Technische Universität München
Fakultät für Informatik
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Assignment 12
January 14, 2019

Efficient Algorithms and Data Structures I

*Deadline: January 21, 2019, 10:15 am in the **Efficient Algorithms** mailbox.*

Homework 1 (4 Points)

The edge connectivity of an undirected graph is the minimum number of edges that must be removed to disconnect the graph.

(a) Show that the edge connectivity of any n-node graph is at most $n-1$.

(b) Show how to determine the edge connectivity of an undirected graph G of n vertices by running a black-box maximum-flow algorithm on at most n flow networks.

Homework 2 (4 Points)

Prove or disprove the following statements.

(a) There is a constant $c > 1$ so that for any $n \geq 4$ there exists a network on n nodes with a single maximum flow and at least c^n minimum cuts.

(b) There is a constant $c > 1$ so that for any $n \geq 4$ there exists a network on n nodes with a single minimum cut and at least c^n integral maximum flows.

Homework 3 (5 Points)

Given s and t in a network $G = (V, E)$, we want to identify a minimum (s,t)-cut $A \subset V$ that minimizes the number of edges leaving A among all minimum (s,t)-cuts. The edges in the network have integer capacities.
Show how to rescale the capacities of G such that a single max-flow/min-cut computation yields the desired cut.

Homework 4 (7 Points)

The department has s students $S_1, \dots, S_s$ looking for a thesis topic. There are t open thesis topics $T_1, \dots, T_t$. For each thesis topic, there is at least one student qualified for it and each student is qualified for at least one topic. Each topic is offered by exactly one of the r research groups $R_1, \dots, R_r$.
Each student must pick a thesis topic he is qualified for. No two students may pick the same topic and research group i can take care of at most u_i students.
Given full information about students, topics and research groups, is there a way to distribute the thesis topics?

(a) Show how to formulate the above problem as a Maximum Flow Problem. Explain the different elements of your construction.

(b) Given an integral maximum flow in your network, show how to determine whether a distribution scheme as desired is possible and prove that your method works.

Tutorial Exercise 1

Show that there always exists a sequence of at most m augmenting paths that compute the maximum flow in a network of m edges.

Tutorial Exercise 2

Let f be a flow in a network, and let α be a real number. The scalar flow product, denoted $\alpha \cdot f$, is a function from $V \times V$ to R defined by

$$(\alpha \cdot f)(u,v) = \alpha \cdot f(u,v)$$

Prove that the feasible flows in a network form a convex set. That is, show that if f_1 and f_2 are flows, then so is $\alpha f_1 + (1-\alpha) f_2$ for $0 \leq \alpha \leq 1$.

Tutorial Exercise 3

An unknown attacker has targeted a flow network of n nodes, source s and target t and unit capacities on all edges. The attacker has separated nodes s and t by removing k edges, where k is the capacity of a minimum cut separating s and t.
Detective Dirk wants to find out which edges have been removed by the attacker. He knows how the network looked like before the attack, but he does not know how it looks like after the attack. However, Dirk can *ping* vertices in the damaged network. A ping for vertex v is successful, if the damaged network still contains a path from s to v. Dirk wants to work as fast as possible, so he aims to minimize the number of pings.

(a) Suppose that the network is a simple path from s to t. Show how $\mathcal{O}(\log n)$ pings are sufficient to find the damaged edge.

(b) Dirk conjectures that there exists an algorithm needing only $\mathcal{O}(k \log n)$ pings in an arbitrary network. Show that Dirk is right by giving such an algorithm. Prove correctness of your algorithm and prove that it reaches the required time bound.

> Shortly after the "iron curtain" fell in 1990, an American and a Russian, who had both worked on the development of weapons, met. The American asked: "When you developed the Bomb, how were you able to perform such an enormous amount of computing with your weak computers?" The Russian responded: "We used better algorithms."
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> – Y. Dinitz