i)$ for all nodes i
found $\leftarrow$ false
unlabel all nodes;
give an even label to r and initialize *list* $\leftarrow \{r\}$
while *list* $\neq \emptyset$ **do**
    delete a node i from *list*
    examine(i, *found*)
    if *found* = true **then return**

If you found augmenting path
abort and start from next root.

Algorithm 51 examine(i, $found$)

```
for all j ∈ Ā(i) do
    if j is even then contract(i, j) and return
    if j is unmatched then
        q ← j;
        pred(q) ← i;
        found ← true;
        return
    if j is matched and unlabeled then
        pred(j) ← i;
        pred(mate(j)) ← j;
        add mate(j) to list
```

Examine the neighbours of a node i

Algorithm 51 examine(i, *found*)

```
for all j ∈ Ā(i) do
    if j is even then contract(i, j) and return
    if j is unmatched then
        q ← j;
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        found ← true;
        return
    if j is matched and unlabeled then
        pred(j) ← i;
        pred(mate(j)) ← j;
        add mate(j) to list
```

For all neighbours j do...

Algorithm 51 examine(i, $found$)

```
for all j ∈ Ā(i) do
    if j is even then contract(i, j) and return
    if j is unmatched then
        q ← j;
        pred(q) ← i;
        found ← true;
        return
    if j is matched and unlabeled then
        pred(j) ← i;
        pred(mate(j)) ← j;
        add mate(j) to list
```

You have found a blossom...

Algorithm 51 examine($i, found$)

for all $j \in \bar{A}(i)$ **do**
    if j is even **then** contract(i, j) and **return**
    if j is unmatched **then**
        $q \leftarrow j$;
        $\text{pred}(q) \leftarrow i$;
        $found \leftarrow \text{true}$;
        return
    if j is matched and unlabeled **then**
        $\text{pred}(j) \leftarrow i$;
        $\text{pred}(\text{mate}(j)) \leftarrow j$;
        add $\text{mate}(j)$ to *list*

You have found a free node which gives you an augmenting path.

Algorithm 51 examine(i, $found$)

```
for all j ∈ Ā(i) do
    if j is even then contract(i, j) and return
    if j is unmatched then
        q ← j;
        pred(q) ← i;
        found ← true;
        return
    if j is matched and unlabeled then
        pred(j) ← i;
        pred(mate(j)) ← j;
        add mate(j) to list
```

If you find a matched node that is not in the tree you grow...

Algorithm 51 examine(i, *found*)

```
for all j ∈ Ā(i) do
    if j is even then contract(i, j) and return
    if j is unmatched then
        q ← j;
        pred(q) ← i;
        found ← true;
        return
    if j is matched and unlabeled then
        pred(j) ← i;
        pred(mate(j)) ← j;
        add mate(j) to list
```

mate(j) is a new node from which you can grow further.

Algorithm 52 contract(i, j)

1: trace pred-indices of i and j to identify a blossom B
2: create new node b and set $\bar{A}(b) \leftarrow \cup_{x \in B} \bar{A}(x)$
3: label b even and add to *list*
4: update $\bar{A}(j) \leftarrow \bar{A}(j) \cup \{b\}$ for each $j \in \bar{A}(b)$
5: form a circular double linked list of nodes in B
6: delete nodes in B from the graph

Contract blossom identified by nodes i and j

Algorithm 52 contract(i, j)

1: trace pred-indices of i and j to identify a blossom B
2: create new node b and set $\bar{A}(b) \leftarrow \cup_{x \in B} \bar{A}(x)$
3: label b even and add to *list*
4: update $\bar{A}(j) \leftarrow \bar{A}(j) \cup \{b\}$ for each $j \in \bar{A}(b)$
5: form a circular double linked list of nodes in B
6: delete nodes in B from the graph

Get all nodes of the blossom.

Time: $\mathcal{O}(m)$

Algorithm 52 contract(i, j)

1: trace pred-indices of i and j to identify a blossom B
2: create new node b and set $\bar{A}(b) \leftarrow \cup_{x \in B} \bar{A}(x)$
3: label b even and add to *list*
4: update $\bar{A}(j) \leftarrow \bar{A}(j) \cup \{b\}$ for each $j \in \bar{A}(b)$
5: form a circular double linked list of nodes in B
6: delete nodes in B from the graph

Identify all neighbours of b.

Time: $\mathcal{O}(m)$ (how?)

Algorithm 52 contract(i, j)

trace pred-indices of i and j to identify a blossom B
create new node b and set $\bar{A}(b) \leftarrow \cup_{x \in B} \bar{A}(x)$
label b even and add to *list*
update $\bar{A}(j) \leftarrow \bar{A}(j) \cup \{b\}$ for each $j \in \bar{A}(b)$
form a circular double linked list of nodes in B
delete nodes in B from the graph

b will be an even node, and it has unexamined neighbours.

Algorithm 52 contract(i, j)

1: trace pred-indices of i and j to identify a blossom B
2: create new node b and set $\bar{A}(b) \leftarrow \cup_{x \in B} \bar{A}(x)$
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5: form a circular double linked list of nodes in B
6: delete nodes in B from the graph

Every node that was adjacent to a node in B is now adjacent to b

Algorithm 52 contract(i, j)

trace pred-indices of i and j to identify a blossom B
create new node b and set $\bar{A}(b) \leftarrow \cup_{x \in B} \bar{A}(x)$
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update $\bar{A}(j) \leftarrow \bar{A}(j) \cup \{b\}$ for each $j \in \bar{A}(b)$
form a circular double linked list of nodes in B
delete nodes in B from the graph

Only for making a blossom expansion easier.

Algorithm 52 contract(i, j)

1: trace pred-indices of i and j to identify a blossom B
2: create new node b and set $\bar{A}(b) \leftarrow \cup_{x \in B} \bar{A}(x)$
3: label b even and add to *list*
4: update $\bar{A}(j) \leftarrow \bar{A}(j) \cup \{b\}$ for each $j \in \bar{A}(b)$
5: form a circular double linked list of nodes in B
6: delete nodes in B from the graph

Only delete links from nodes not in B to B.

When expanding the blossom again we can recreate these links in time $\mathcal{O}(m)$.

Analysis

- A contraction operation can be performed in time $\mathcal{O}(m)$. Note, that any graph created will have at most m edges.

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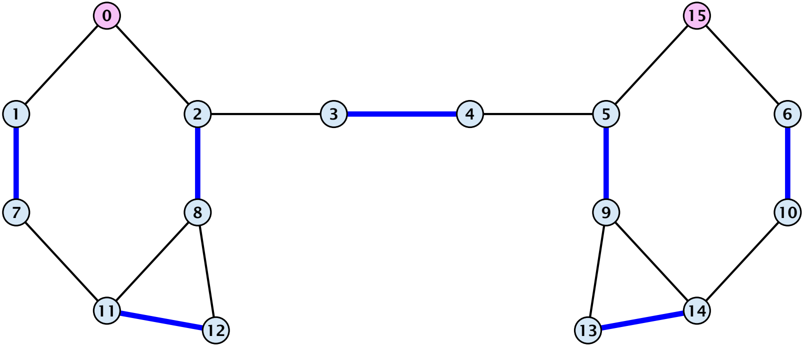
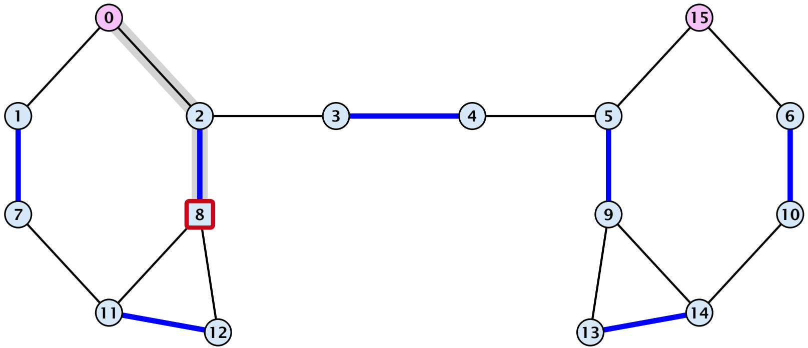
- There are at most n contractions as each contraction reduces the number of vertices.
- The expansion can trivially be done in the same time as needed for all contractions.
- An augmentation requires time $\mathcal{O}(n)$. There are at most n of them.
- In total the running time is at most

$$n \cdot (\mathcal{O}(mn) + \mathcal{O}(n)) = \mathcal{O}(mn^2) .$$

Example: Blossom Algorithm



Example: Blossom Algorithm



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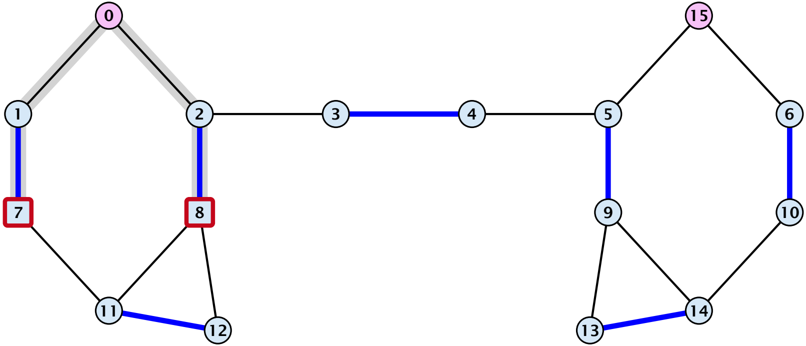

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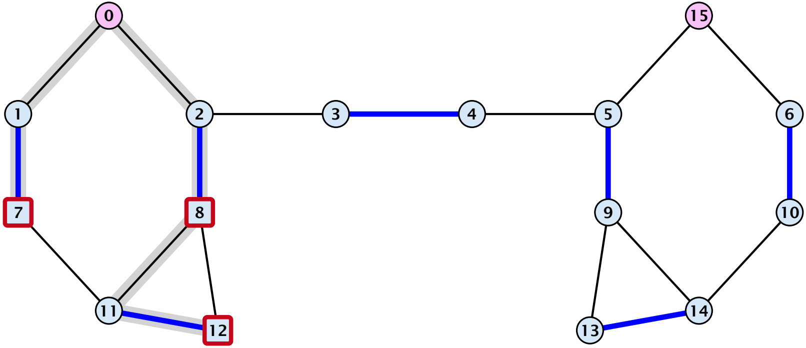

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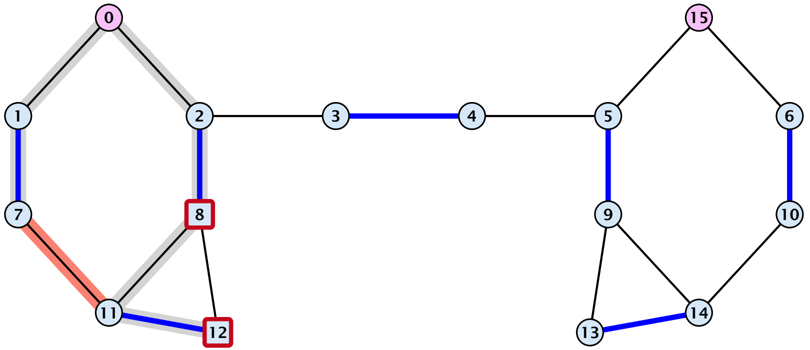

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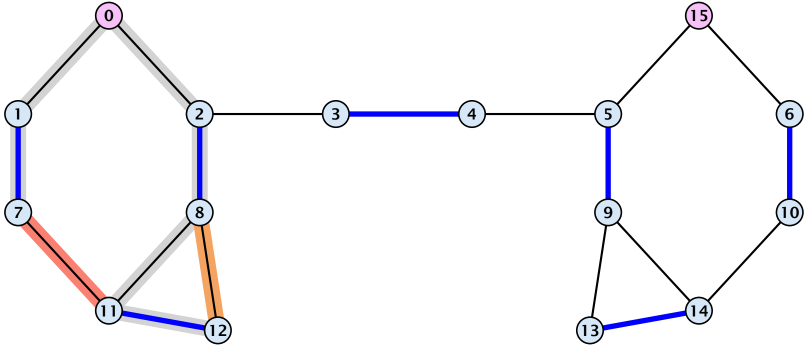

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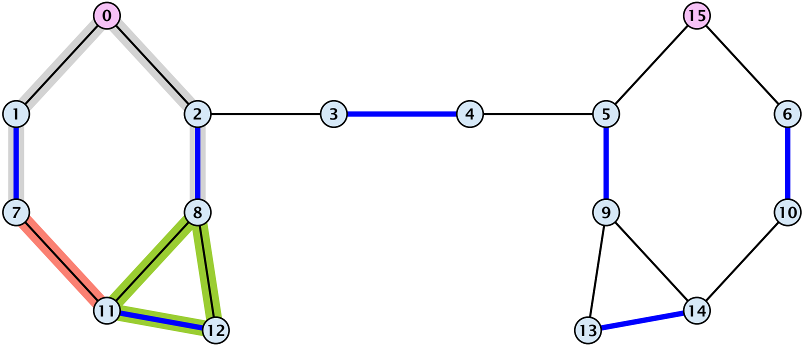

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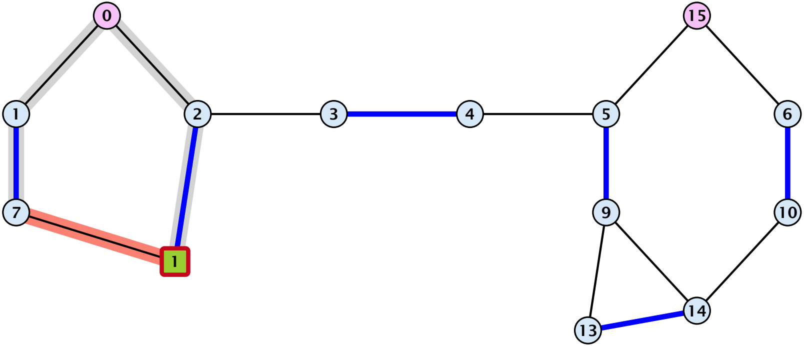

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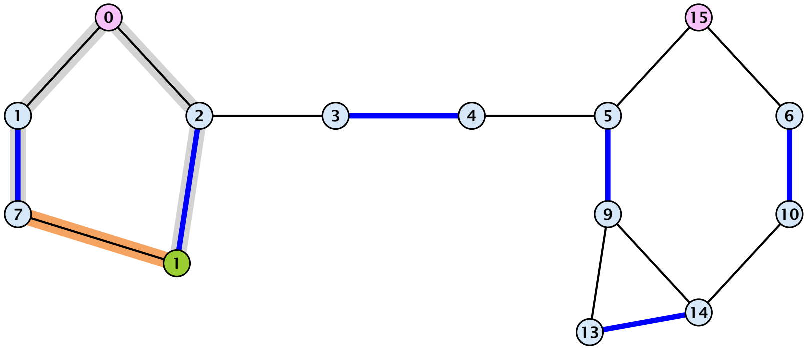

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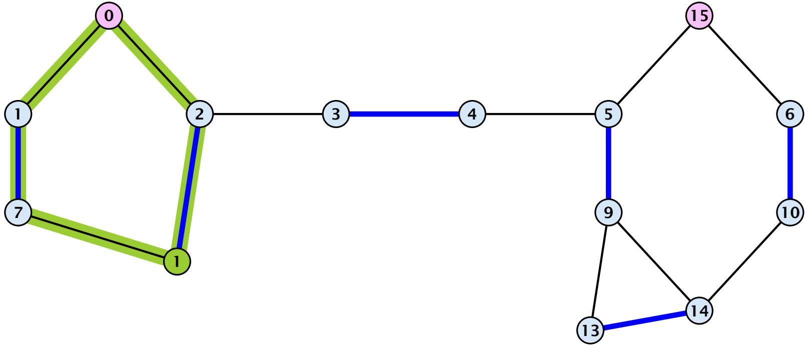

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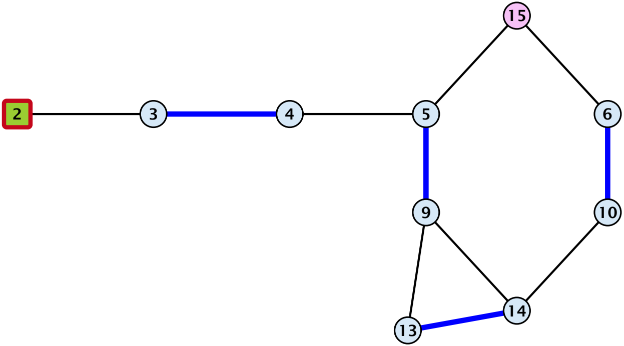

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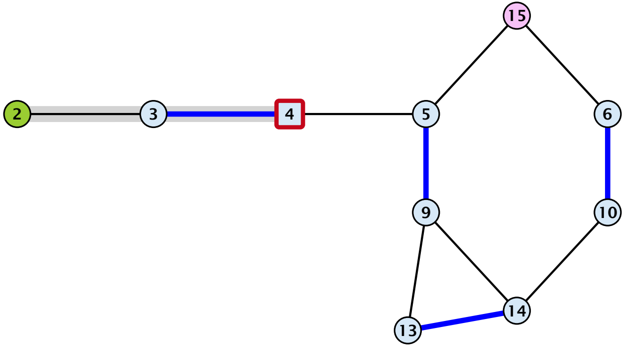

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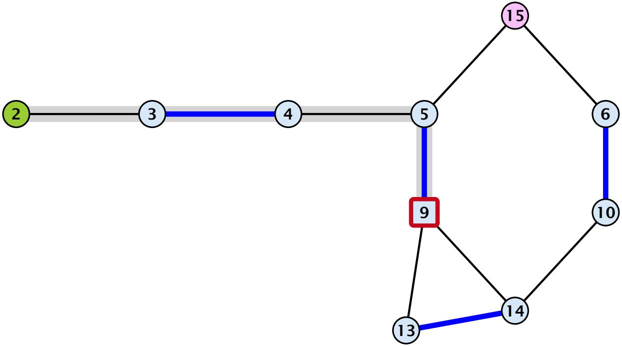

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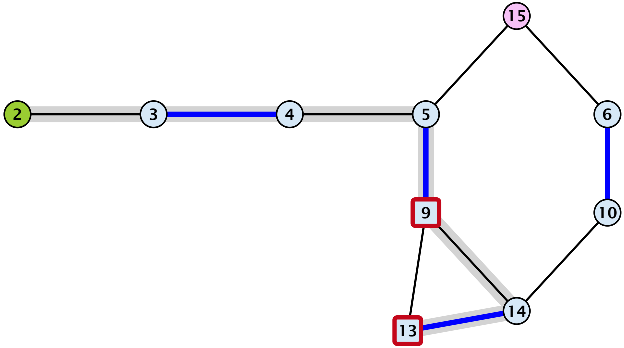

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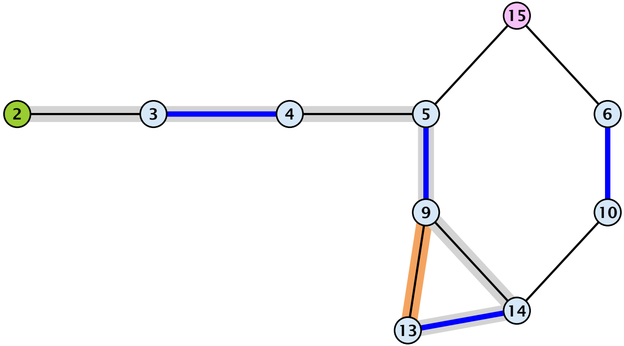

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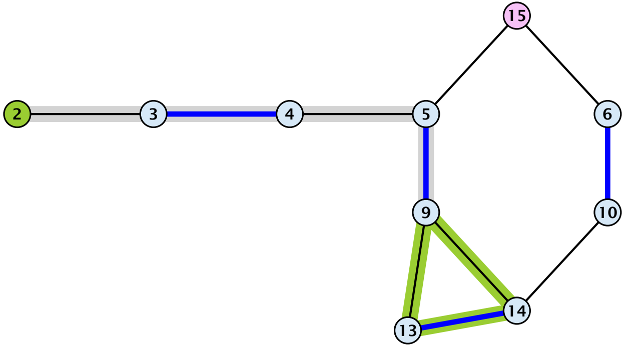

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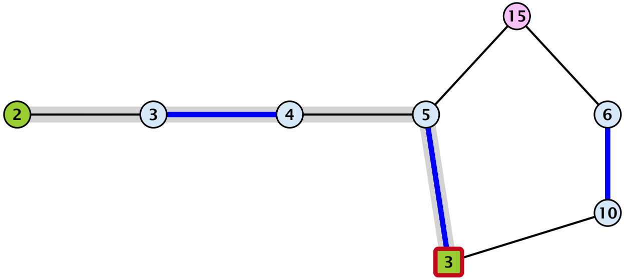

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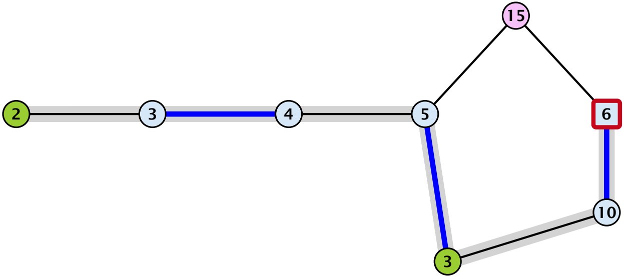

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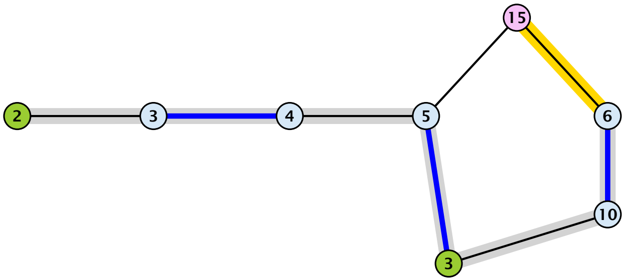

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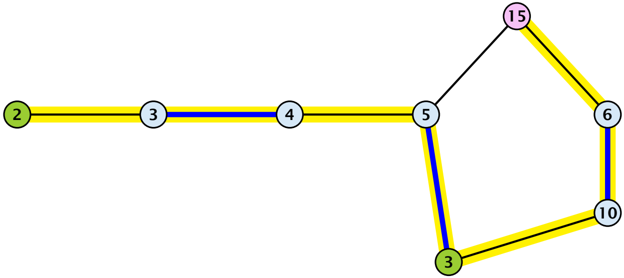

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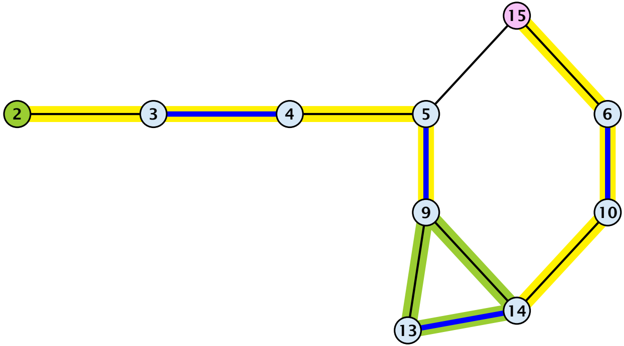

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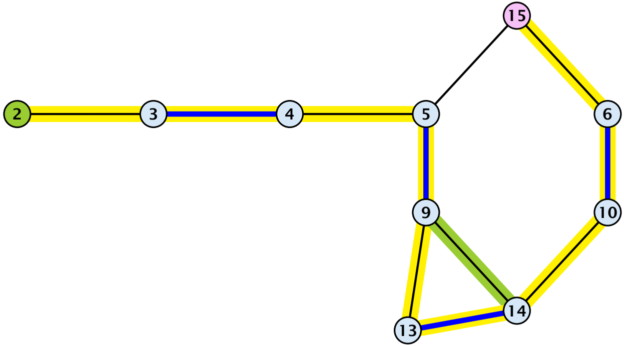

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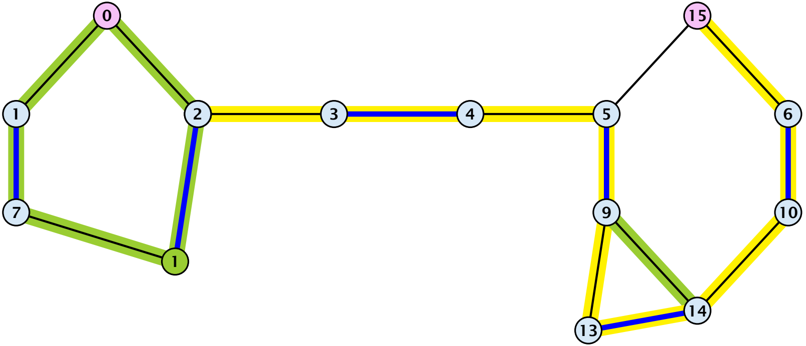

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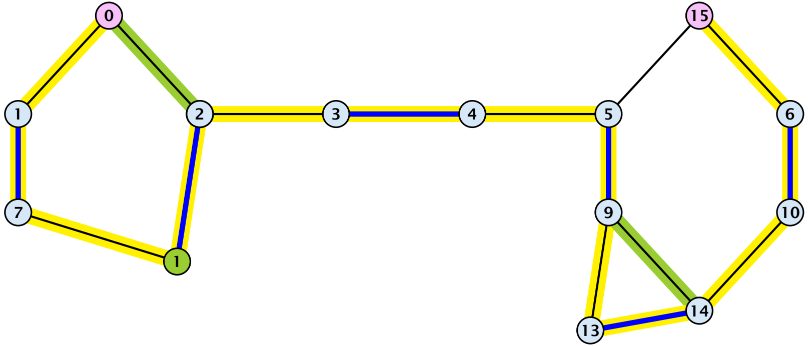

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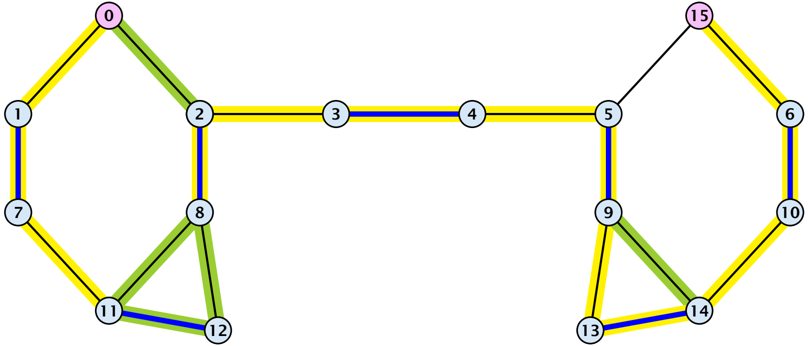

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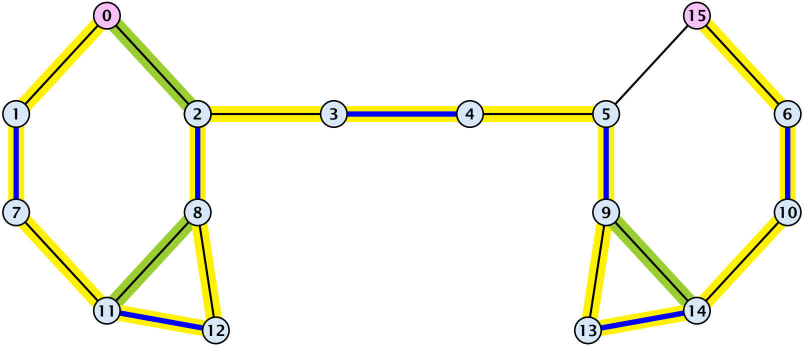

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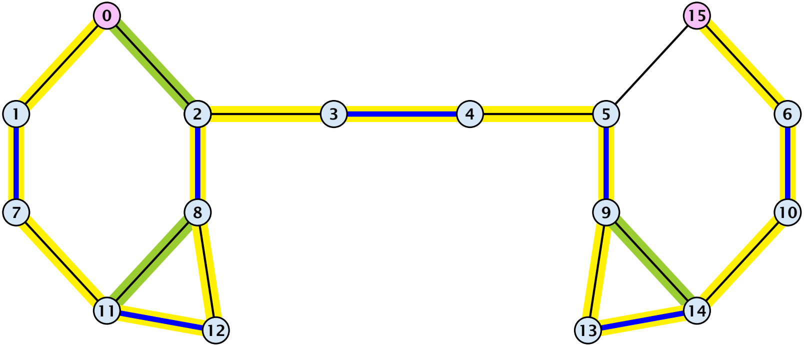

A Fast Matching Algorithm

Algorithm 53 Bimatch-Hopcroft-Karp(G)

1: $M \leftarrow \emptyset$
2: **repeat**
3: let $\mathcal{P} = \{P_1, \ldots, P_k\}$ be maximal set of
4: vertex-disjoint, shortest augmenting path w.r.t. M.
5: $M \leftarrow M \oplus (P_1 \cup \cdots \cup P_k)$
6: **until** $\mathcal{P} = \emptyset$
7: **return** M

We call one iteration of the repeat-loop a phase of the algorithm.

Analysis Hopcroft-Karp

Lemma 12

Given a matching M and a matching M^* with $|M^*| - |M| \geq 0$. There exist $|M^*| - |M|$ vertex-disjoint augmenting path w.r.t. M.

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- Similar to the proof that a matching is optimal iff it does not contain an augmenting path.

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- Consider the graph $G = (V, M \oplus M^*)$, and mark edges in this graph blue if they are in M and red if they are in M^*.

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- The connected components of G are cycles and paths.

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- Consider the graph $G = (V, M \oplus M^*)$, and mark edges in this graph blue if they are in M and red if they are in M^*.
- The connected components of G are cycles and paths.
- The graph contains $k \stackrel{\text{def}}{=} |M^*| - |M|$ more red edges than blue edges.

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- The connected components of G are cycles and paths.
- The graph contains $k \stackrel{\text{def}}{=} |M^*| - |M|$ more red edges than blue edges.
- Hence, there are at least k components that form a path starting and ending with a red edge. These are augmenting paths w.r.t. M.

Analysis Hopcroft-Karp

- Let $P_1, \dots, P_k$ be a maximal collection of vertex-disjoint, shortest augmenting paths w.r.t. M (let $\ell = |P_i|$).

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Lemma 13

The set $A \stackrel{\text{def}}{=} M \oplus (M' \oplus P) = (P_1 \cup \dots \cup P_k) \oplus P$ *contains at least* $(k+1)\ell$ *edges.*

Analysis Hopcroft-Karp

Proof.

- The set describes exactly the symmetric difference between matchings M and $M' \oplus P$.

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- Each of these paths is of length at least ℓ.

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P is of length at least $\ell + 1$. This shows that the length of a shortest augmenting path increases between two phases of the Hopcroft-Karp algorithm.

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- Otherwise, at least one edge from P coincides with an edge from paths $\{P_1, \dots, P_k\}$.
- This edge is not contained in A.
- Hence, $|A| \leq k\ell + |P| - 1$.
- The lower bound on $|A|$ gives $(k+1)\ell \leq |A| \leq k\ell + |P| - 1$, and hence $|P| \geq \ell + 1$.

Analysis Hopcroft-Karp

If the shortest augmenting path w.r.t. a matching M has ℓ edges then the cardinality of the maximum matching is of size at most $|M| + \frac{|V|}{\ell+1}$.

Analysis Hopcroft-Karp

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Proof.
The symmetric difference between M and M^* contains $|M^*| - |M|$ vertex-disjoint augmenting paths. Each of these paths contains at least $\ell + 1$ vertices. Hence, there can be at most $\frac{|V|}{\ell+1}$ of them.

Analysis Hopcroft-Karp

Lemma 15
The Hopcroft-Karp algorithm requires at most $2\sqrt{|V|}$ *phases.*

Analysis Hopcroft-Karp

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Proof.

- After iteration $\lfloor\sqrt{|V|}\rfloor$ the length of a shortest augmenting path must be at least $\lfloor\sqrt{|V|}\rfloor + 1 \geq \sqrt{|V|}$.
- Hence, there can be at most $|V|/(\sqrt{|V|} + 1) \leq \sqrt{|V|}$ additional augmentations.

Analysis Hopcroft-Karp

Lemma 16
One phase of the Hopcroft-Karp algorithm can be implemented in time $\mathcal{O}(m)$.

construct a "level graph" G':

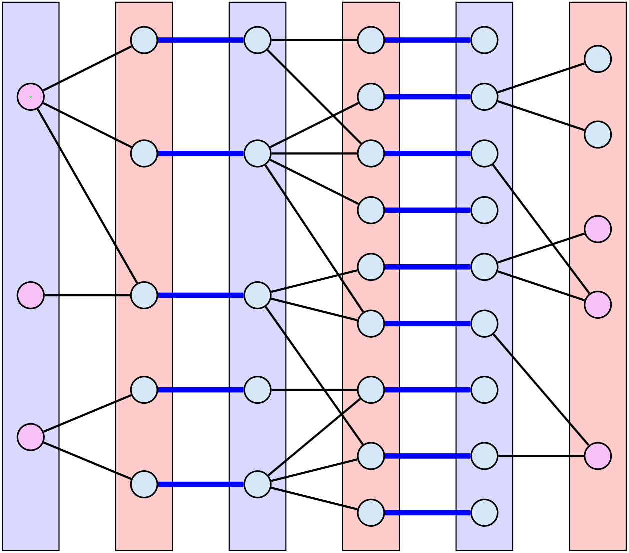
- construct Level 0 that includes all free vertices on left side L
- construct Level 1 containing all neighbors of Level 0
- construct Level 2 containing matching neighbors of Level 1
- construct Level 3 containing all neighbors of Level 2
- ...
- stop when a level (apart from Level 0) contains a free vertex

can be done in time $\mathcal{O}(m)$ by a modified BFS

Analysis Hopcroft-Karp

- a shortest augmenting path must go from Level 0 to the last layer constructed
- it can only use edges between layers
- construct a maxim**al** set of vertex disjoint augmenting path connecting the layers
- for this, go forward until you either reach a free vertex or you reach a "dead end" v
- if you reach a free vertex delete the augmenting path and all incident edges from the graph
- if you reach a dead end backtrack and delete v together with its incident edges

Analysis Hopcroft-Karp



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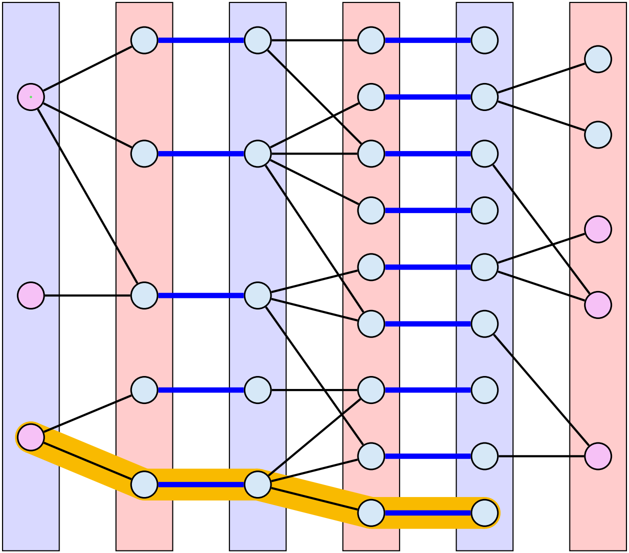

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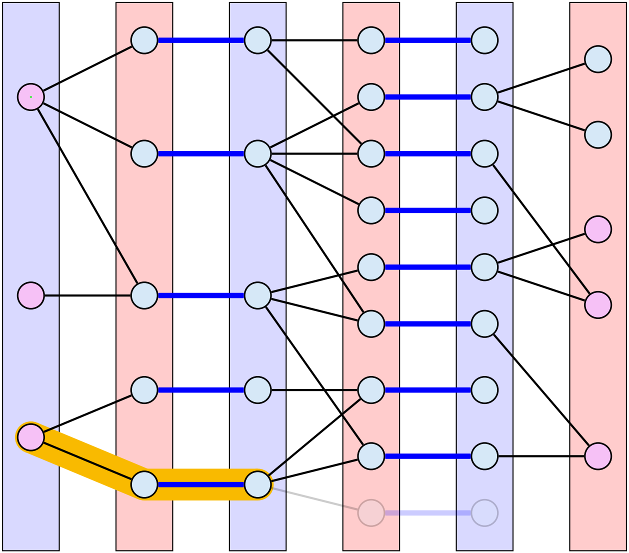

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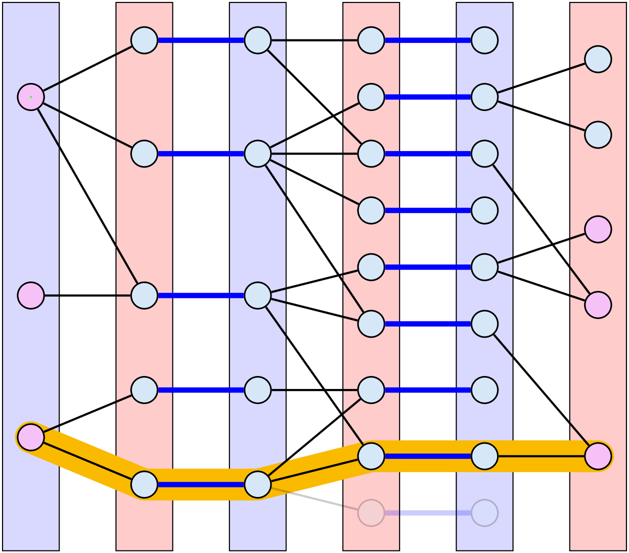

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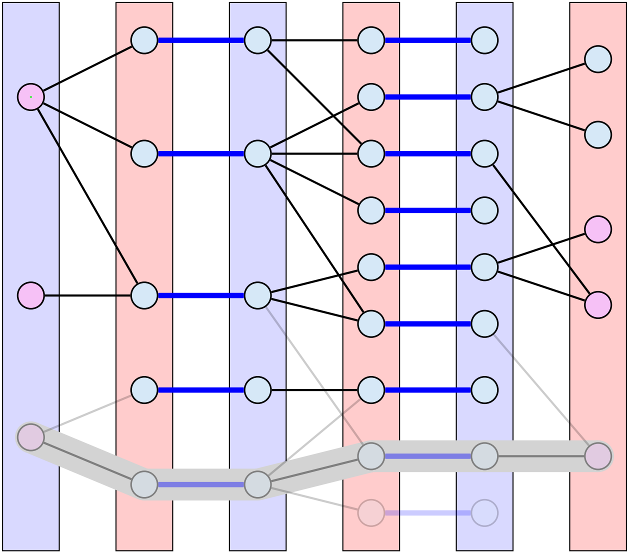

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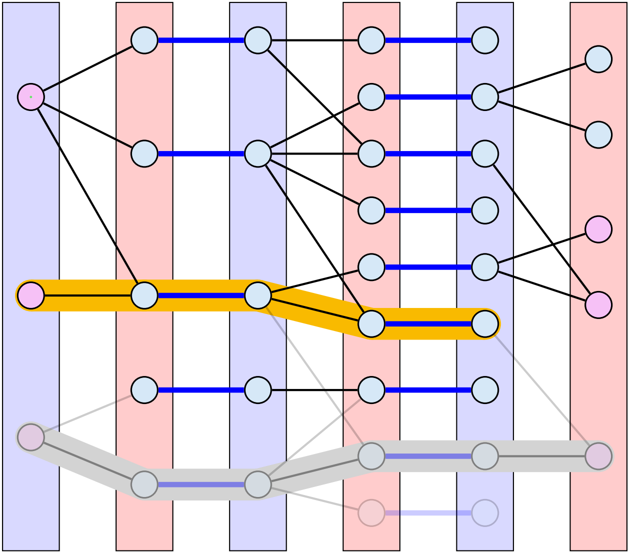

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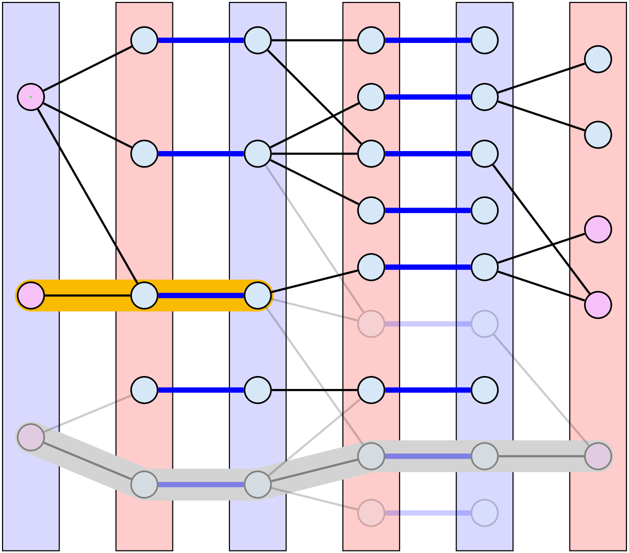

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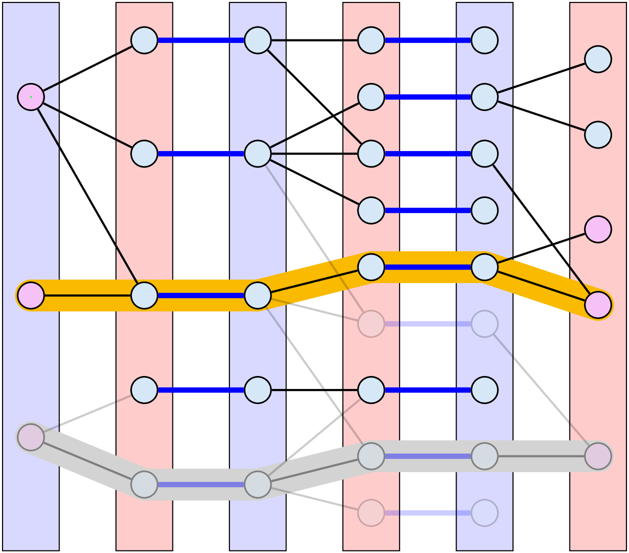

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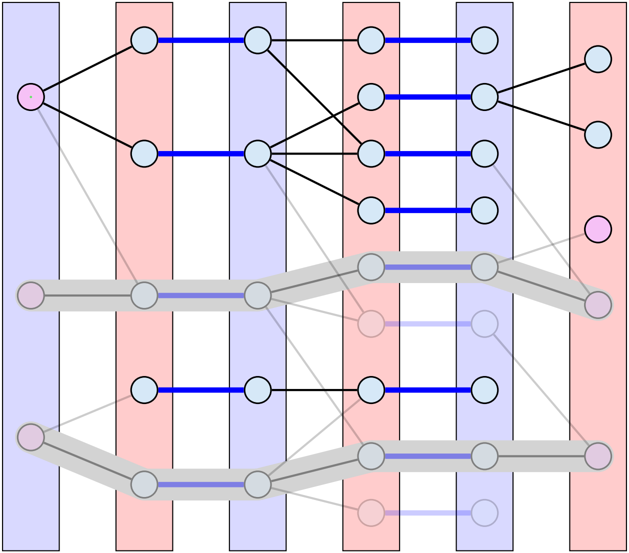

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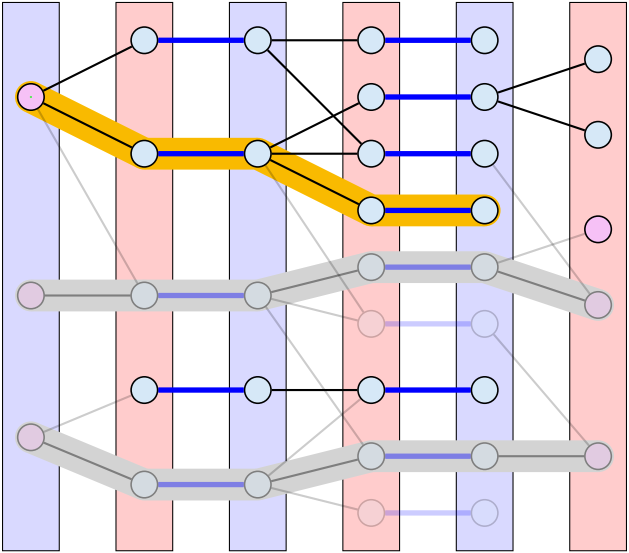

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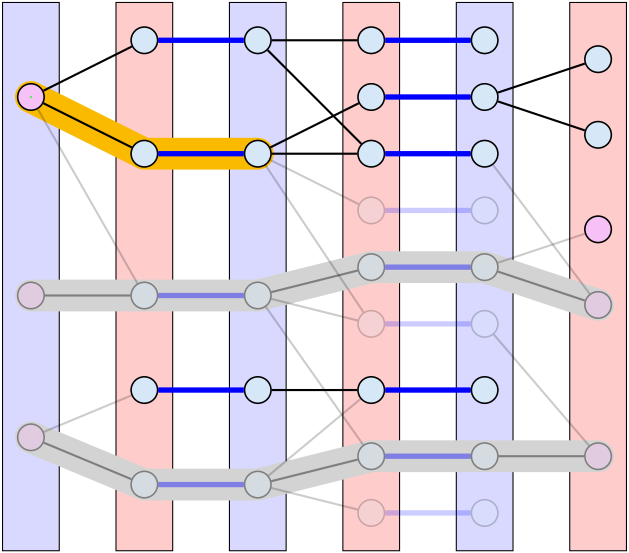

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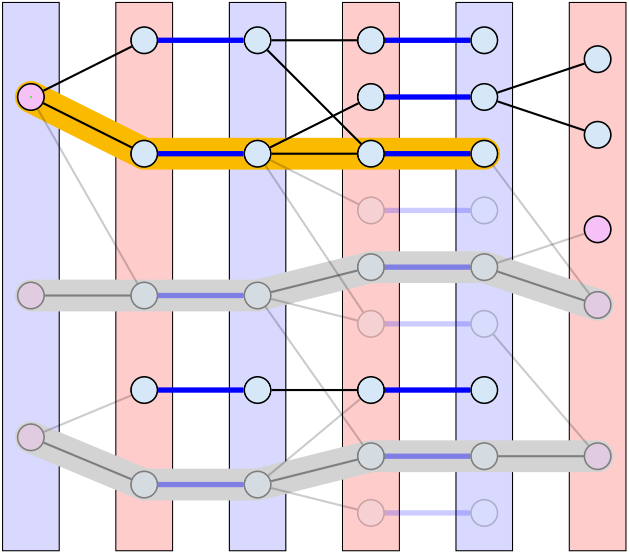

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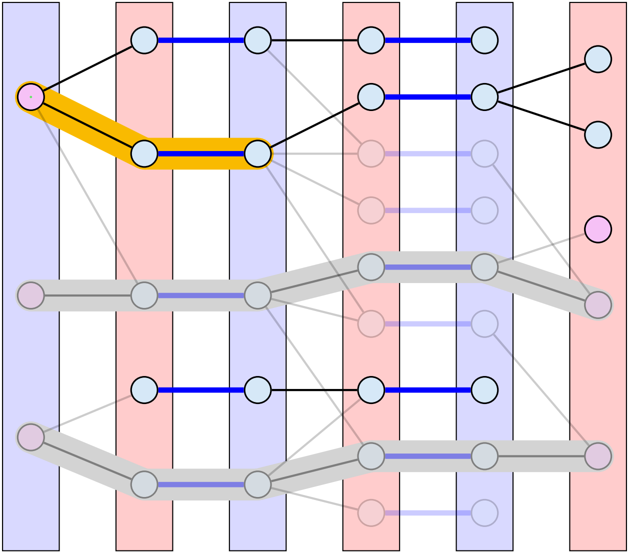

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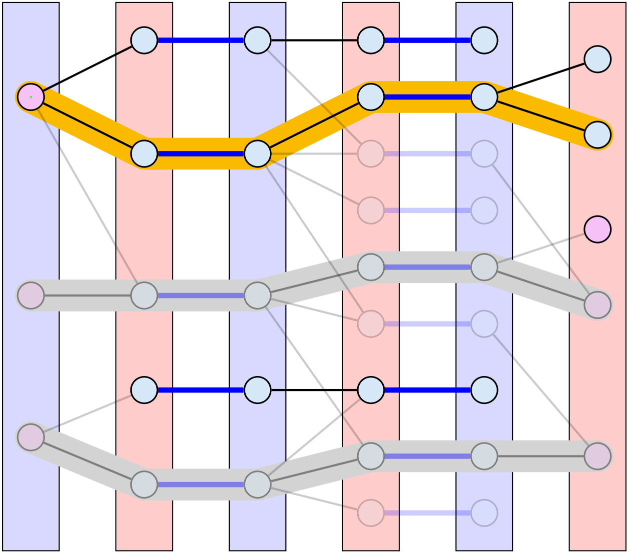

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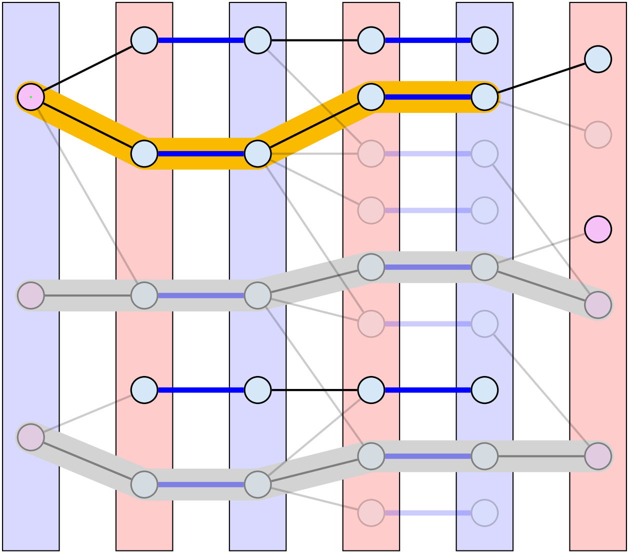

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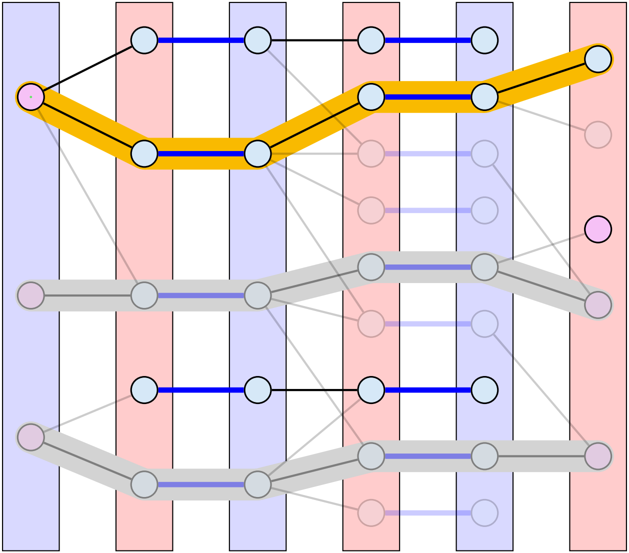

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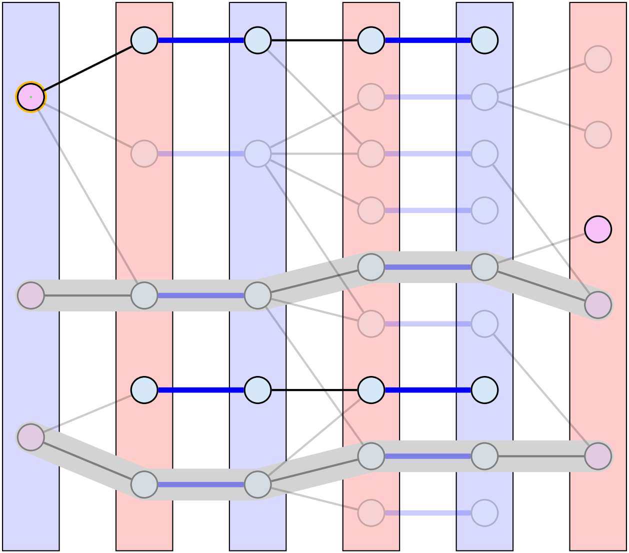

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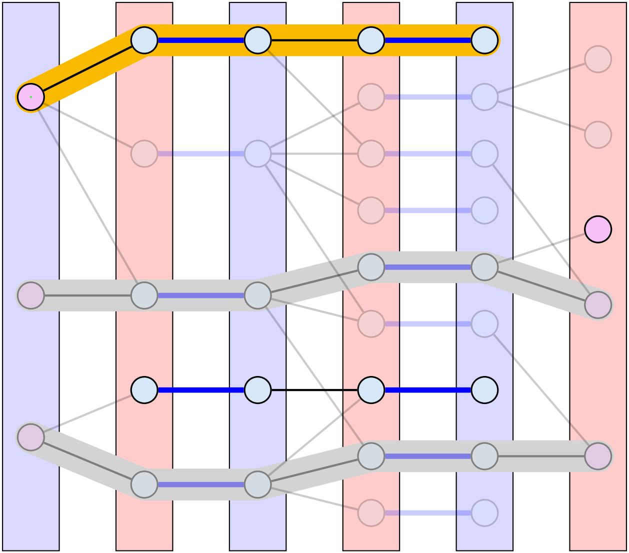

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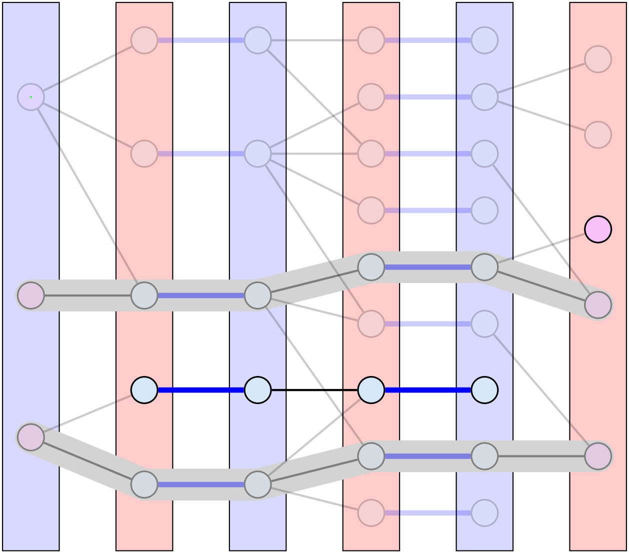

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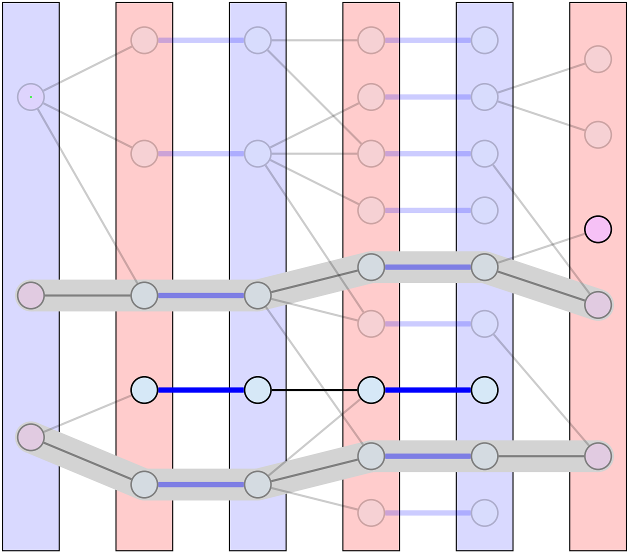

Analysis: Shortest Augmenting Path for Flows

cost for searches during a phase is $\mathcal{O}(mn)$

- a search (successful or unsuccessful) takes time $\mathcal{O}(n)$
- a search deletes at least one edge from the level graph

there are at most n phases

Time: $\mathcal{O}(mn^2)$.

Analysis for Unit-capacity Simple Networks

cost for searches during a phase is $\mathcal{O}(m)$

- an edge/vertex is traversed at most twice

need at most $\mathcal{O}(\sqrt{n})$ phases

- after $\sqrt{n}$ phases there is a cut of size at most $\sqrt{n}$ in the residual graph
- hence at most $\sqrt{n}$ additional augmentations required

Time: $\mathcal{O}(m\sqrt{n})$.

21 Gomory Hu Trees

Given an undirected, weighted graph $G = (V, E, c)$ a cut-tree $T = (V, F, w)$ is a tree with edge-set F and capacities w that fulfills the following properties.

1. **Equivalent Flow Tree:** For any pair of vertices $s, t \in V$, $f(s, t)$ in G is equal to $f_T(s, t)$.
2. **Cut Property:** A minimum s-t cut in T is also a minimum cut in G.

Here, $f(s, t)$ is the value of a maximum s-t flow in G, and $f_T(s, t)$ is the corresponding value in T.

Overview of the Algorithm

The algorithm maintains a partition of V, (sets $S_1, \ldots, S_t$), and a spanning tree T on the vertex set $\{S_1, \ldots, S_t\}$.

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In the end this gives a tree on the vertex set V.

Details of the Split-operation

- Select S_i that contains at least two nodes a and b.

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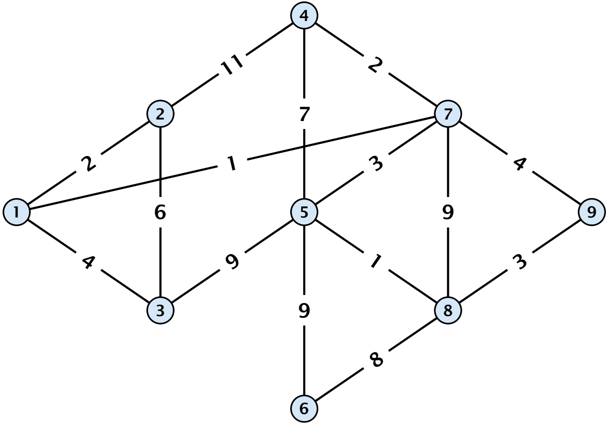
- Replace an edge $\{S_i, S_x\}$ by $\{S_i^a, S_x\}$ if $S_x \subset A$ and by $\{S_i^b, S_x\}$ if $S_x \subset B$.

Example: Gomory-Hu Construction



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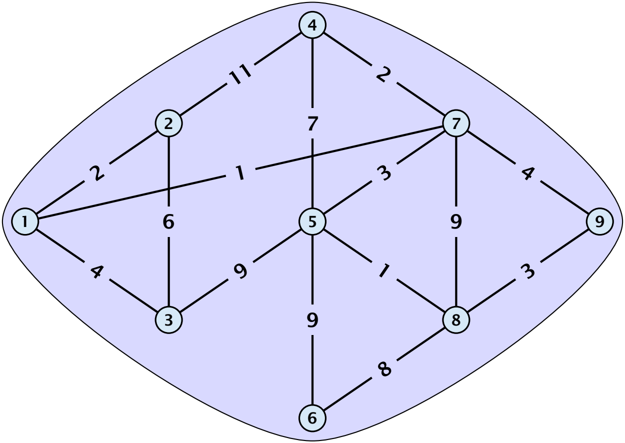

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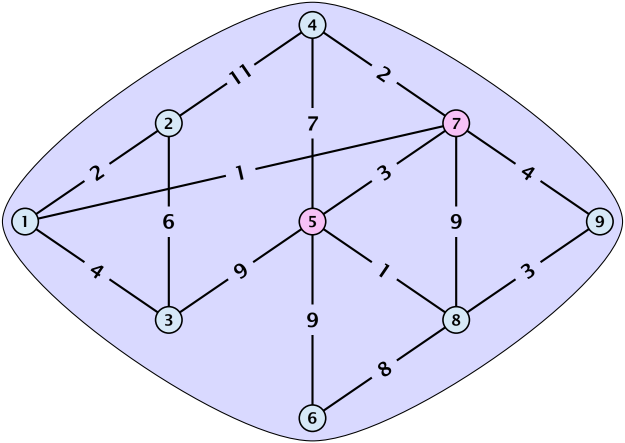

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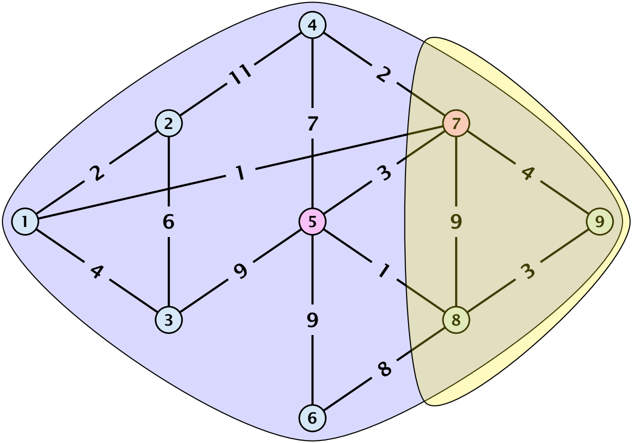

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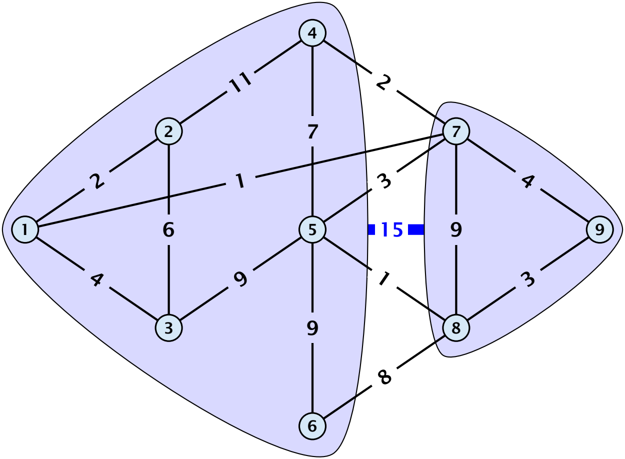

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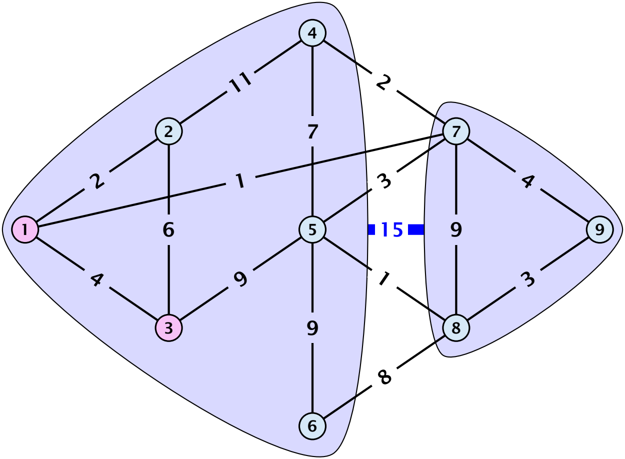

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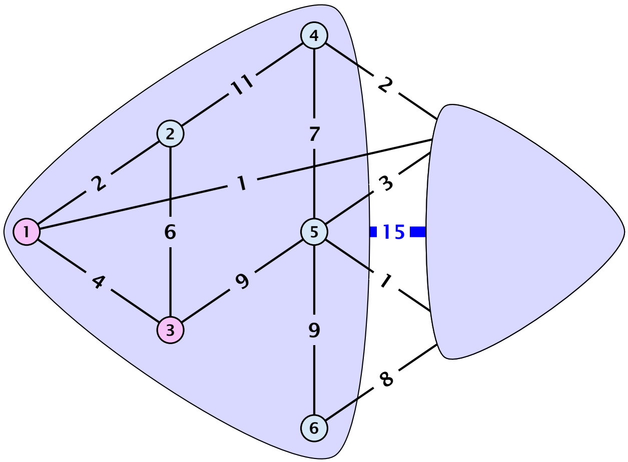

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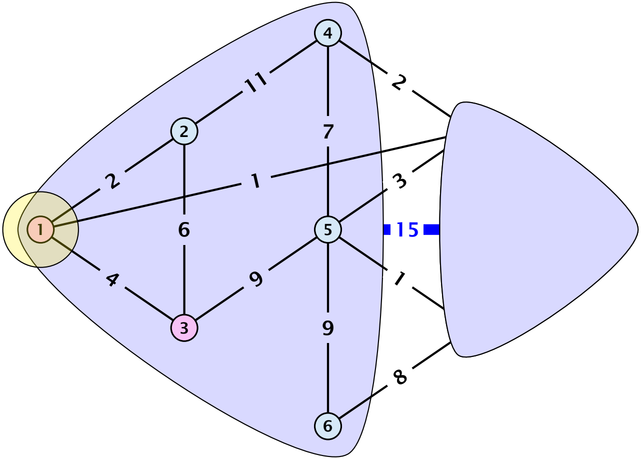

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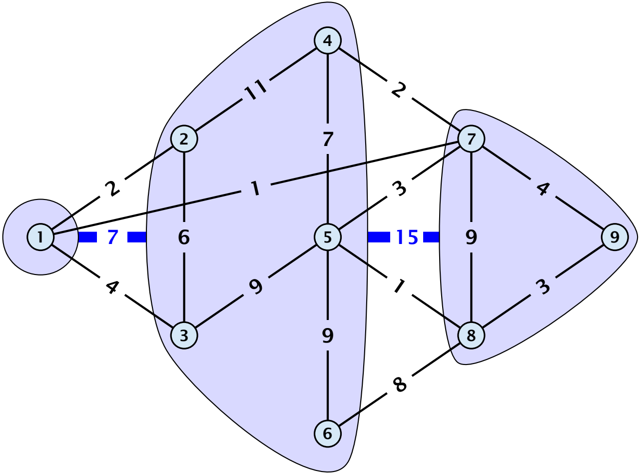

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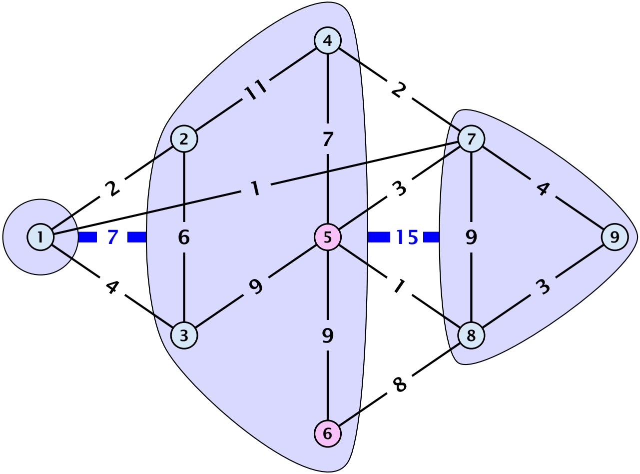

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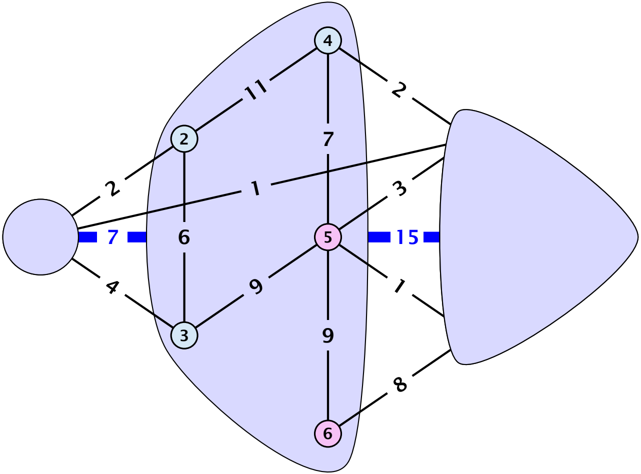

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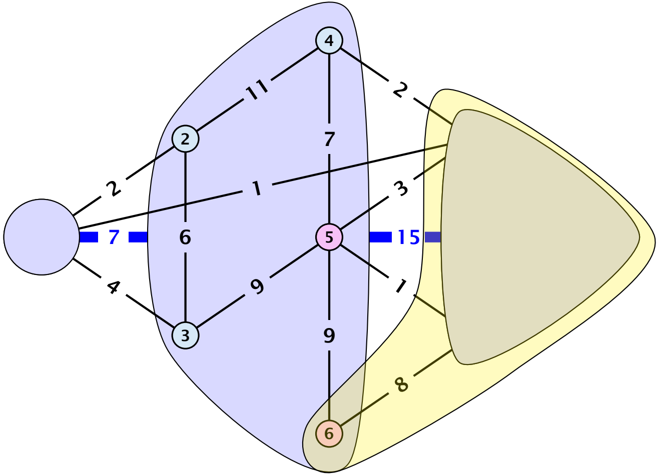

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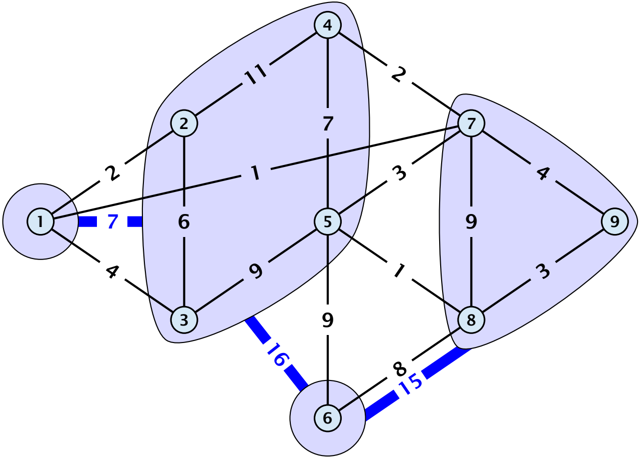

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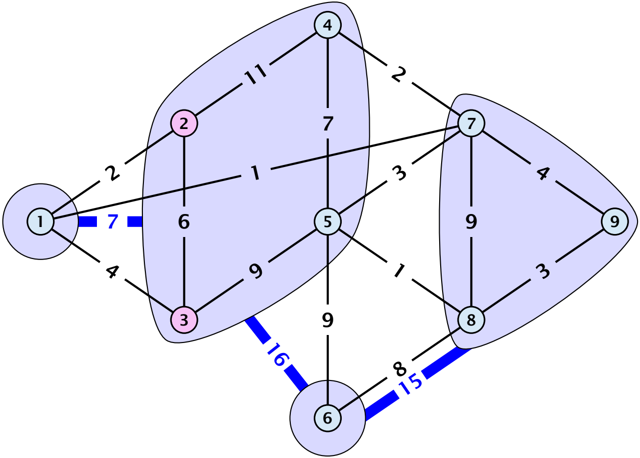

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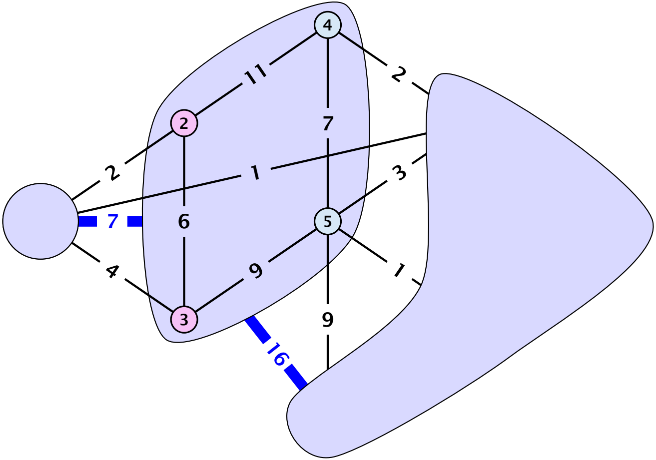

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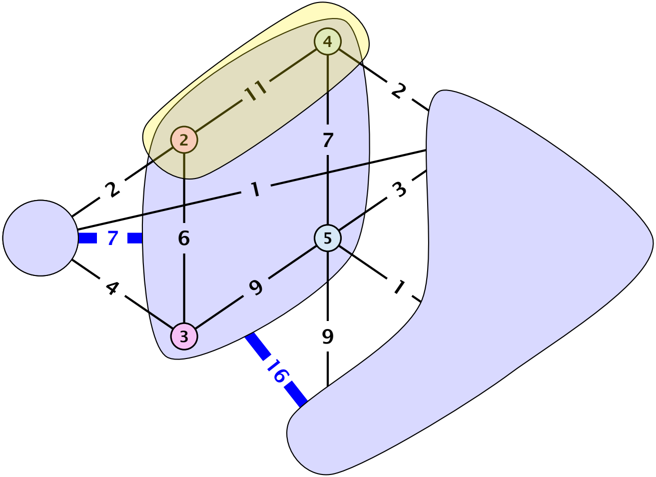

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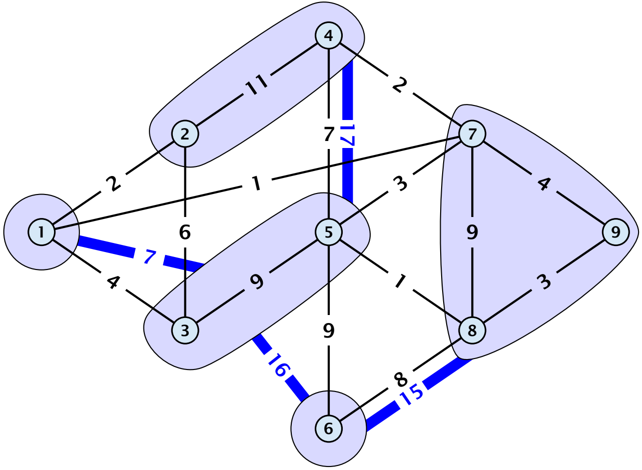

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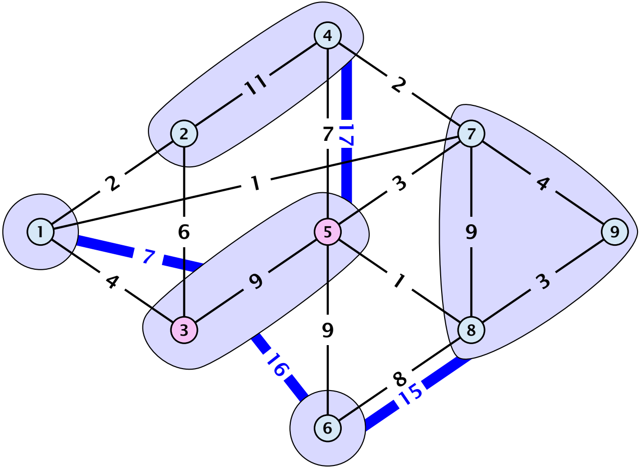

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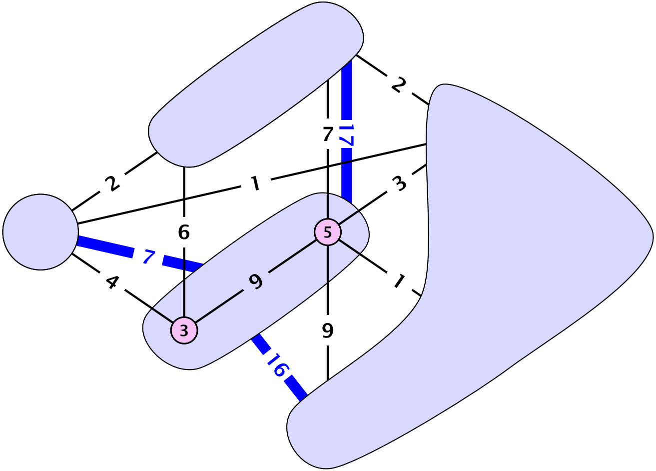

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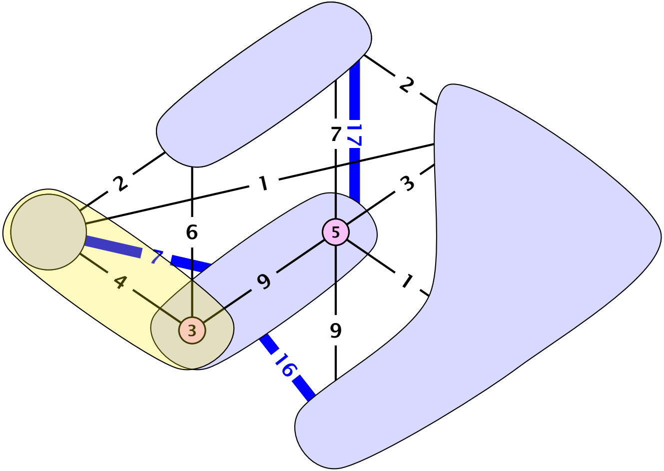

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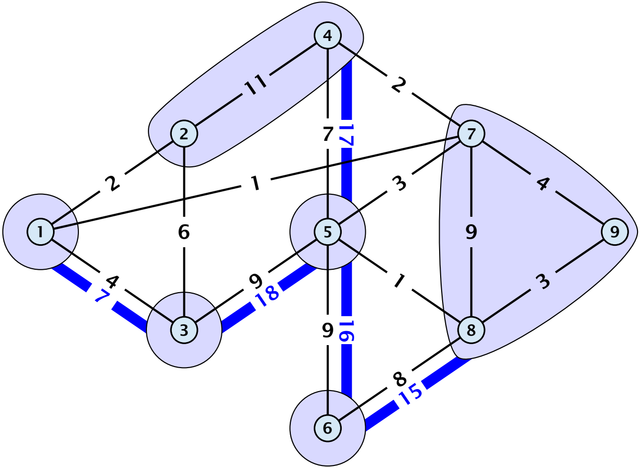

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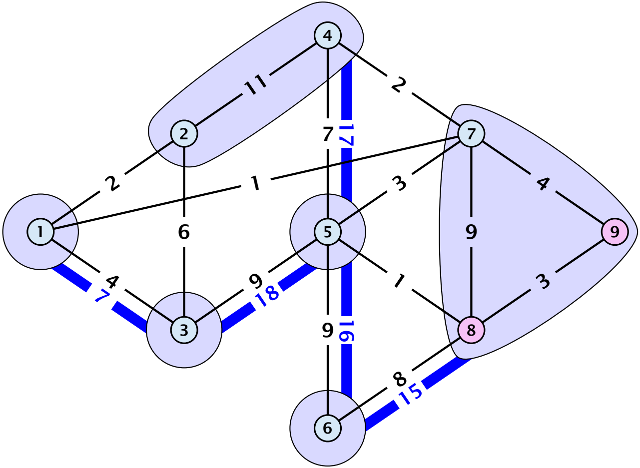

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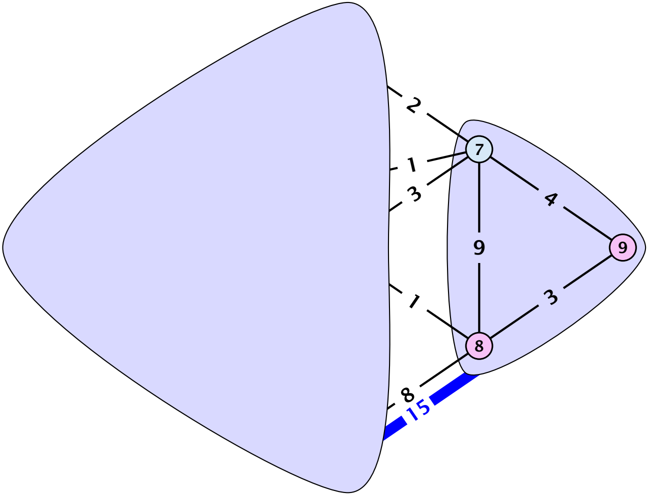

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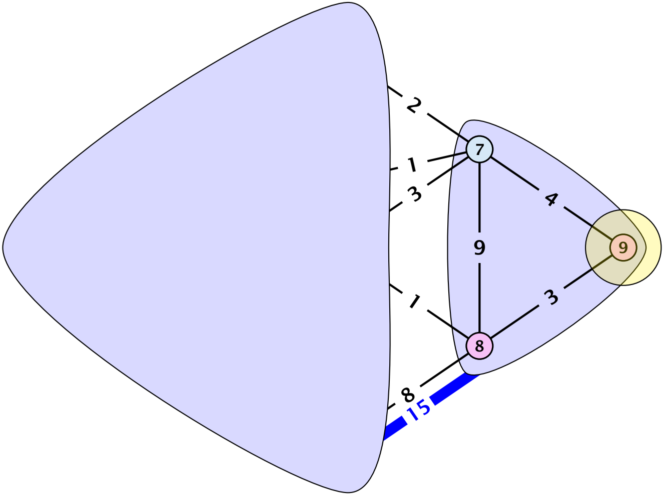

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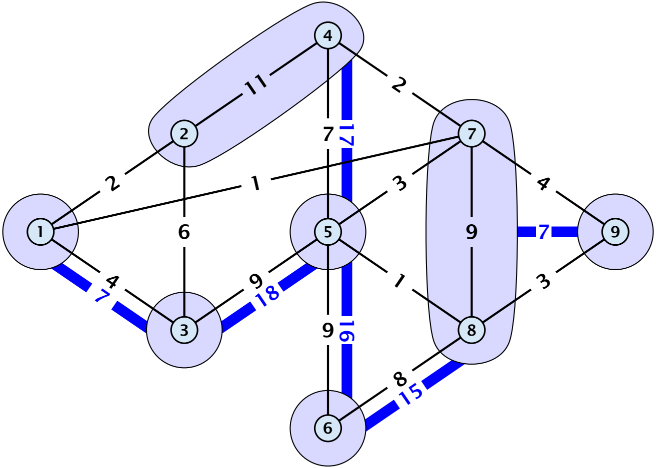

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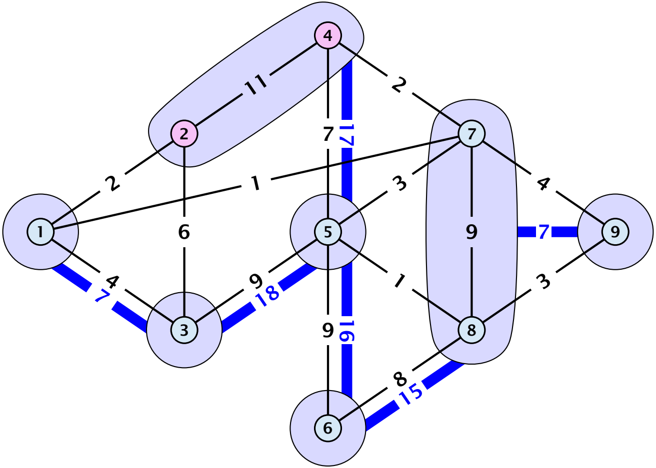

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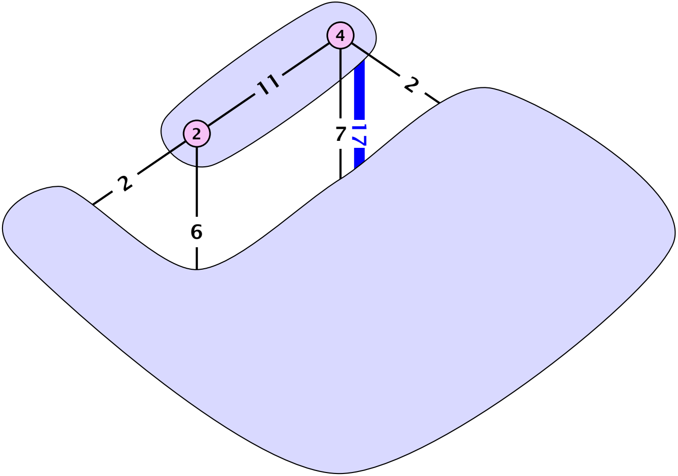

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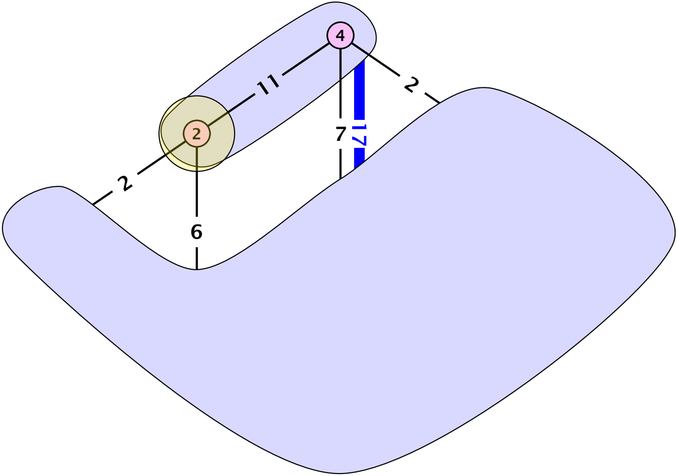

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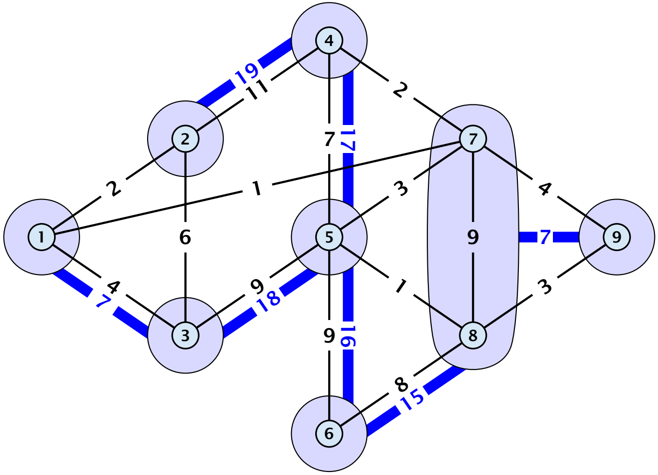

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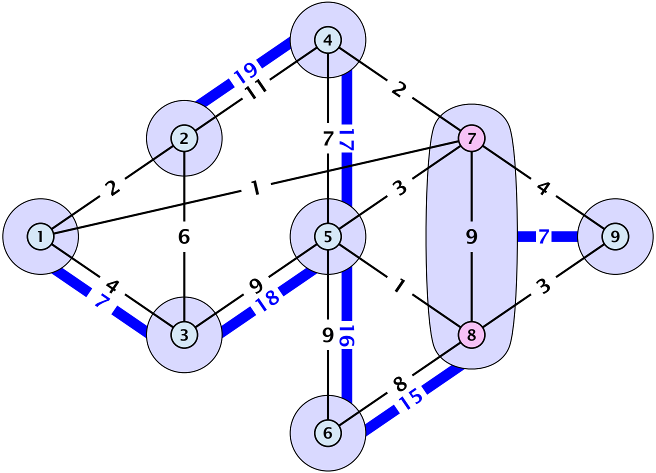

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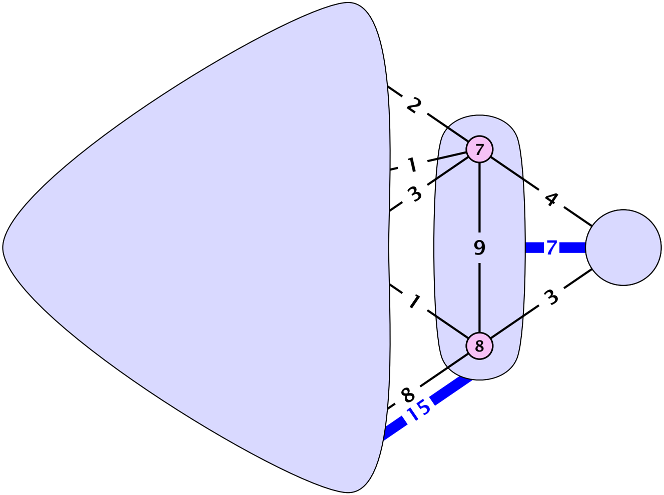

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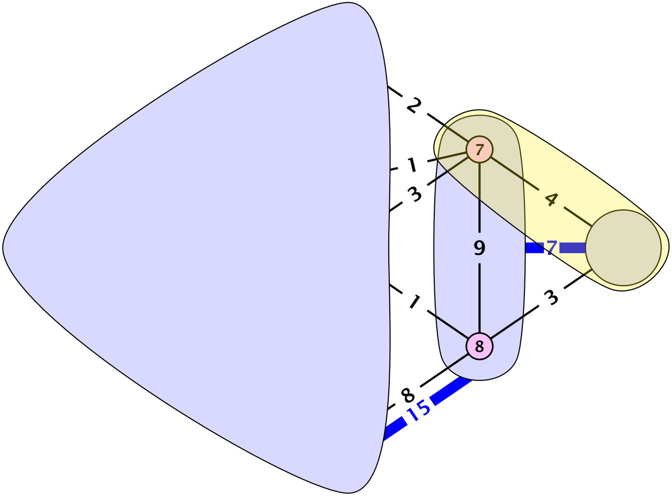

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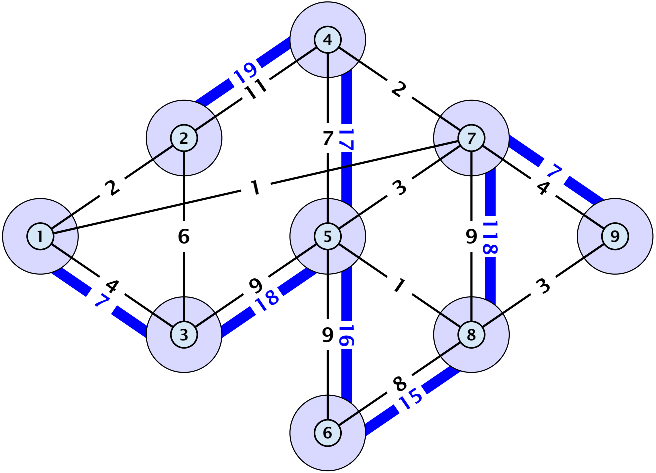

Analysis

Lemma 17
For nodes $s, t, x \in V$ *we have* $f(s,t) \geq \min\{f(s,x), f(x,t)\}$

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Lemma 18
For nodes $s, t, x_1, \ldots, x_k \in V$ *we have*
$f(s,t) \geq \min\{f(s,x_1), f(x_1,x_2), \ldots, f(x_{k-1},x_k), f(x_k,t)\}$

Lemma 19

Let S be some minimum r-s cut for some nodes $r, s \in V$ ($s \in S$), and let $v, w \in S$. Then there is a minimum v-w-cut T with $T \subset S$.

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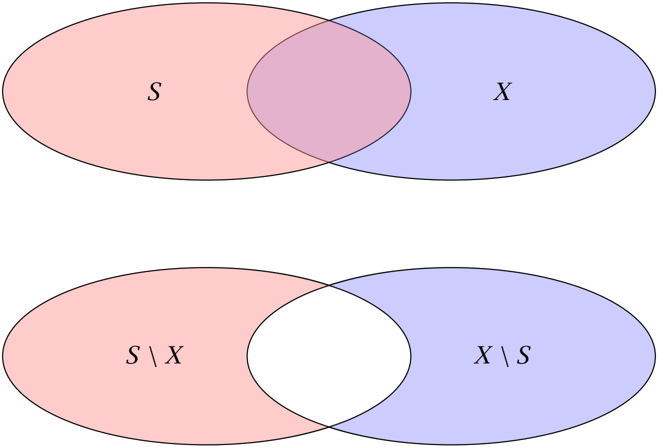

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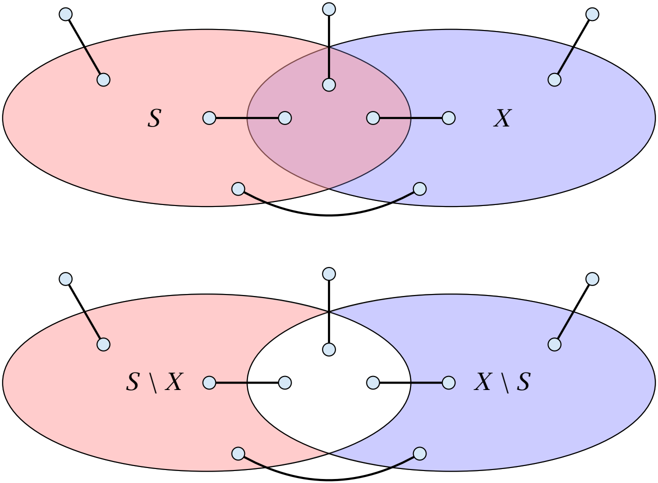

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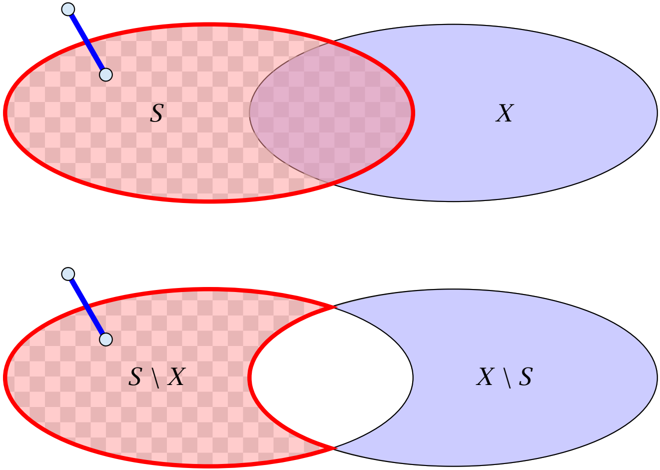

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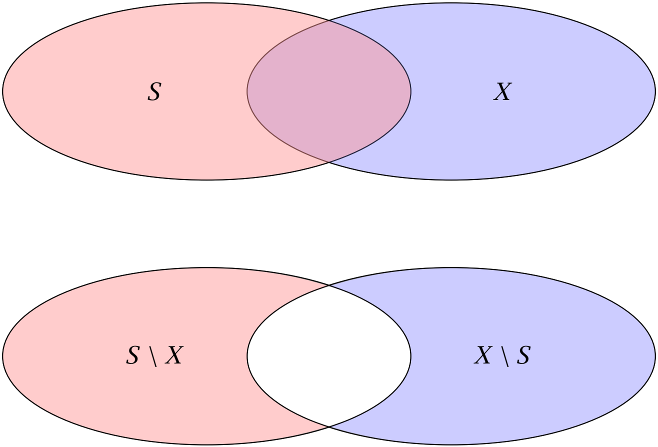

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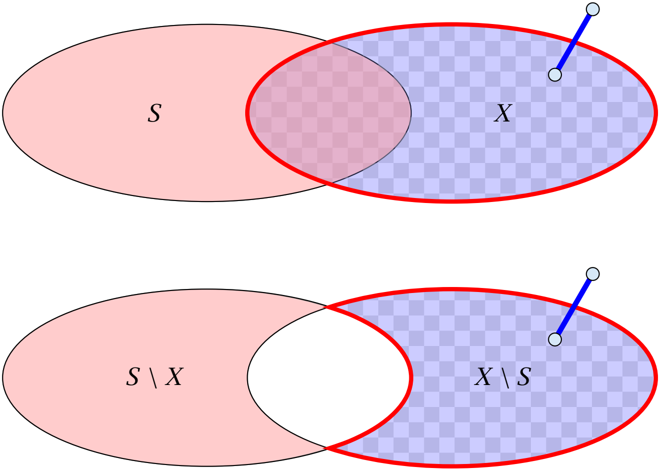

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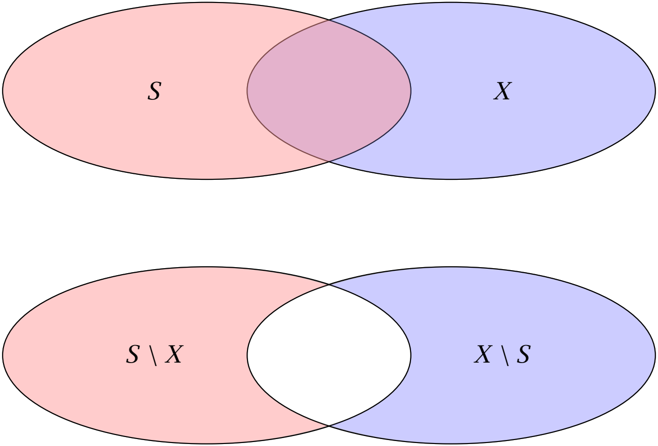

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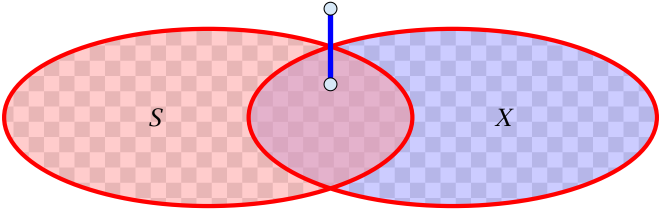

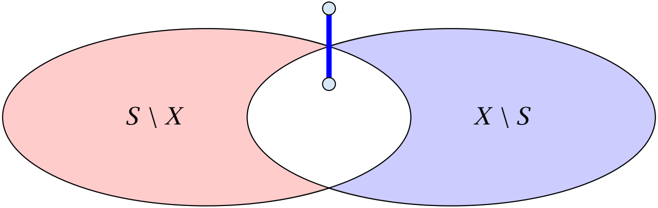

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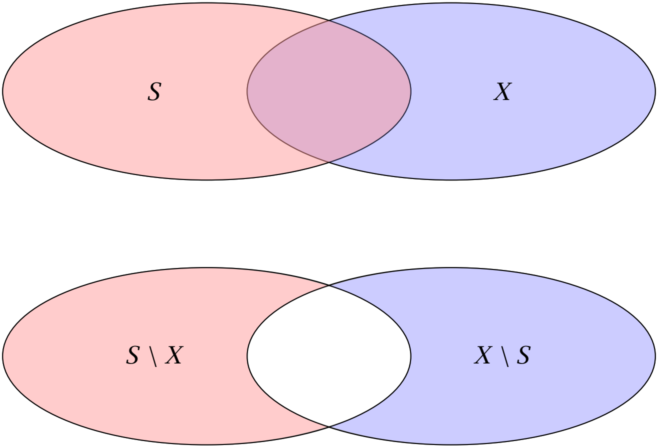

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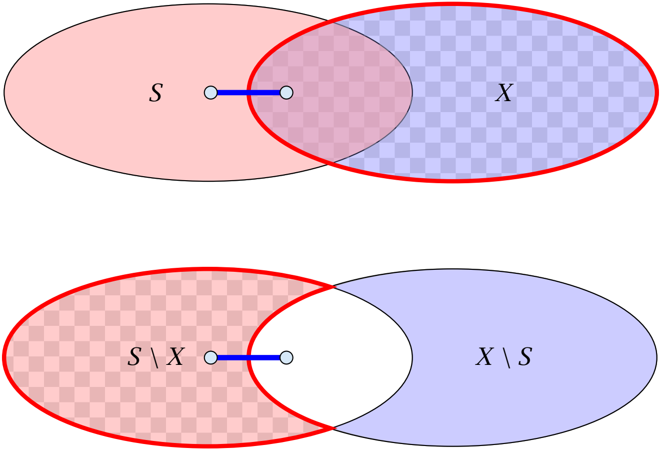

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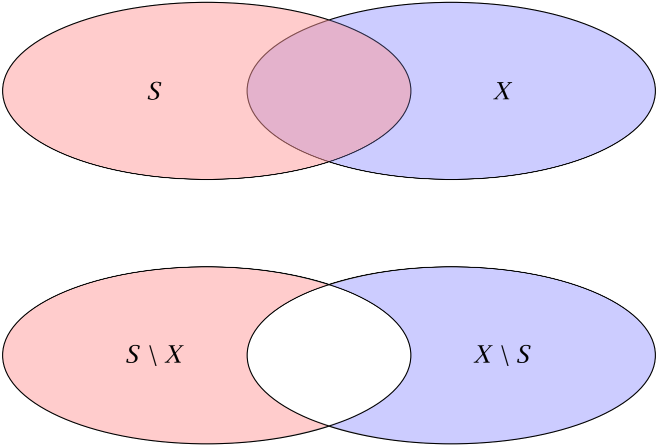

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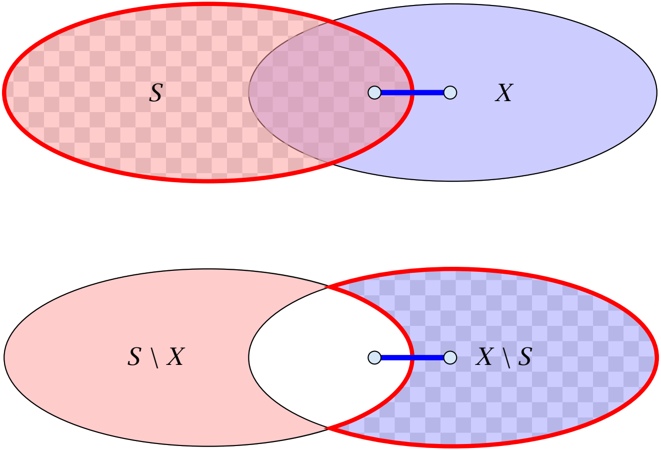

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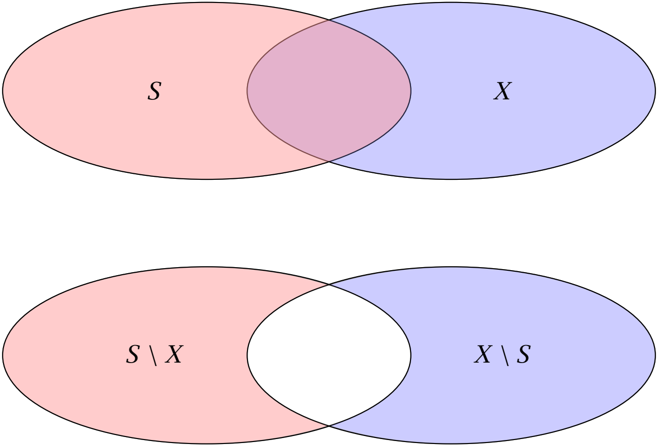

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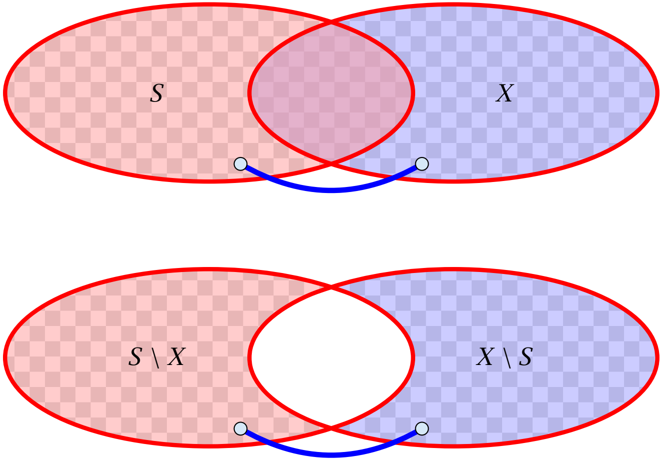

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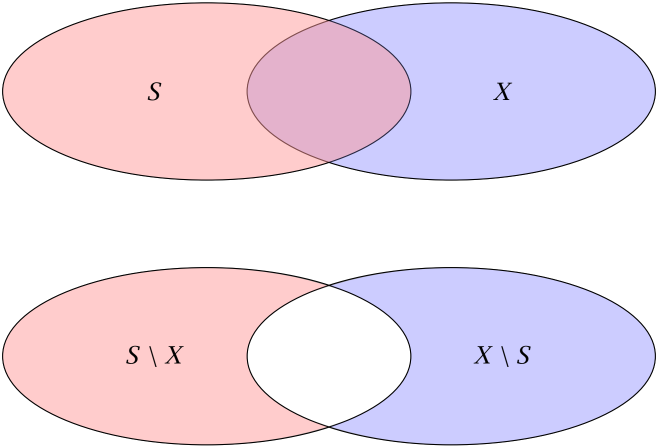

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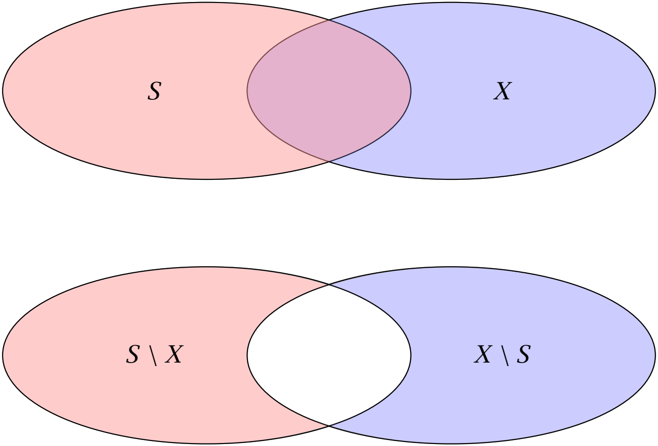

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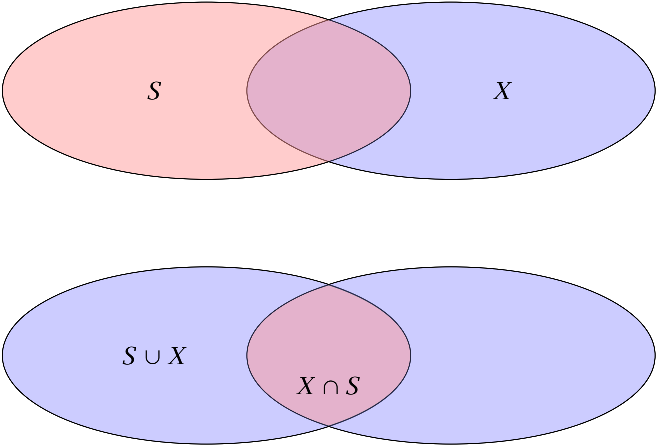

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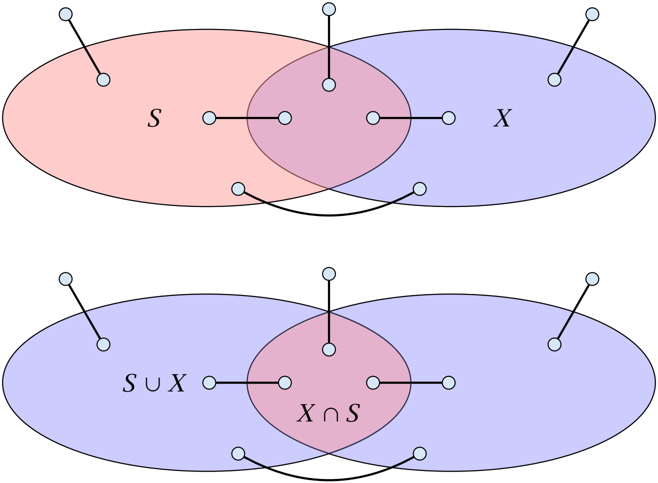

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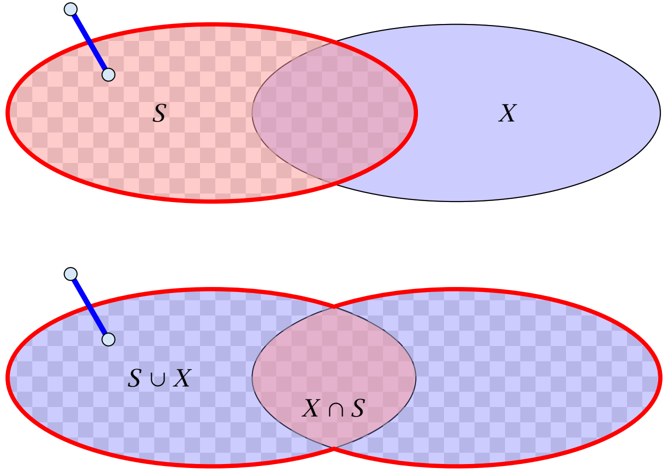

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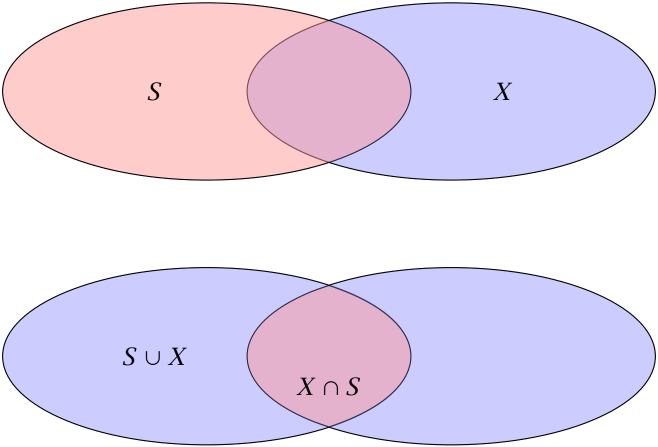

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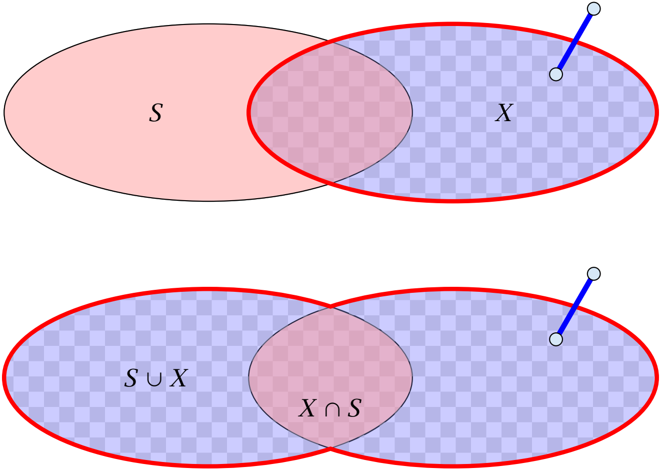

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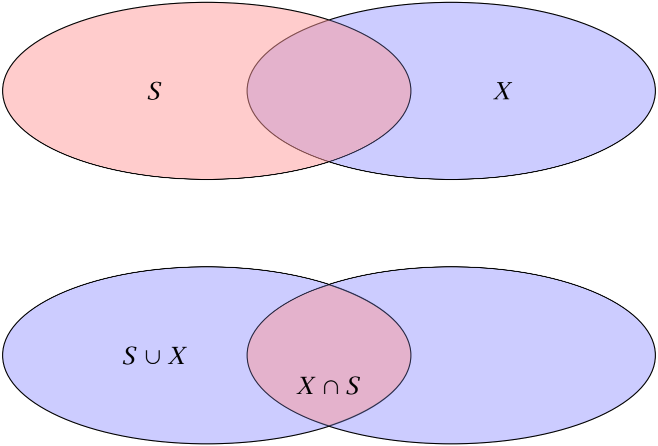

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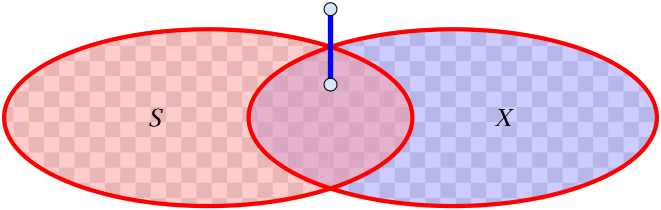

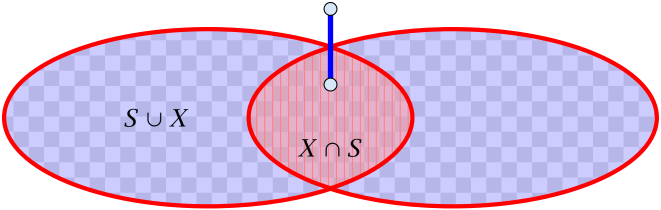

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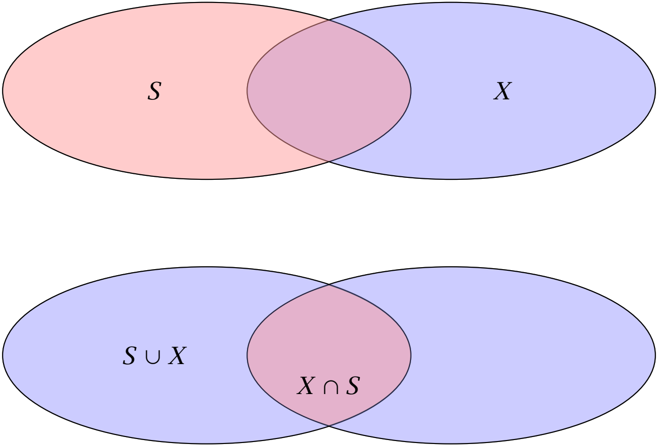

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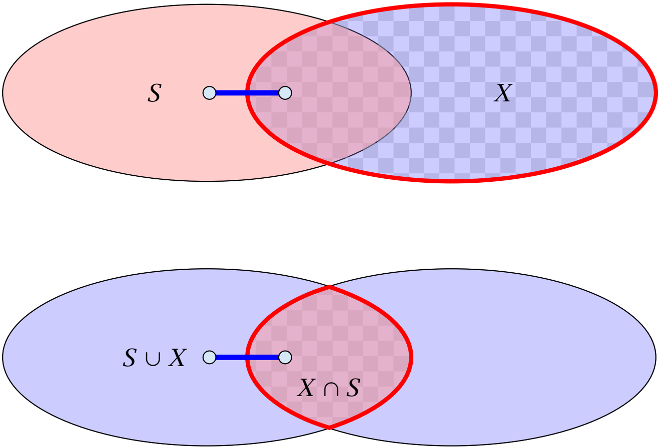

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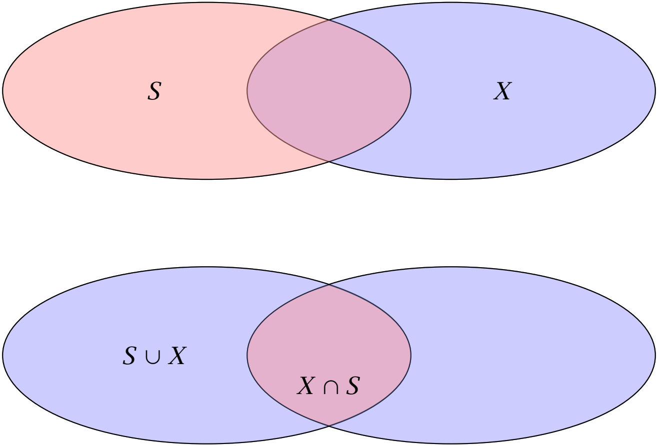

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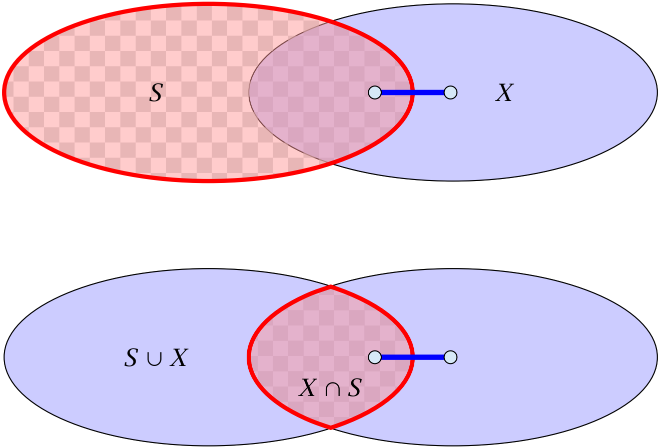

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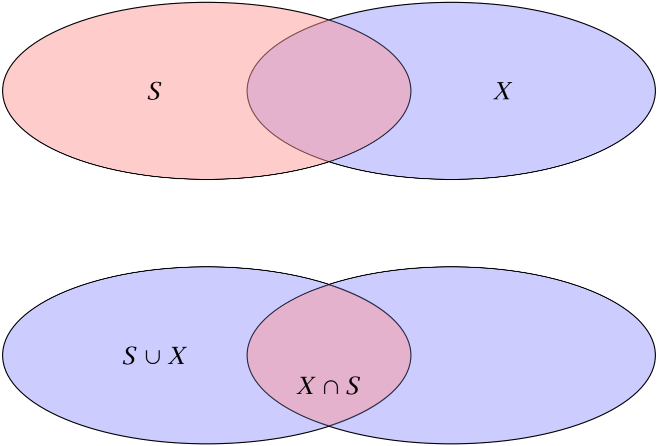

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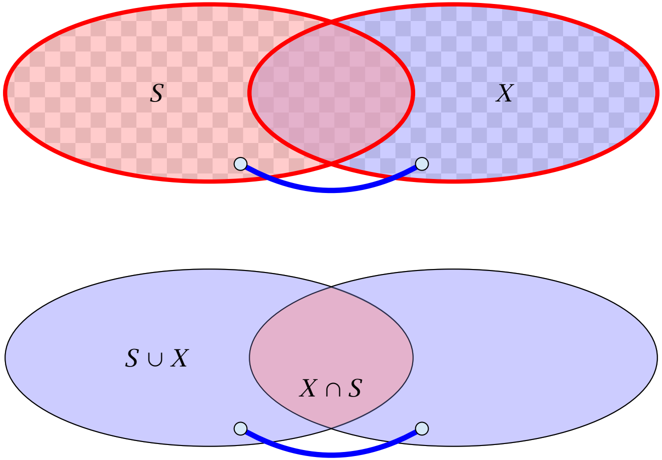

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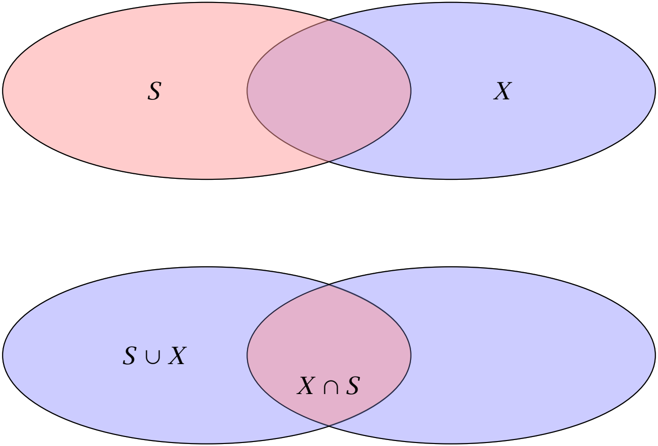

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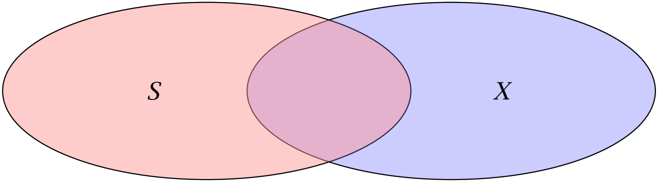

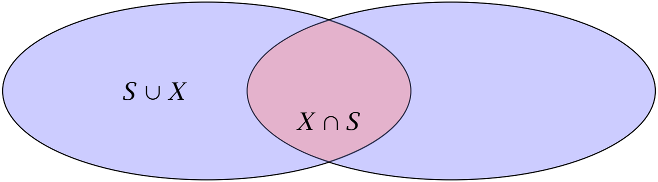

Analysis

Lemma 19 tells us that if we have a graph $G = (V, E)$ and we contract a subset $X \subset V$ that corresponds to some mincut, then the value of $f(s,t)$ does not change for two nodes $s, t \notin X$.

We will show (later) that the connected components that we contract during a split-operation each correspond to some mincut and, hence, $f_H(s,t) = f(s,t)$, where $f_H(s,t)$ is the value of a minimum s-t mincut in graph H.

Analysis

Invariant [existence of representatives]:
For any edge $\{S_i, S_j\}$ in T, there are vertices $a \in S_i$ and $b \in S_j$ such that $w(S_i, S_j) = f(a, b)$ and the cut defined by edge $\{S_i, S_j\}$ is a minimum a-b cut in G.

Analysis

We first show that the invariant implies that at the end of the algorithm T is indeed a cut-tree.

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$$\begin{aligned} f_T(s,t) &= \min_{i \in \{0,\dots,k-1\}} \{w(x_i, x_{i+1})\} \\ &= \min_{i \in \{0,\dots,k-1\}} \{f(x_i, x_{i+1})\} \le f(s,t) \ . \end{aligned}$$

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We first show that the invariant implies that at the end of the algorithm T is indeed a cut-tree.

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- Let $\{x_j, x_{j+1}\}$ be the edge with minimum weight on the path.
- Since by the invariant this edge induces an s-t cut with capacity $f(x_j, x_{j+1})$ we get $f(s,t) \leq f(x_j, x_{j+1}) = f_T(s,t)$.

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- Hence, $f_T(s,t) = f(s,t)$ (flow equivalence).
- The edge $\{x_j, x_{j+1}\}$ is a mincut between s and t in T.
- By invariant, it forms a cut with capacity $f(x_j, x_{j+1})$ in G (which separates s and t).
- Since, we can send a flow of value $f(x_j, x_{j+1})$ btw. s and t, this is an s-t mincut (cut property).

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After the split we have to choose representatives for all edges. For the new edge $\{S_i^a, S_i^b\}$ with capacity $w(S_i^a, S_i^b) = f_H(a,b)$ we can simply choose a and b as representatives.

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Consider an edge $\{X, S_i\}$, and suppose that before the split it used representatives $x \in X$, and $s \in S_i$. Assume that this edge is replaced by $\{X, S_i^a\}$ in the new tree (the case when it is replaced by $\{X, S_i^b\}$ is analogous).

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If $s \in S_i^a$ we can keep x and s as representatives.

Otherwise, we choose x and a as representatives. We need to show that $f(x, a) = f(x, s)$.

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The set B forms a mincut separating a from b. Contracting all nodes in this set gives a new graph G' where the set B is represented by node v_B. Because of Lemma 19 we know that $f'(x,a) = f(x,a)$ as $x, a \notin B$.

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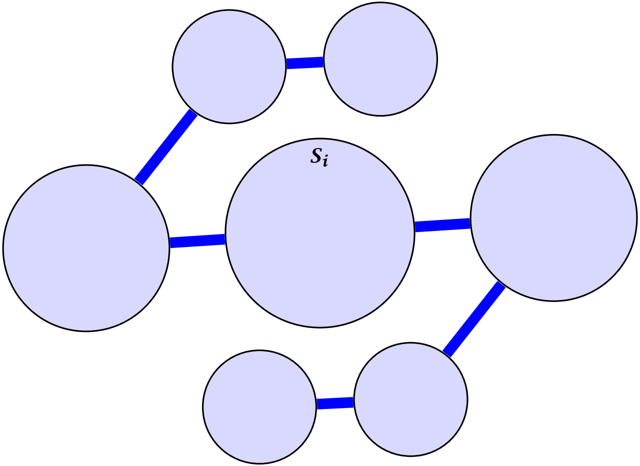
We further have $f'(x, a) \ge \min\{f'(x, v_B), f'(v_B, a)\}$.

Since $s \in B$ we have $f'(v_B, x) \ge f(s, x)$.

Also, $f'(a, v_B) \ge f(a, b) \ge f(x, s)$ since the a-b cut that splits S_i into S_i^a and S_i^b also separates s and x.

Analysis



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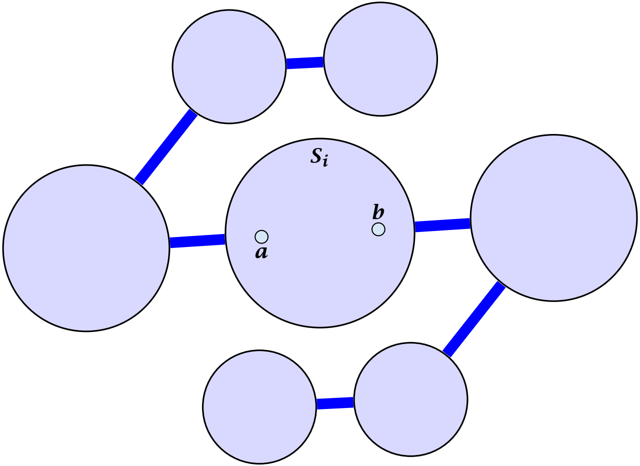

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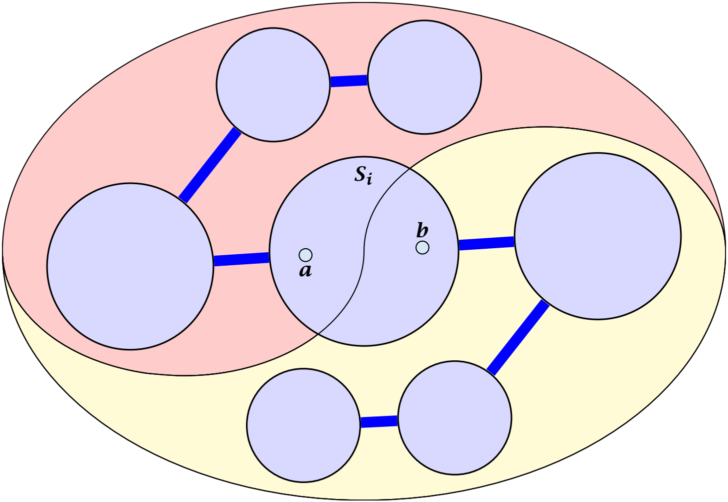

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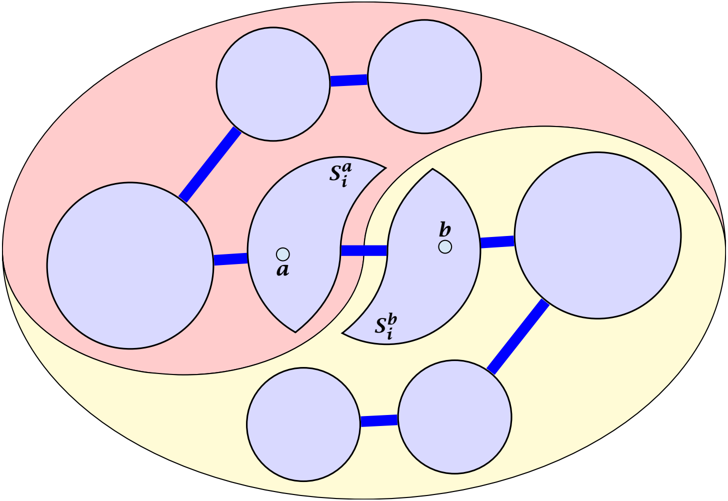

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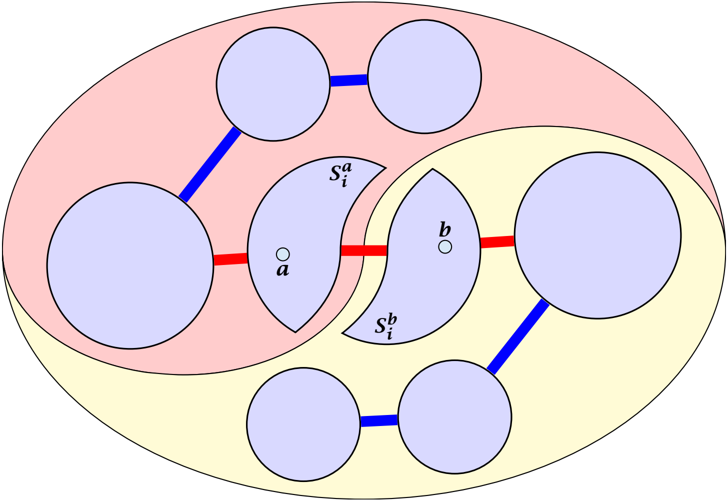

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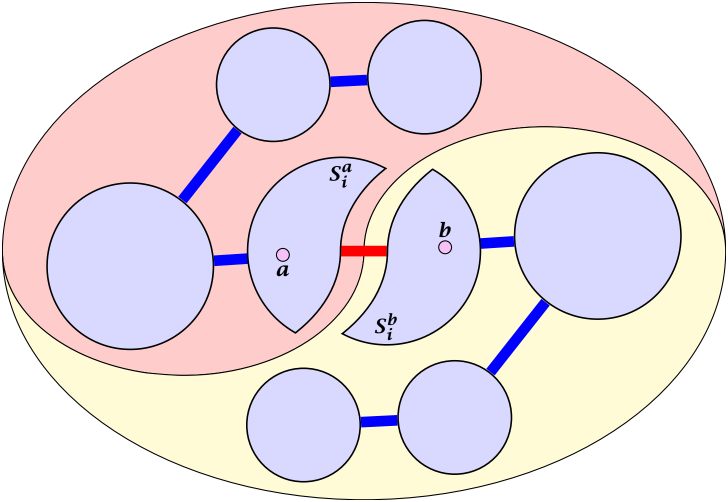

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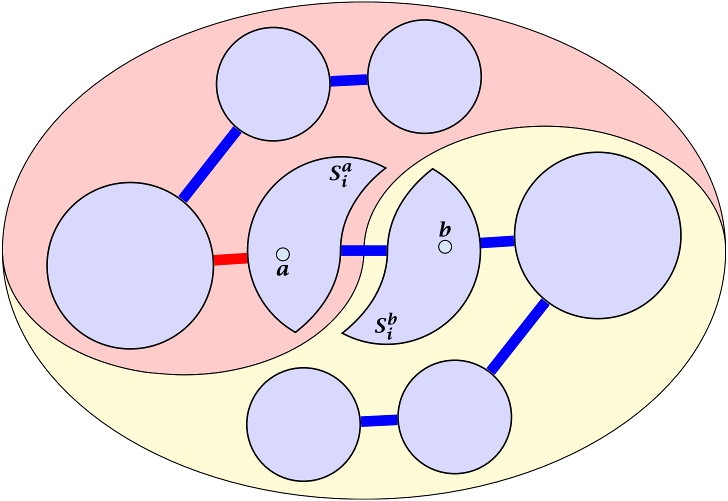

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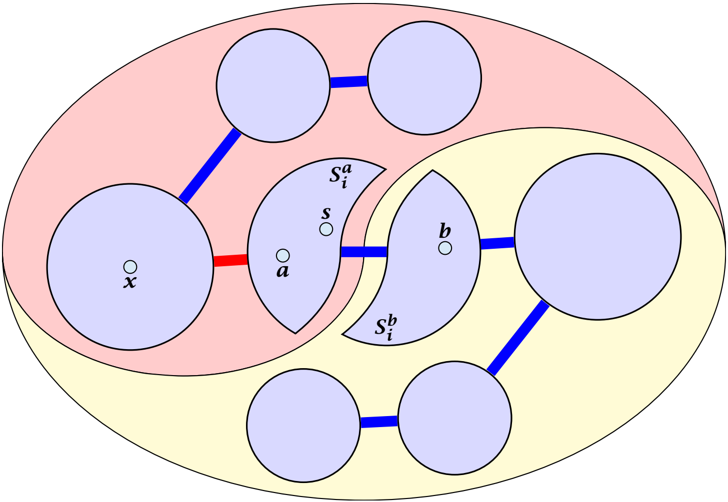

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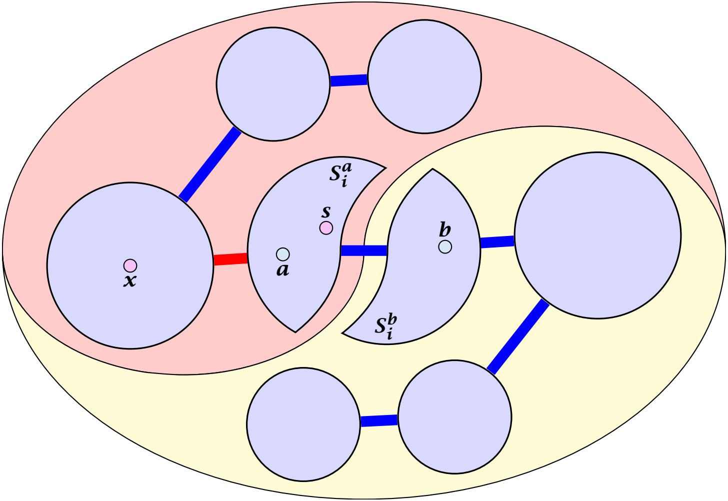

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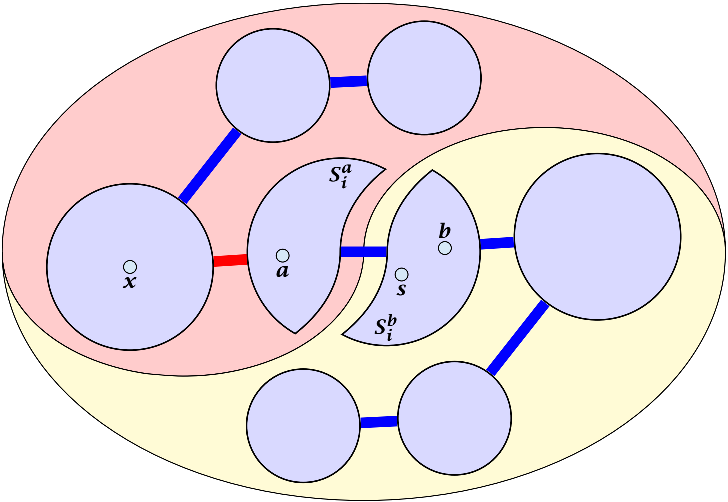

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