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Winter term 2019/20
Assignment 05
November 11, 2019

Efficient Algorithms and Data Structures I

*Deadline: November 18, 10:15 am in the **Efficient Algorithms** mailbox.*

Homework 1 (6 Points)

Solve the following recurrence relations using generating functions:

(a) $a_n = -2a_{n-1} - a_{n-2}$ for $n \geq 2$ with $a_0 = 1$ and $a_1 = -1$.

(b) $a_n = a_{n-1} + 4^{n+1}$ for $n \geq 1$ with $a_0 = 0$.

Homework 2 (5 Points)

Let $f_0 = 0$ and $f_n = 1/n$ for $n > 0$. The harmonic number h_n is defined as $\sum_{i=0}^{n} f_i$. Use the tables on generating functions (p. 102 and 103) to determine the following.

(a) Give a closed-form expression for $F(z) = \sum_{n\geq 0} f_n z^n$.

(b) Use (a) to determine $\sum_{n\geq 0} h_n/3^n$.

Homework 3 (4 Points)

Give tight asymptotic bounds for the following recurrence relation:

$$T(n) = T(\sqrt{n}) + 1$$

Homework 4 (5 Points)

The biologist Andrew is on a research mission in the jungle. He notices that the unicorns living there love to feast on the leaves of the full binary search trees growing in the jungle. In these trees, a node is either a leaf or it has two children. Andrew wants to understand how many types of full binary search trees with $n + 1$ leaves exist.

(a) Let b_n be the number of full rooted binary search trees with $n+1$ leaves (conveniently named $0, 1, \ldots, n$). We set $b_0 = 1$. Show that for $n \geq 1$,

$$b_n = \sum_{k=0}^{n-1} b_k b_{n-1-k} \quad .$$

(b) Let $B(z) = \sum_{n\geq 0} b_n z^n$. Show that $B(z) = zB(z)^2 + 1$.

(c) Solve the above polynomial for $B(z)$ using the fact that $\lim_{z\to 0} B(z)$ must be 1 (i.e. only one of the two possible solutions is valid).

(d) Show that the number of full rooted binary search trees with $n+1$ leaves is $\frac{1}{n+1}\binom{2n}{n}$.

Hint: Use the equality

$$\sqrt{1-4z} = -2\left(-\frac{1}{2} + \sum_{n\geq 1} \frac{1}{n}\binom{2(n-1)}{n-1} z^n\right) \ .$$

Tutorial Exercise 1

(a) Solve the recurrence

$$g_0 = 1;$$
$$g_n = \sum_{i=1}^{n} i \cdot g_{n-i} \text{for } n \geq 1.$$

Hint: Use the fact that
$$\sum_{n\geq 0} F_{2n} z^n = \frac{z}{1-3z+z^2} \ ,$$
where F_n is the nth Fibonacci number (with $F_0 = 0$).

(Extra) Prove the hint!

Tutorial Exercise 2

The *depth* of a node v in a binary search tree is the number of edges on the shortest path from v to the root of the tree.
Show that there exists a binary search tree with n nodes with height in $\omega(\log(n))$ and average depth in $\mathcal{O}(\log(n))$.

Tutorial Exercise 3

In this exercise, we show that the *rotation distance* between binary trees of n nodes is $\mathcal{O}(n)$. For trees T_1 and T_2 over the same nodes, the rotation distance is defined as the number of rotations needed to transform tree T_1 into tree T_2.

(a) A *right-linear chain* is a tree in which every internal node including the root has no left child. Show that any binary tree of n nodes can be transformed into a right-linear chain using at most n rotations.

(b) Conclude that the rotation distance is only $\mathcal{O}(n)$.

A generating function is a clothesline on which we hang up a sequence of numbers for display.

- H. Wilf