Technische Universität München
Fakultät für Informatik
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Winter term 2018/19
Assignment 4
November 5, 2017

# Efficient Algorithms and Data Structures I

*Deadline: November 12, 10:15 am in the **Efficient Algorithms** mailbox.*

## Homework 1 (4 Points)

Solve the following recurrence relation using the characteristic polynomial:

$$a_n = 3a_{n-2} + 2a_{n-3} \text{ for } n \geq 3 \qquad \text{with } a_0 = 3, a_1 = 2, a_2 = 11.$$

## Homework 2 (5 Points)

Calculate the value of $\sum_{i=1}^{n} i^2$ by setting up a recurrence relation; transforming it into a homogeneous relation via the method developed in the lecture and then solving this relation via the characteristic polynomial.

## Homework 3 (5 Points)

This exercise will provide an alternative method for analyzing homogeneous linear recurrences. Consider the following recurrence:

$$a_n = 6a_{n-1} - 11a_{n-2} + 6a_{n-3} \qquad \text{for } n \geq 3 \ .$$

with $a_0 = 3$, $a_1 = 6$, and $a_2 = 14$.
Let $b_n = a_{n-1}$ for $n \geq 1$ and $c_n = b_{n-1}$ for $n \geq 2$. Finally, let $\vec{x}_n = (a_{n+2}, b_{n+2}, c_{n+2})^T$ for $n \geq 0$.

1. Determine a matrix $M \in \mathbb{R}^{3\times3}$ such that $\vec{x}_n = M \cdot \vec{x}_{n-1}$ for $n \geq 1$.

2. Let $\lambda$ be an eigenvalue of $M$. Show that $\vec{x}_0$ can be chosen such that $\lambda^n \cdot \vec{x}_0$ is a solution to the recurrence relation derived in the first part.

3. Determine the eigenvalues of $M$ and, for each eigenvalue, determine a corresponding eigenvector.

4. Clearly $x_0 = (14, 6, 3)^T$. Use the eigenvectors and eigenvalues of $M$ to solve the recurrence relation derived in the first part.

5. Determine a closed form for $a_n$.

## Homework 4 (6 Points)

The honey bee Armin randomly snacks on his two favorite flowers, a red rose and a yellow tulip. Whenever he is on the rose, he stays there with probability 0.9 and moves to the tulip with probability 0.1. If he is on the tulip, he stays there with probability 0.8 and moves to the rose with probability 0.2.

1. Let $r_n$ and $t_n$ denote the probability that Armin is after timestep $n$ on the rose and the tulip, respectively. Express $r_n$ and $t_n$ using a two-dimensional recurrence relation for $n \geq 1$, i.e. find a matrix $P$ such that

$$\begin{pmatrix} r_n \\ t_n \end{pmatrix} = P \cdot \begin{pmatrix} r_{n-1} \\ t_{n-1} \end{pmatrix} \; .$$

   Also draw the corresponding Markov chain.

2. Beatrix, the queen of the hive, asks for a closed form of $r_n$ and $t_n$, assuming that Armin starts on the rose. Use the eigenvectors of $P$.

3. Determine the limiting distribution of Armin over the flowers. How is it linked to the eigenvectors of $P$?

**Hint:** Use the approach from Homework 3.

## Tutorial Exercise 1

Solve the following recurrence relation using generating functions:

$$a_n = a_{n-1} + 2^{n-1} \text{ for } n \geq 1 \text{ with } a_0 = 2 \; .$$

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[The PageRank Computation] is es-
sentially the determination of the
limiting distribution of a random
walk on the web graph.
     - Page, Brin, Motwani, Winograd
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