

Preflows

Definition 1

An (s, t) -preflow is a function $f : E \mapsto \mathbb{R}^+$ that satisfies

For each edge $e \in E$

$$0 \leq f(e) \leq c(e)$$

For each vertex $v \in V$

$$\sum_{e \in E^+} f(e) - \sum_{e \in E^-} f(e) \leq b(v)$$

For each vertex $v \in V$

$$\sum_{e \in E^+} f(e) - \sum_{e \in E^-} f(e) = 0$$

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(capacity constraints)

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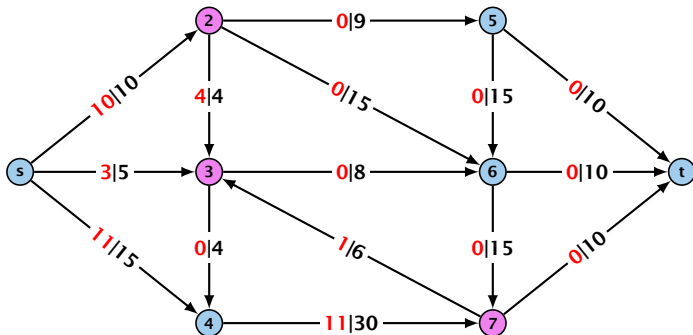
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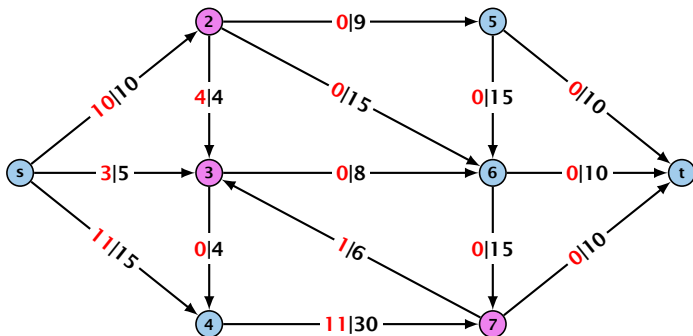
Preflows

Example 2



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A node that has $\sum_{e \in \text{out}(v)} f(e) < \sum_{e \in \text{into}(v)} f(e)$ is called an **active node**.

Preflows

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A **labelling** is a function $\ell : V \rightarrow \mathbb{N}$. It is **valid** for preflow f if

- ▶ $\ell(u) \leq \ell(v) + 1$ for all edges (u, v) in the residual graph G_f (only non-zero capacity edges!!!)

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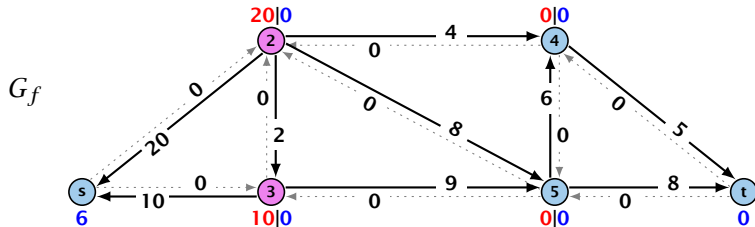
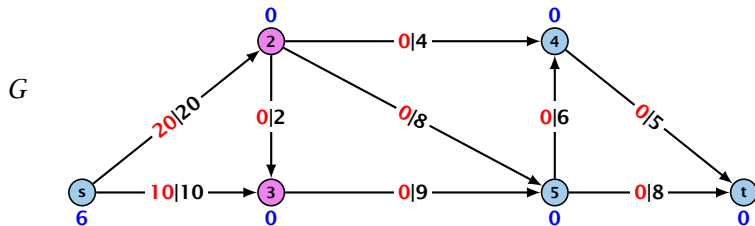
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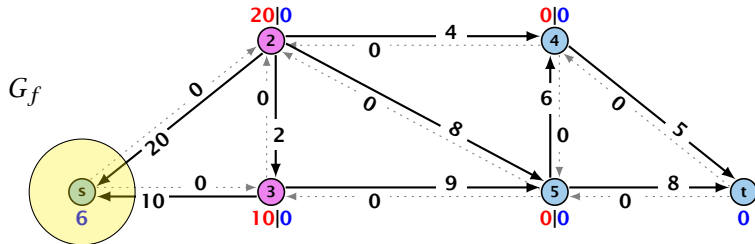
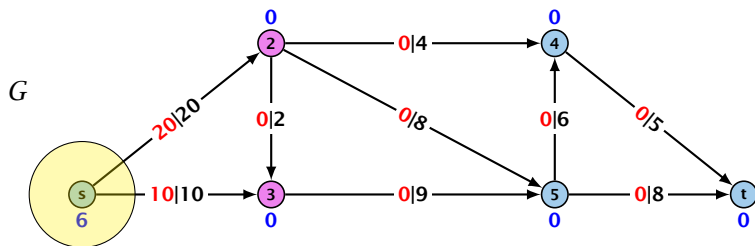
Intuition:

The labelling can be viewed as a height function. Whenever the height from node u to node v decreases by more than 1 (i.e., it goes very steep downhill from u to v), the corresponding edge must be saturated.

Preflows



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Lemma 3

A *preflow* that has a valid labelling saturates a cut.

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Lemma 4

A *flow* that has a valid labelling is a maximum flow.

Push Relabel Algorithms



Push Relabel Algorithms

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- ▶ start with some preflow and some valid labelling

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- ▶ start with some preflow and some valid labelling
- ▶ successively change the preflow while maintaining a valid labelling
- ▶ stop when you have a flow (i.e., no more active nodes)

Changing a Preflow

An arc (u, v) with $c_f(u, v) > 0$ in the residual graph is **admissible** if $\ell(u) = \ell(v) + 1$ (i.e., it goes downwards w.r.t. labelling ℓ).

The push operation

Consider an active node u with **excess flow**

$f(u) = \sum_{e \in \text{into}(u)} f(e) - \sum_{e \in \text{out}(u)} f(e)$ and suppose $e = (u, v)$ is an admissible arc with residual capacity $c_f(e)$.

We can send flow $\min\{c_f(e), f(u)\}$ along e and obtain a new preflow. The old labelling is still valid (!!!).

the arc e is deleted from the residual graph

the node u becomes inactive

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The arc e is removed from the residual graph.

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- ▶ **saturating push** : $\min\{f(u), c_f(e)\} = c_f(e)$
the arc e is deleted from the residual graph
- ▶ **non-saturating push** : $\min\{f(u), c_f(e)\} = f(u)$
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Push Relabel Algorithms



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Consider an active node u that does not have an outgoing admissible arc.

Increasing the label of u by 1 results in a valid labelling.

- ▶ Edges (w, u) incoming to u still fulfill their constraint $\ell(w) \leq \ell(u) + 1$.
- ▶ An outgoing edge (u, w) had $\ell(u) < \ell(w) + 1$ before since it was not admissible. Now: $\ell(u) \leq \ell(w) + 1$.

Push Relabel Algorithms

Intuition:

We want to send flow downwards, since the source has a height/label of n and the target a height/label of 0 . If we see an active node u with an admissible arc we push the flow at u towards the other end-point that has a lower height/label. If we do not have an admissible arc but excess flow into u it should roughly mean that the level/height/label of u should rise. (If we consider the flow to be water then this would be natural.)

Note that the above intuition is very incorrect as the labels are integral, i.e., they cannot really be seen as the height of a node.

Reminder

- ▶ In a **preflow** nodes may not fulfill conservation constraints; a node may have more incoming flow than outgoing flow.
- ▶ Such a node is called **active**.
- ▶ A labelling is **valid** if for every edge (u, v) in the residual graph $\ell(u) \leq \ell(v) + 1$.
- ▶ An arc (u, v) in residual graph is **admissible** if $\ell(u) = \ell(v) + 1$.
- ▶ A **saturating push** along e pushes an amount of $c(e)$ flow along the edge, thereby saturating the edge (and making it disappear from the residual graph).
- ▶ A **non-saturating push** along $e = (u, v)$ pushes a flow of $f(u)$, where $f(u)$ is the **excess flow** of u . This makes u inactive.

Push Relabel Algorithms

Algorithm 3 $\text{maxflow}(G, s, t, c)$

```
1: find initial preflow  $f$ 
2: while there is active node  $u$  do
3:     if there is admiss. arc  $e$  out of  $u$  then
4:          $\text{push}(G, e, f, c)$ 
5:     else
6:          $\text{relabel}(u)$ 
7: return  $f$ 
```

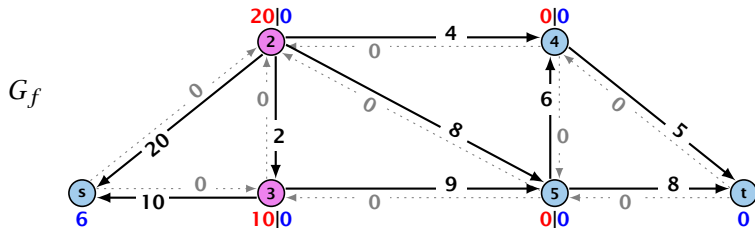
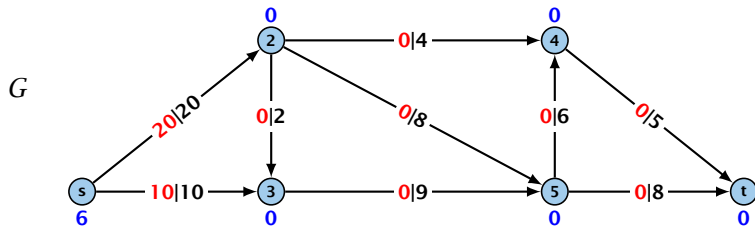
Push Relabel Algorithms

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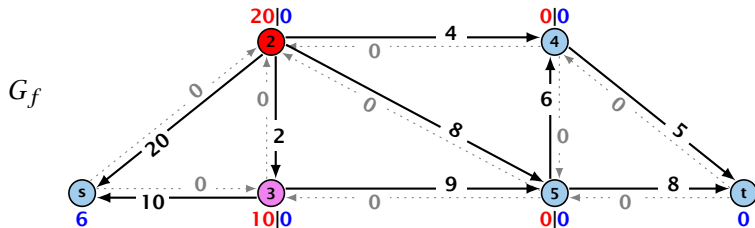
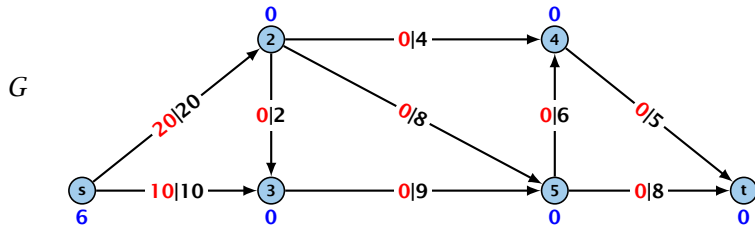
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In the following example we always stick to the same active node u until it becomes inactive but this is not required.

Preflow Push Algorithm



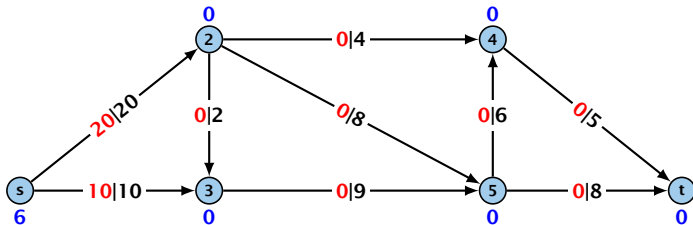
Preflow Push Algorithm



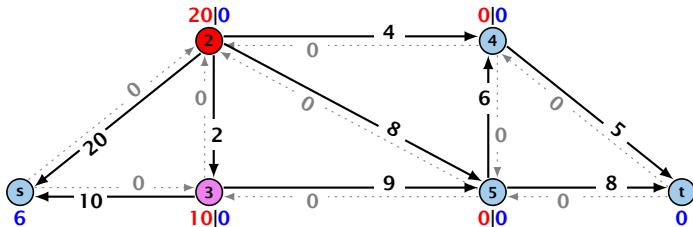
Preflow Push Algorithm

relabel

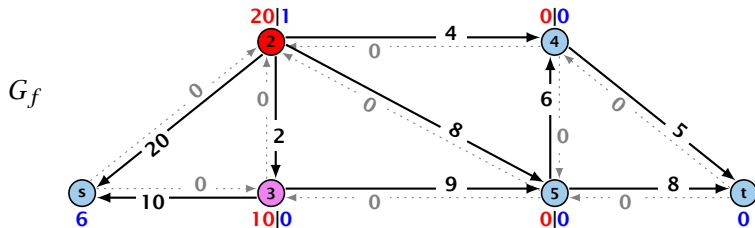
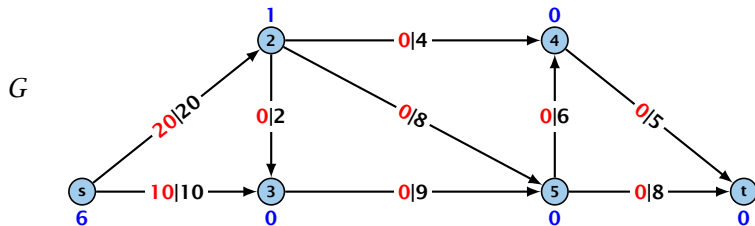
G



G_f



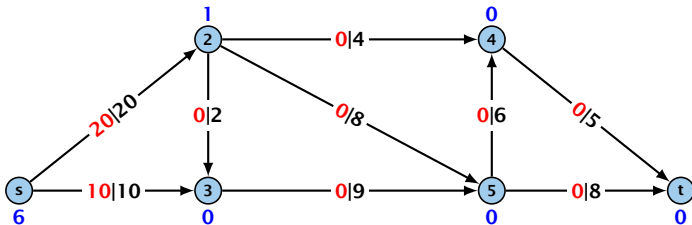
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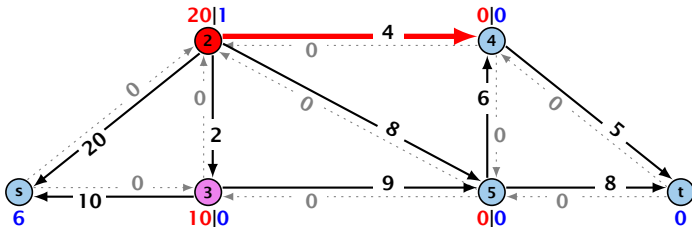
Preflow Push Algorithm

push

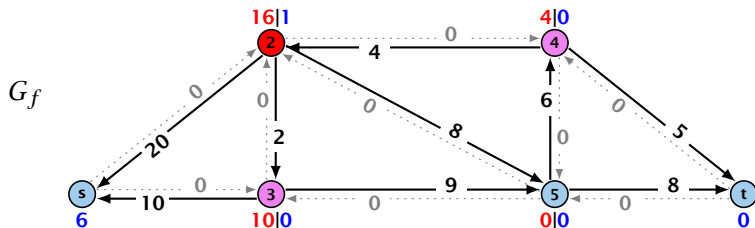
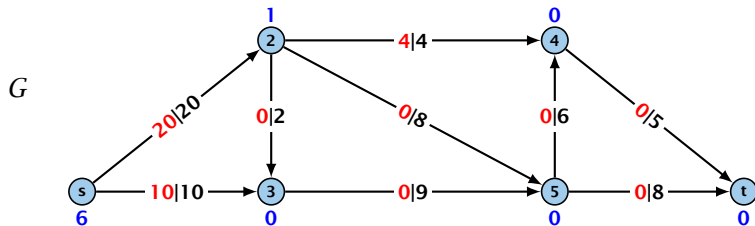
G



G_f



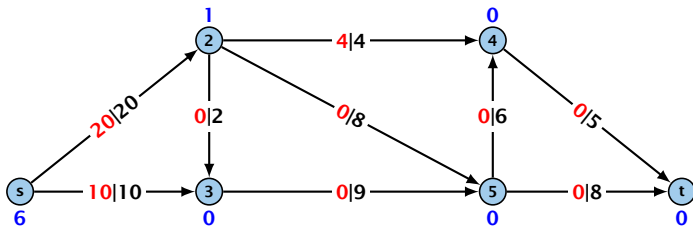
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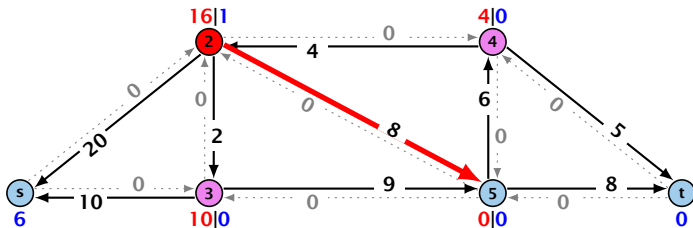
Preflow Push Algorithm

push

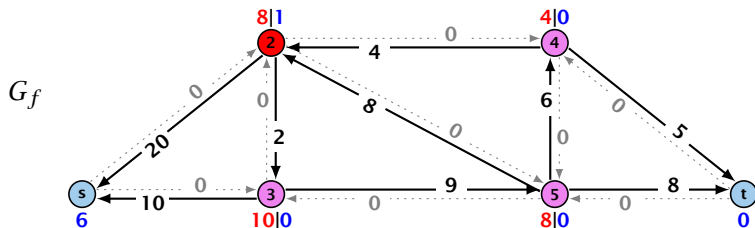
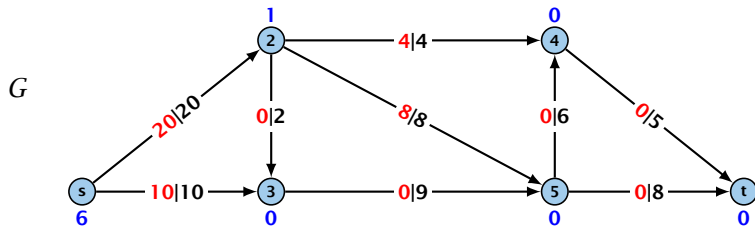
G



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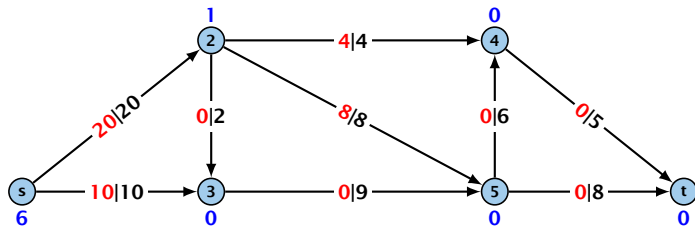
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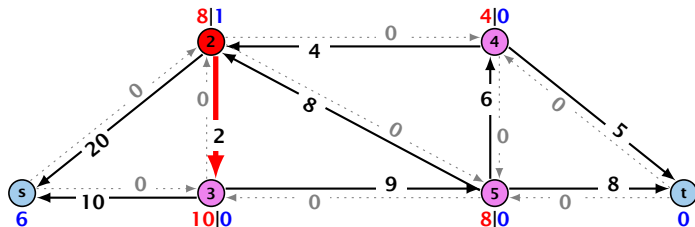
Preflow Push Algorithm

push

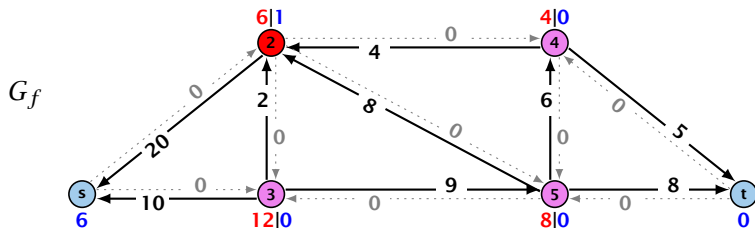
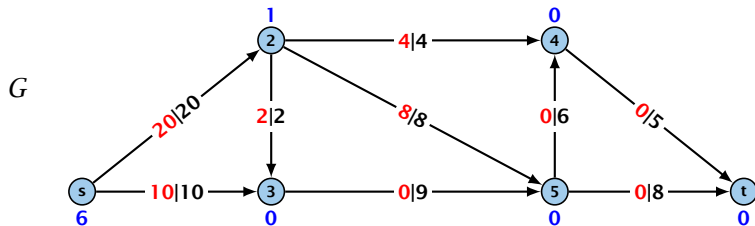
G



G_f



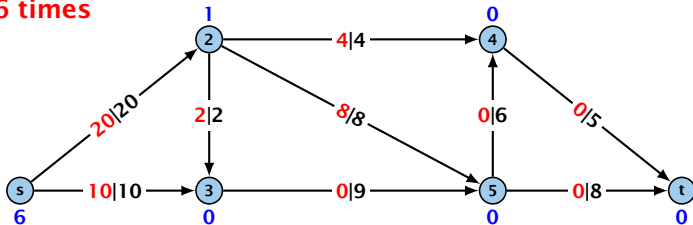
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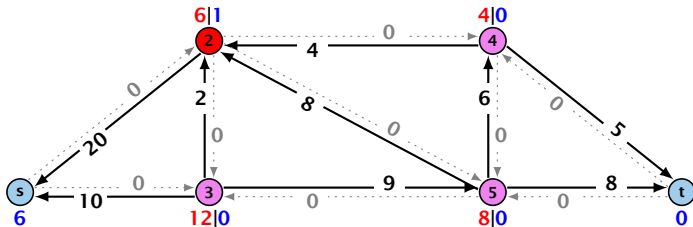
Preflow Push Algorithm

relabel 6 times

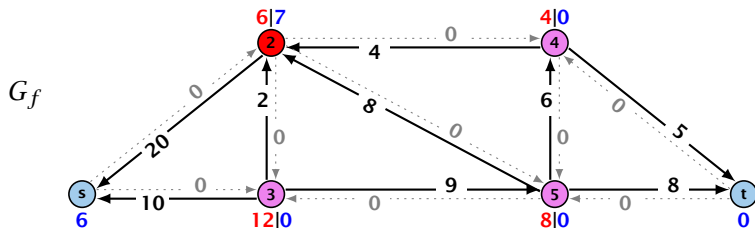
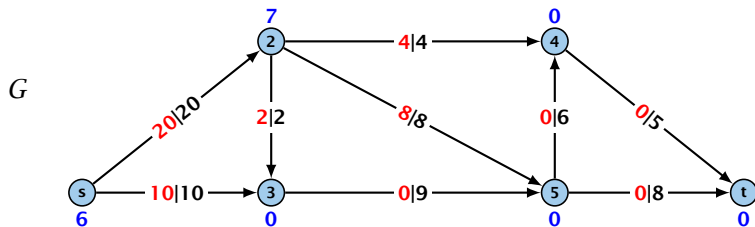
G



G_f



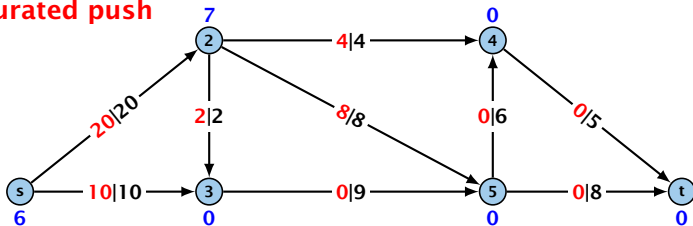
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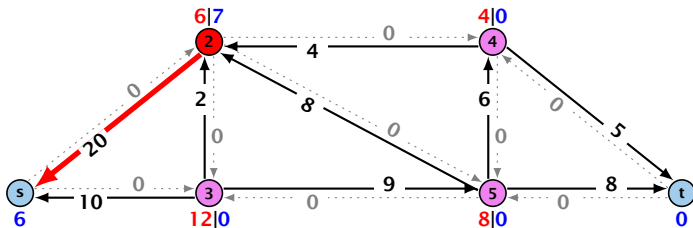
Preflow Push Algorithm

non-saturated push

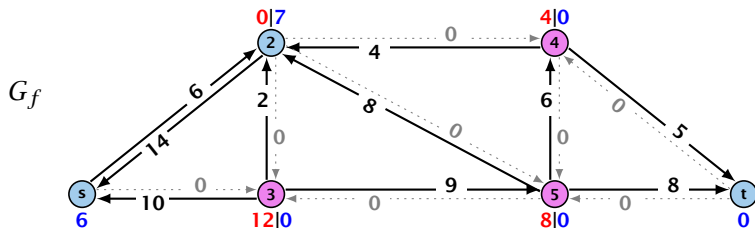
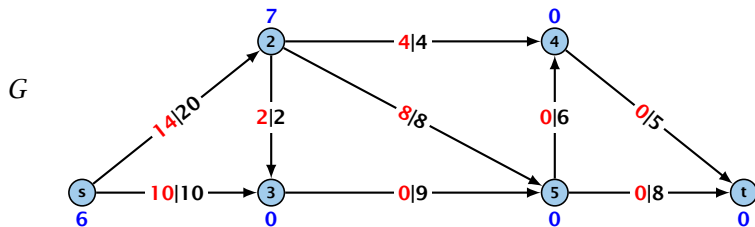
G



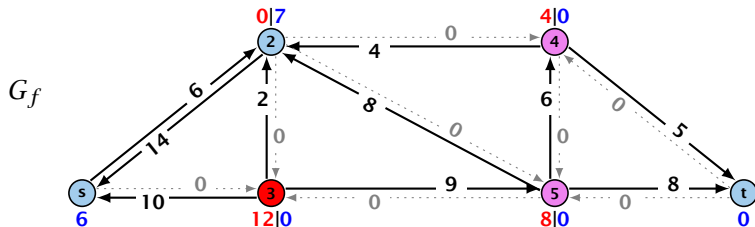
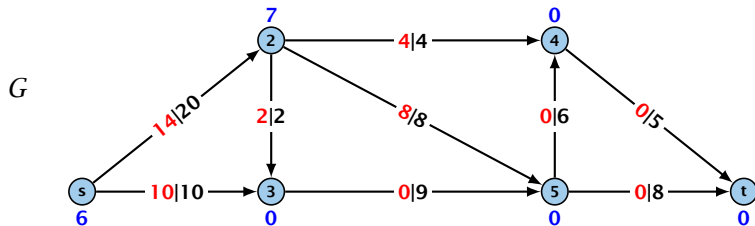
G_f



Preflow Push Algorithm



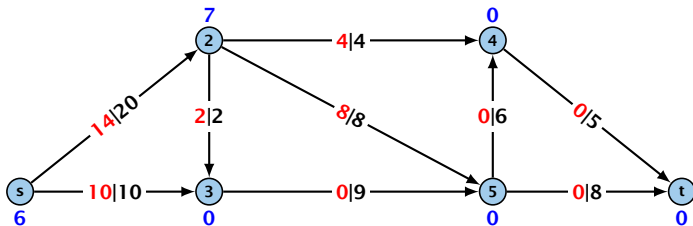
Preflow Push Algorithm



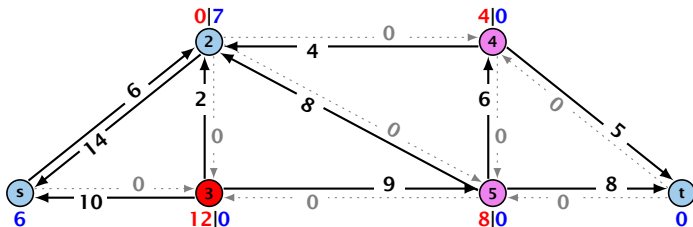
Preflow Push Algorithm

relabel

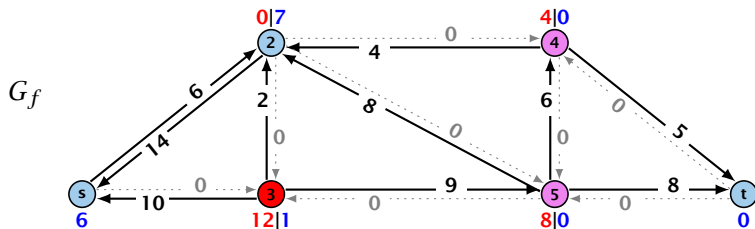
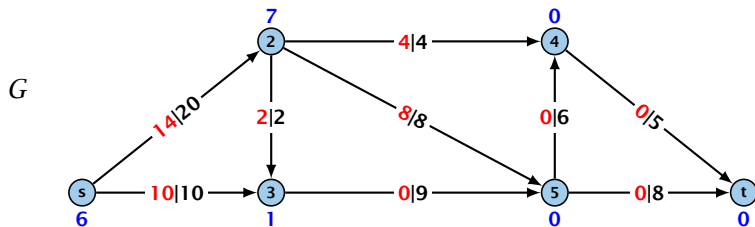
G



G_f

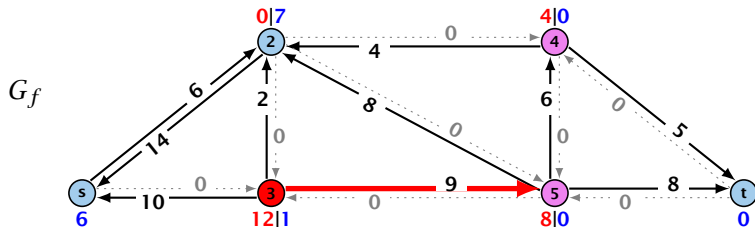
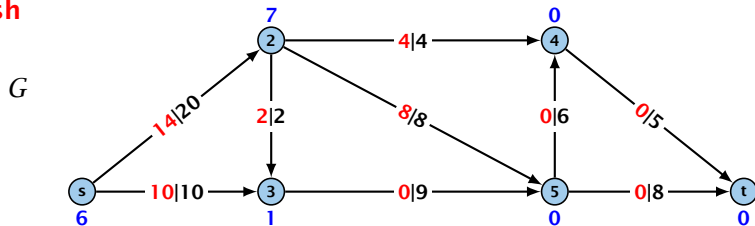


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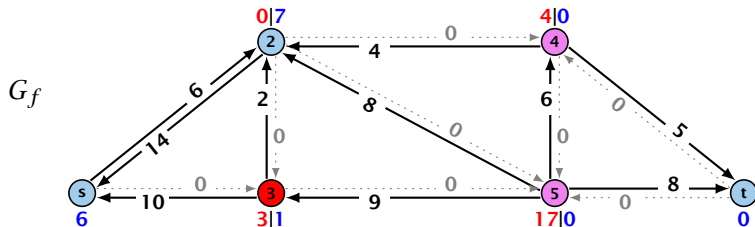
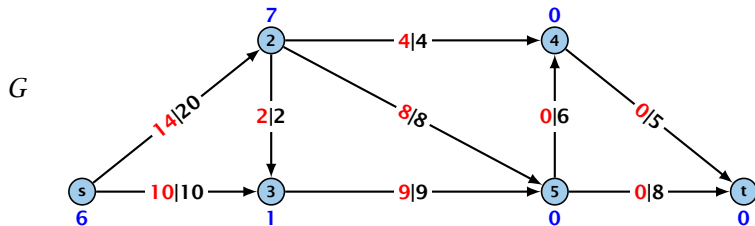


Preflow Push Algorithm

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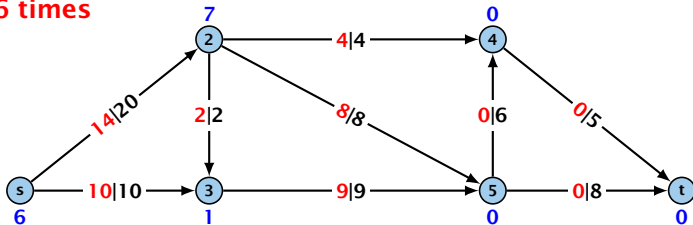
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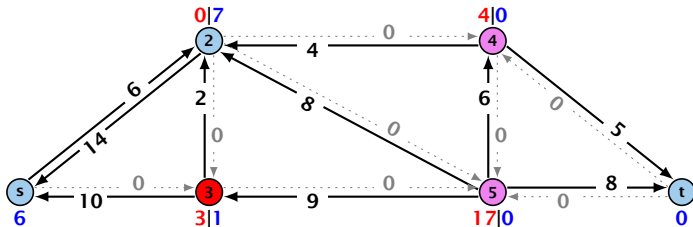
Preflow Push Algorithm

relabel 6 times

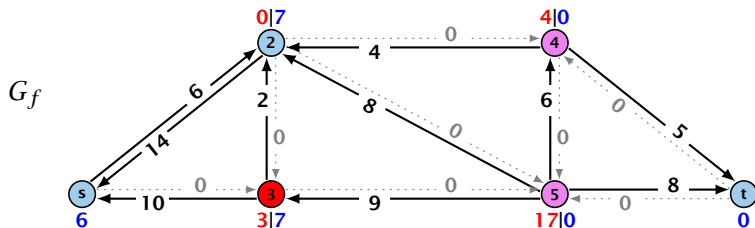
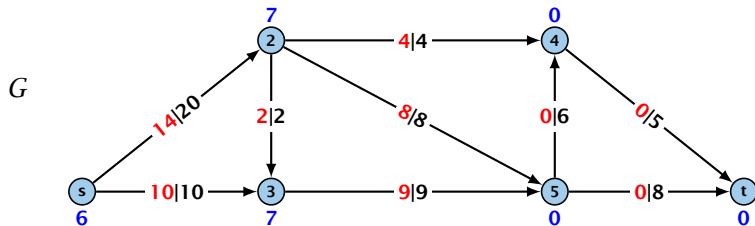
G



G_f

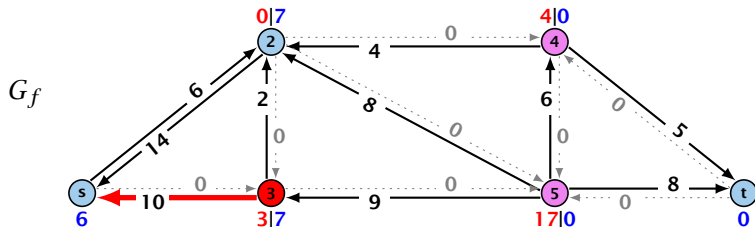
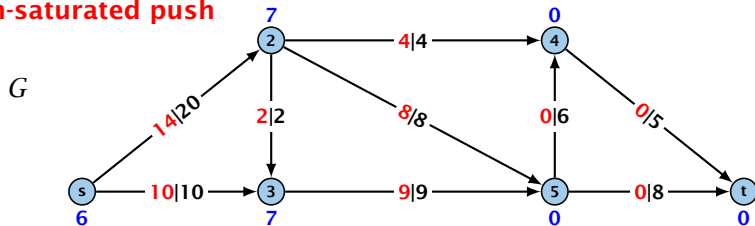


Preflow Push Algorithm

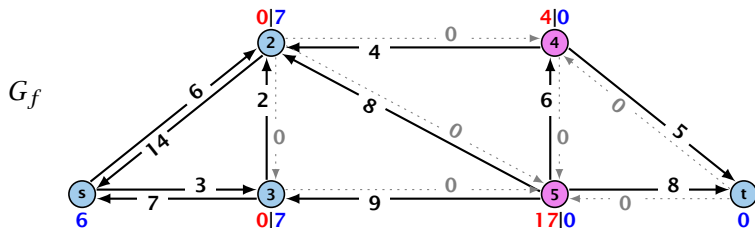
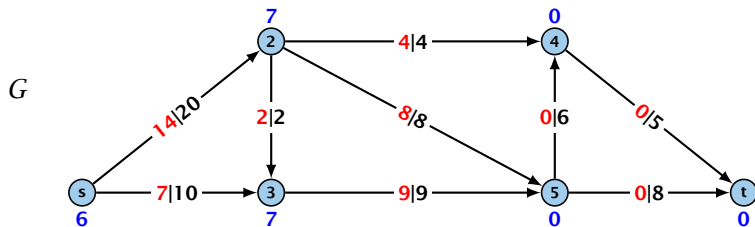


Preflow Push Algorithm

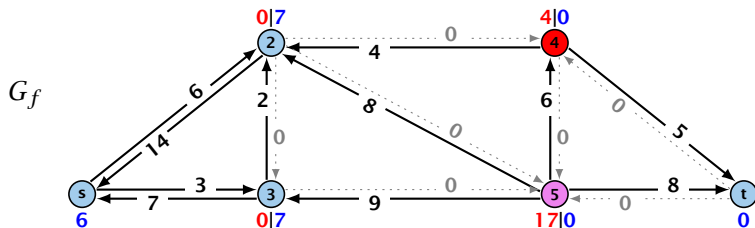
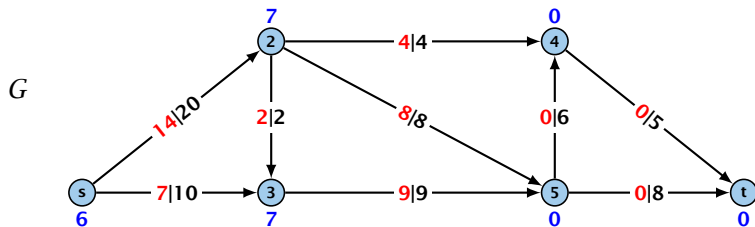
non-saturated push



Preflow Push Algorithm



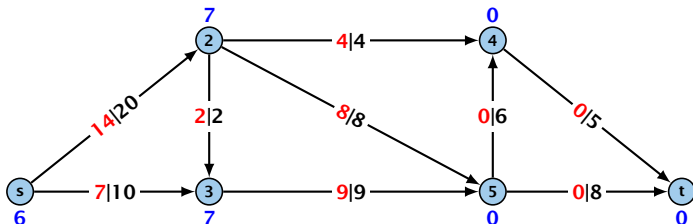
Preflow Push Algorithm



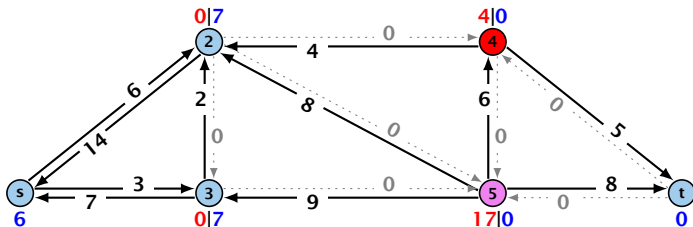
Preflow Push Algorithm

relabel

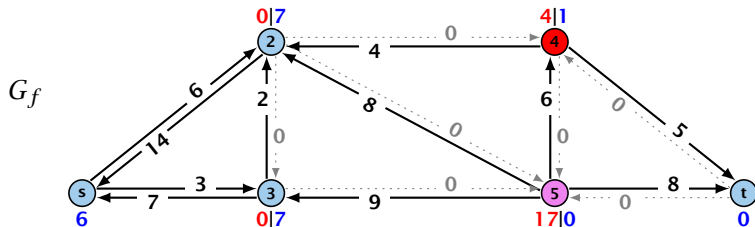
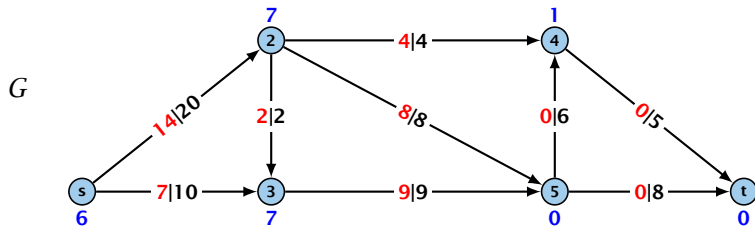
G



G_f

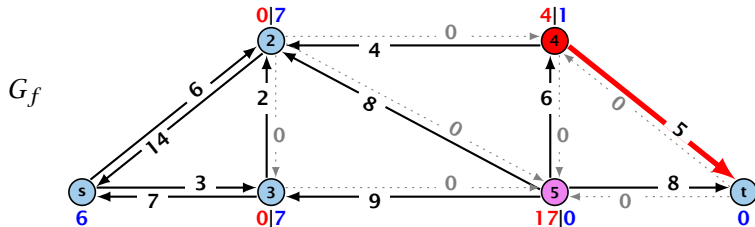
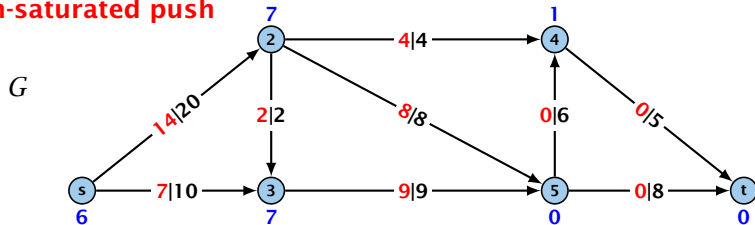


Preflow Push Algorithm

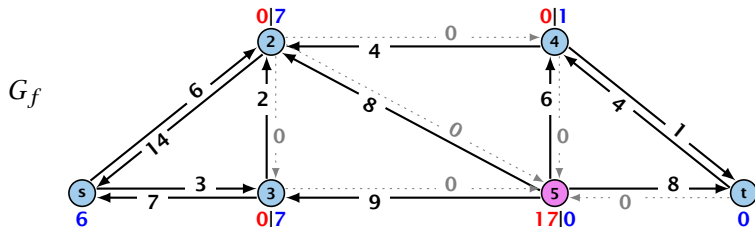
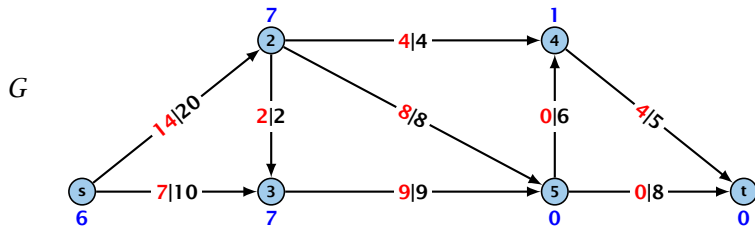


Preflow Push Algorithm

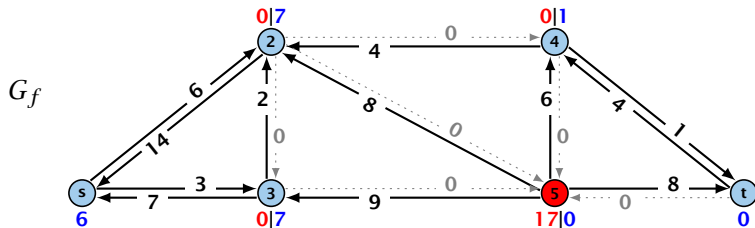
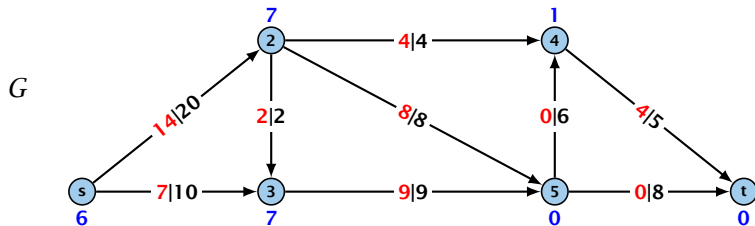
non-saturated push



Preflow Push Algorithm



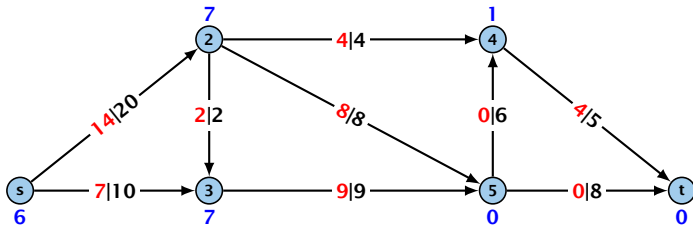
Preflow Push Algorithm



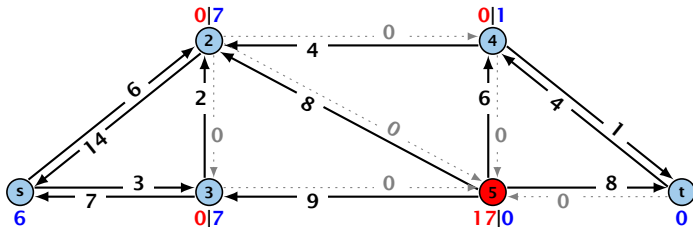
Preflow Push Algorithm

relabel

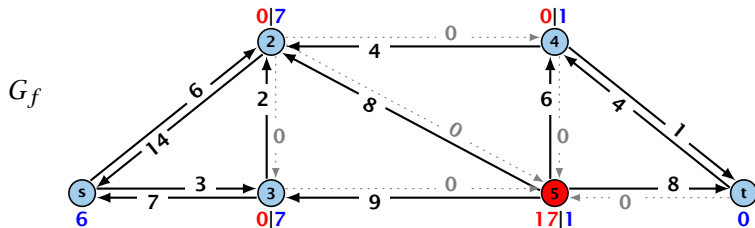
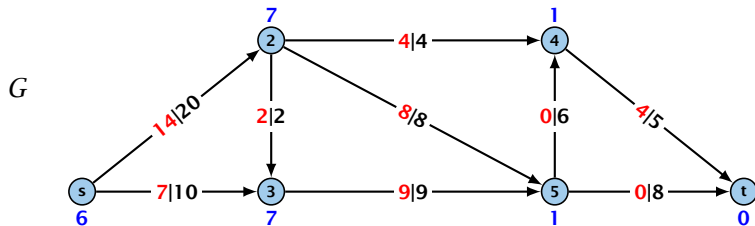
G



G_f



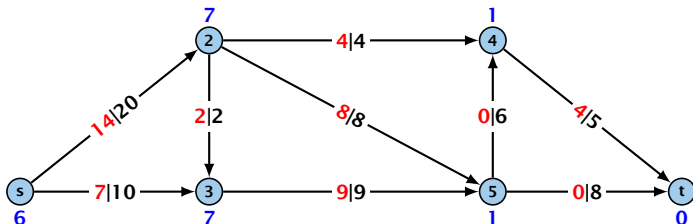
Preflow Push Algorithm



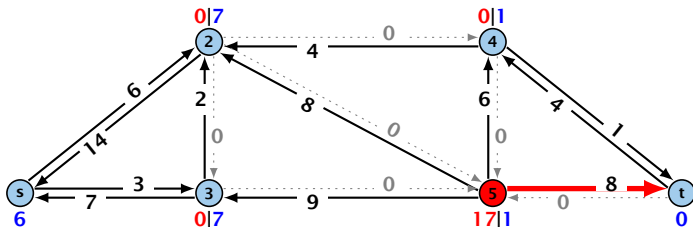
Preflow Push Algorithm

push

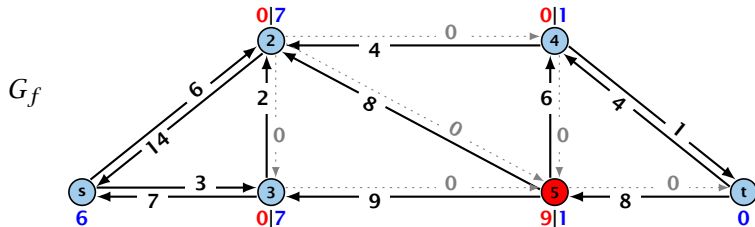
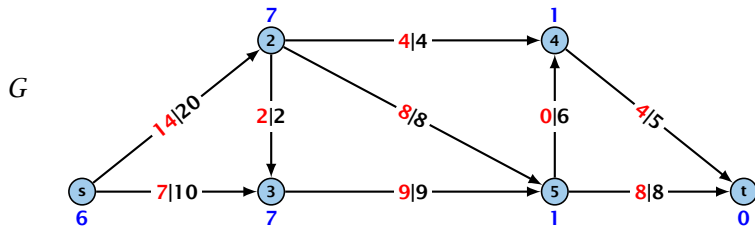
G



G_f



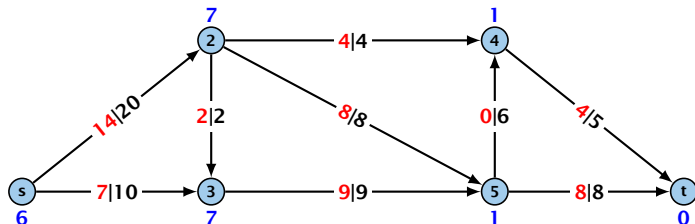
Preflow Push Algorithm



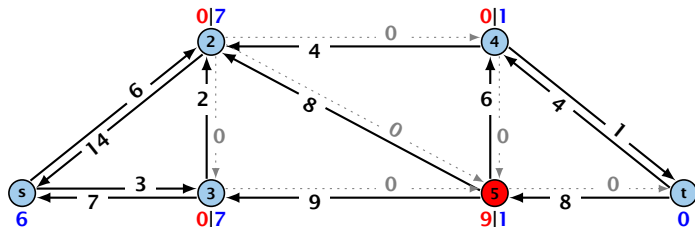
Preflow Push Algorithm

relabel

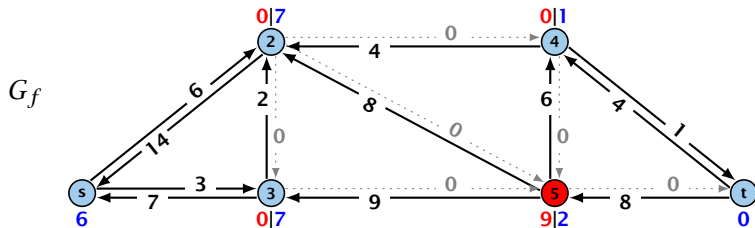
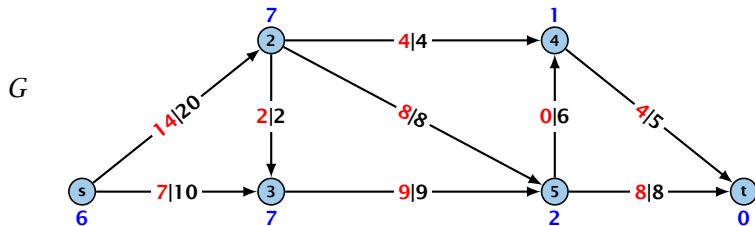
G



G_f



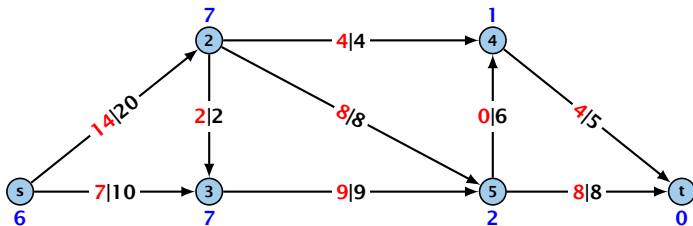
Preflow Push Algorithm



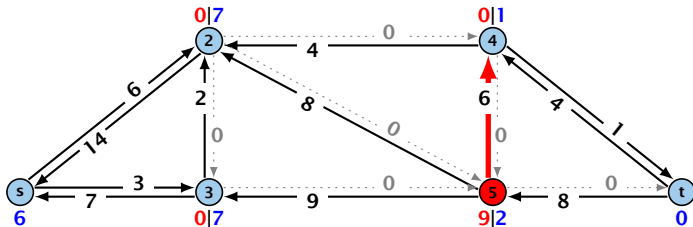
Preflow Push Algorithm

push

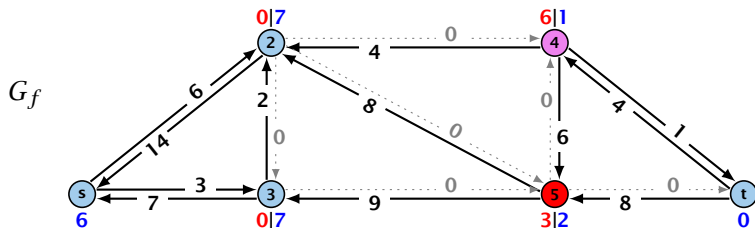
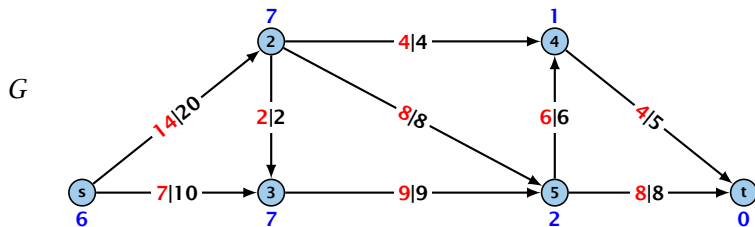
G



G_f



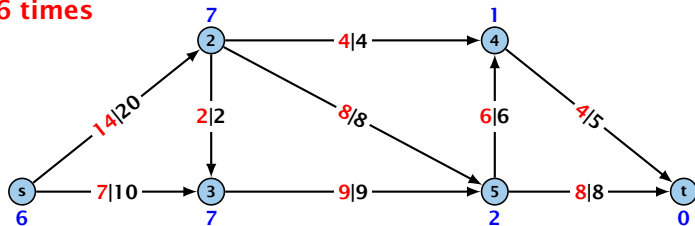
Preflow Push Algorithm



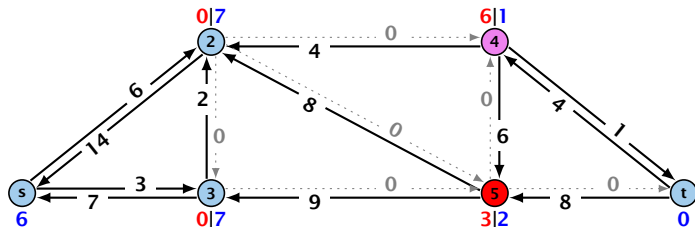
Preflow Push Algorithm

relabel 6 times

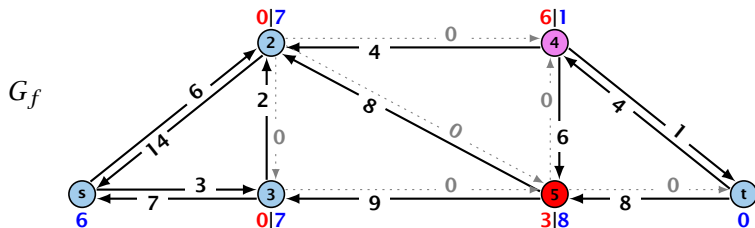
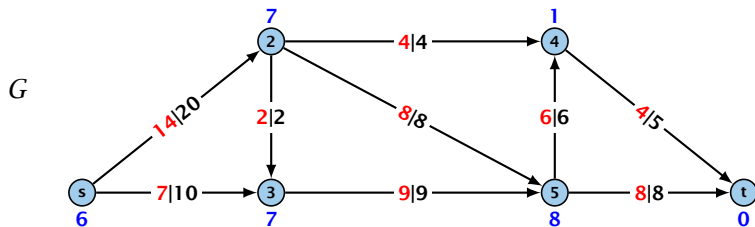
G



G_f

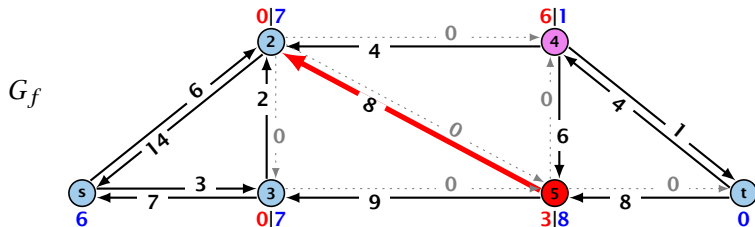
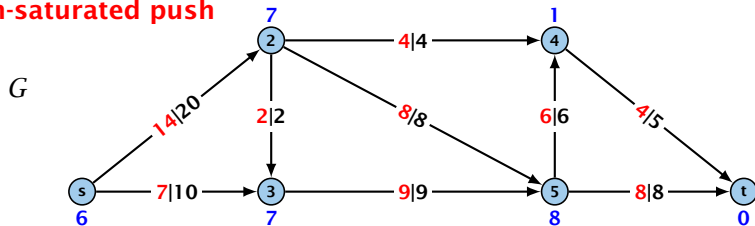


Preflow Push Algorithm

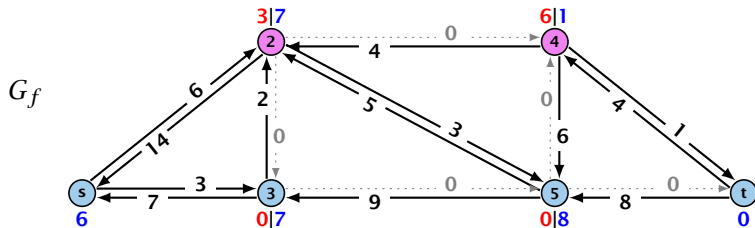
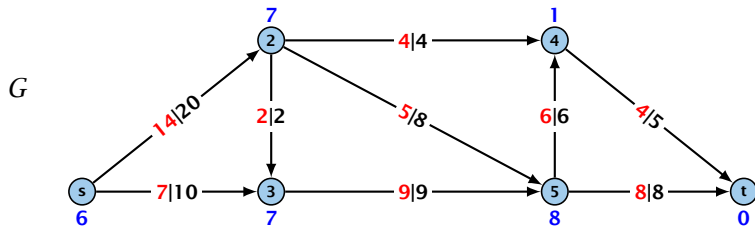


Preflow Push Algorithm

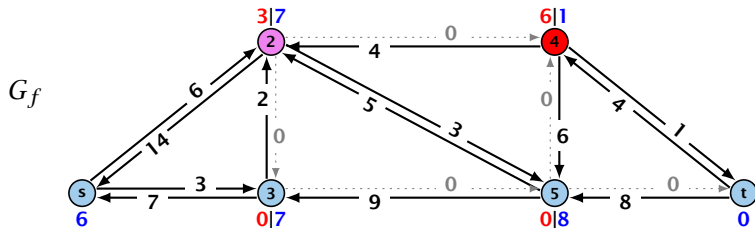
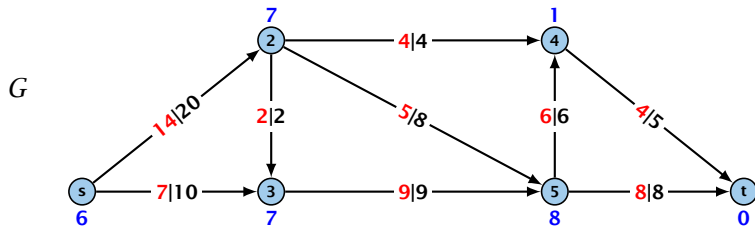
non-saturated push



Preflow Push Algorithm



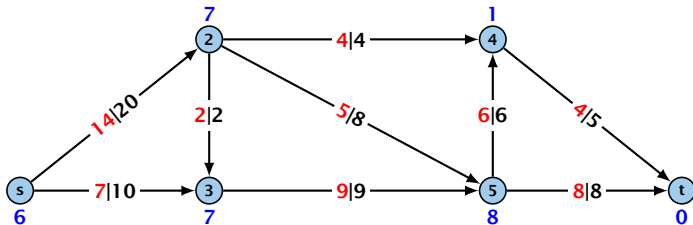
Preflow Push Algorithm



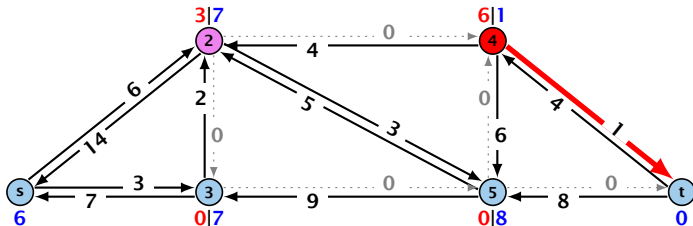
Preflow Push Algorithm

push

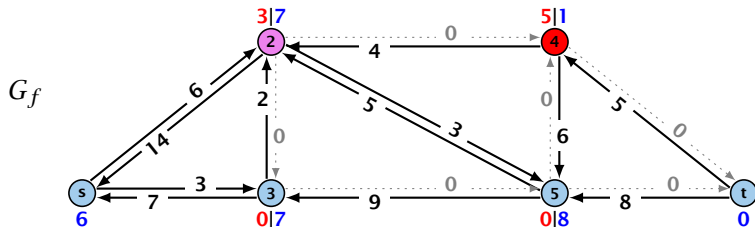
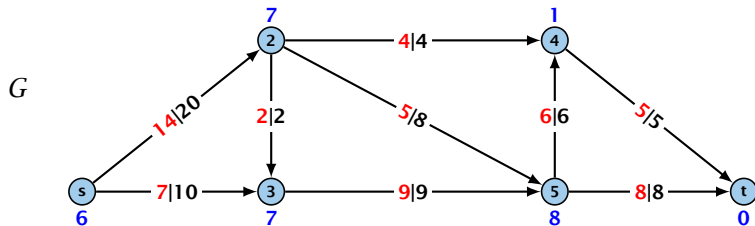
G



G_f



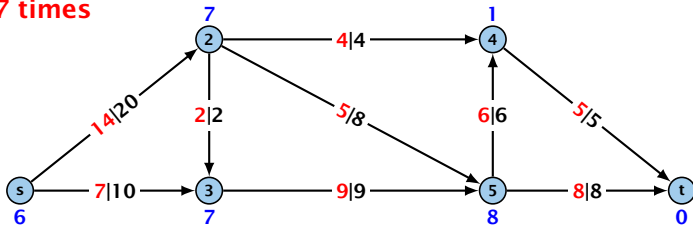
Preflow Push Algorithm



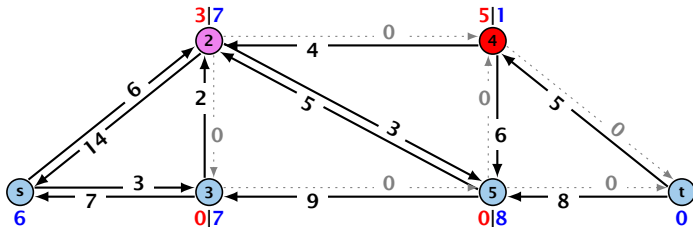
Preflow Push Algorithm

relabel 7 times

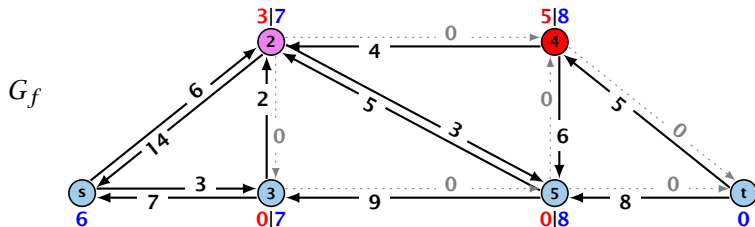
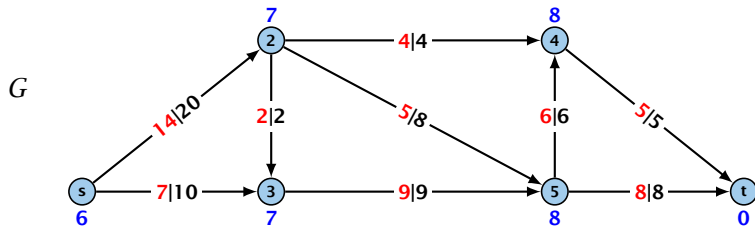
G



G_f



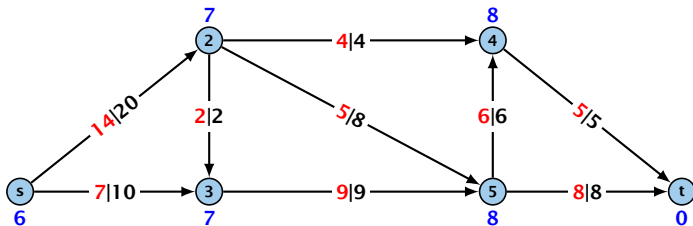
Preflow Push Algorithm



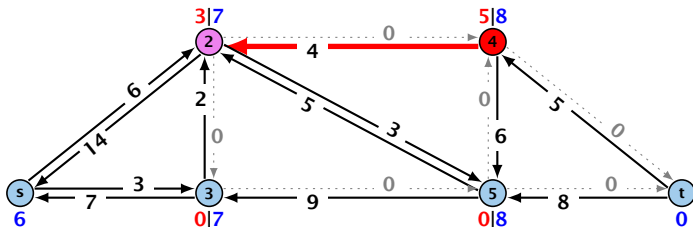
Preflow Push Algorithm

push

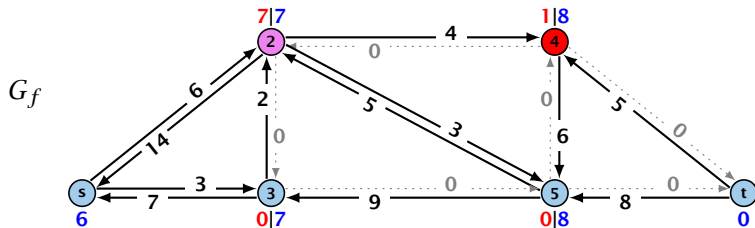
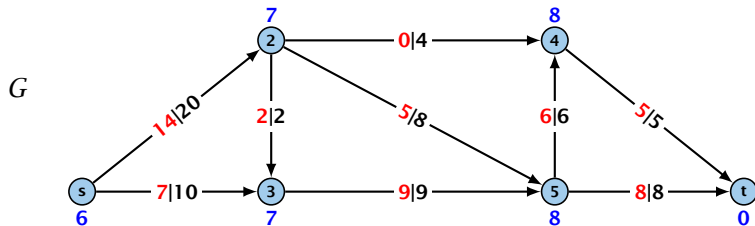
G



G_f

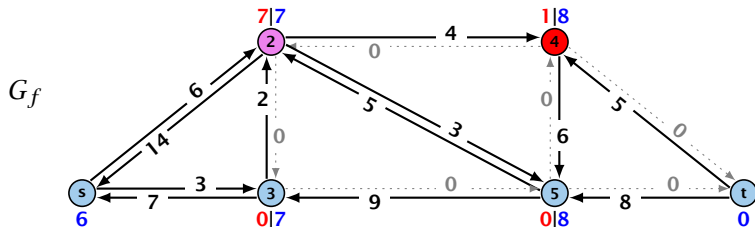
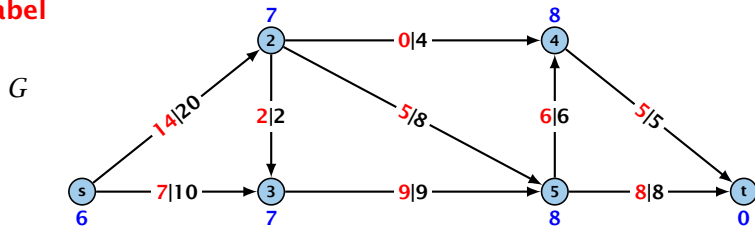


Preflow Push Algorithm

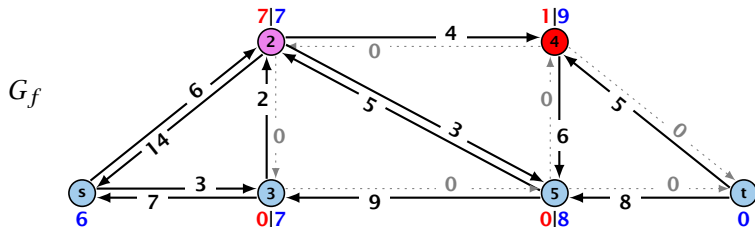
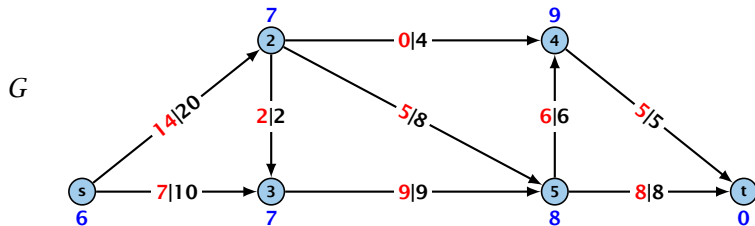


Preflow Push Algorithm

relabel

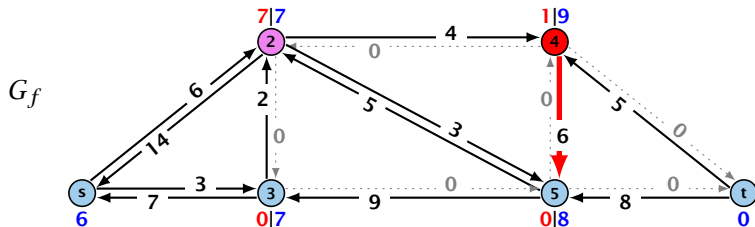
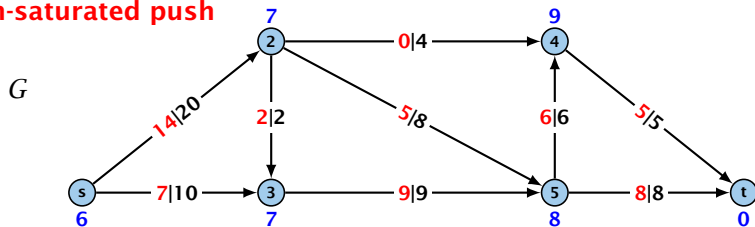


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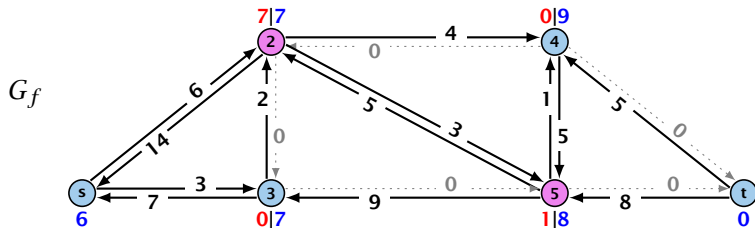
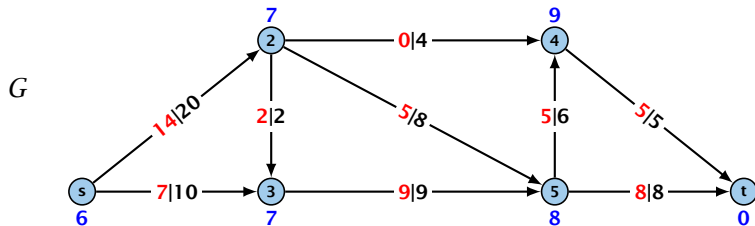


Preflow Push Algorithm

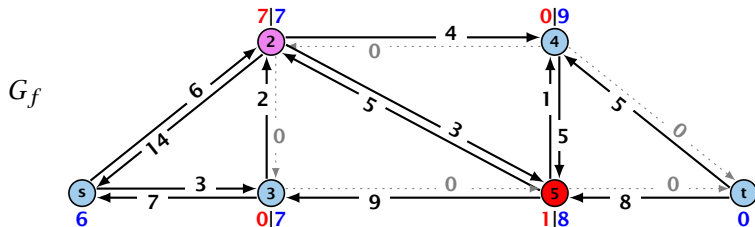
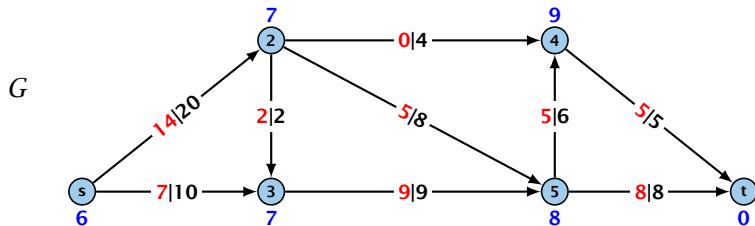
non-saturated push



Preflow Push Algorithm

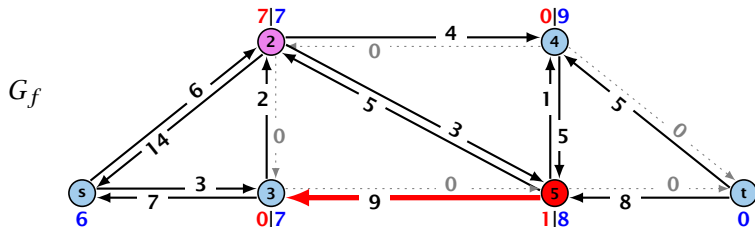
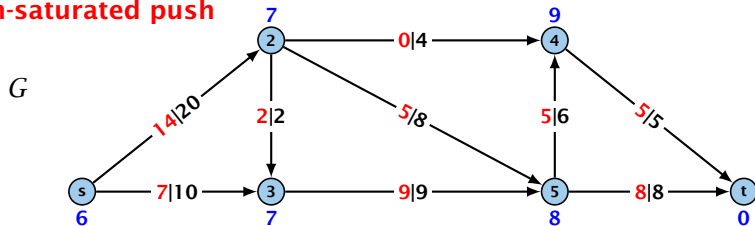


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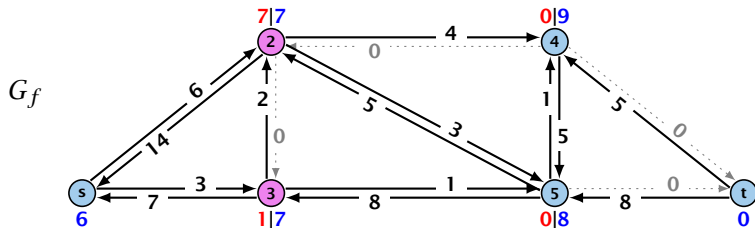
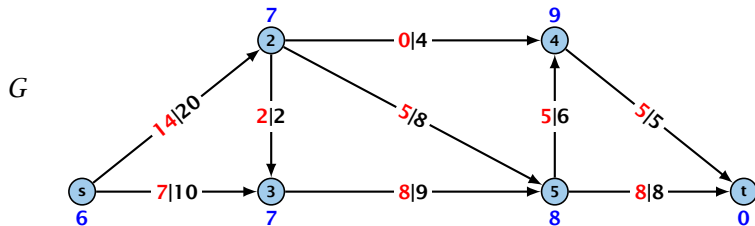


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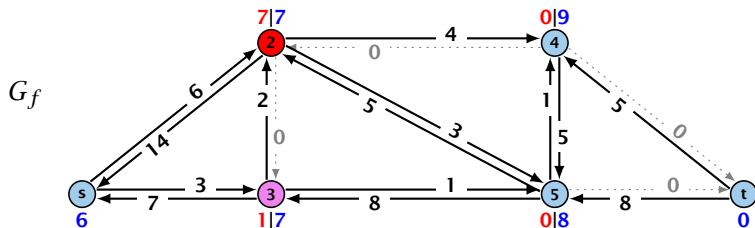
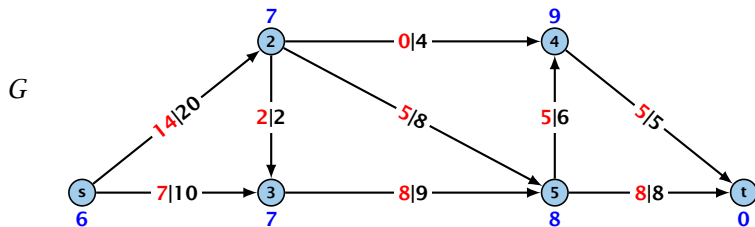
non-saturated push



Preflow Push Algorithm



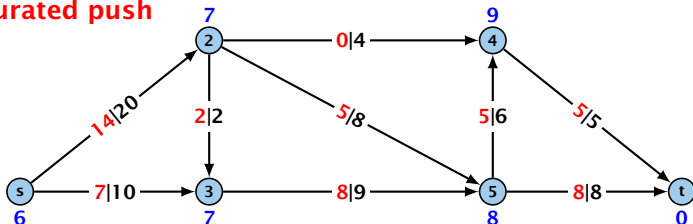
Preflow Push Algorithm



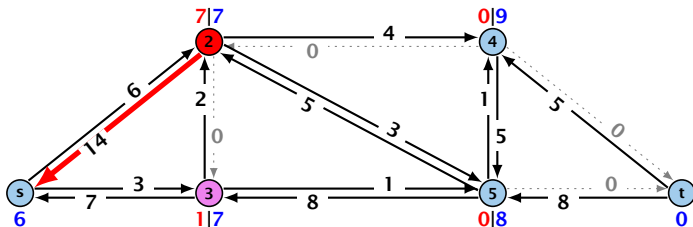
Preflow Push Algorithm

non-saturated push

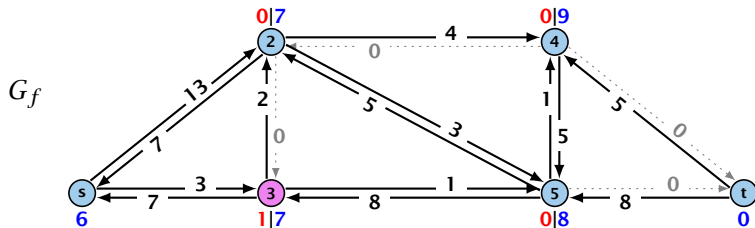
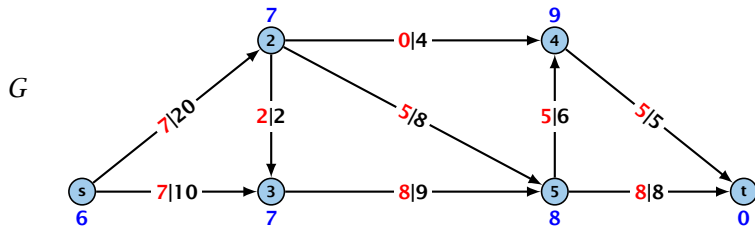
G



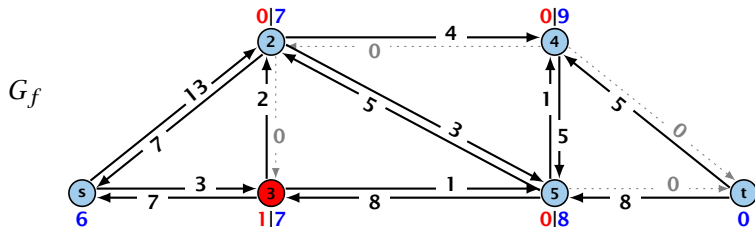
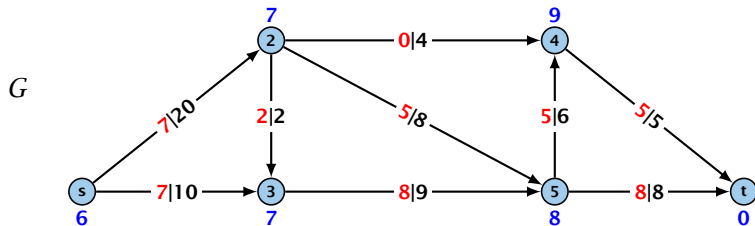
G_f



Preflow Push Algorithm



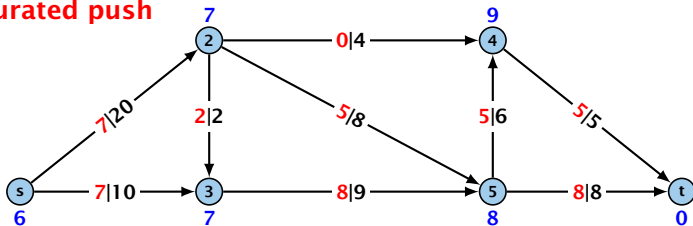
Preflow Push Algorithm



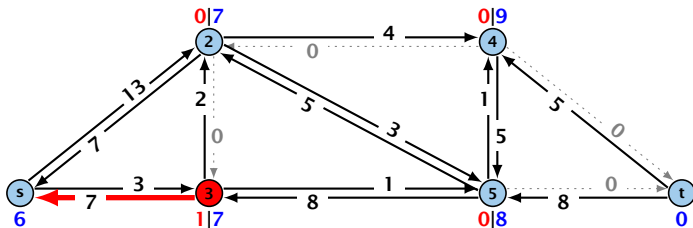
Preflow Push Algorithm

non-saturated push

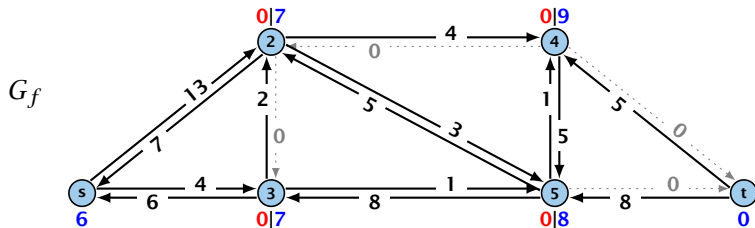
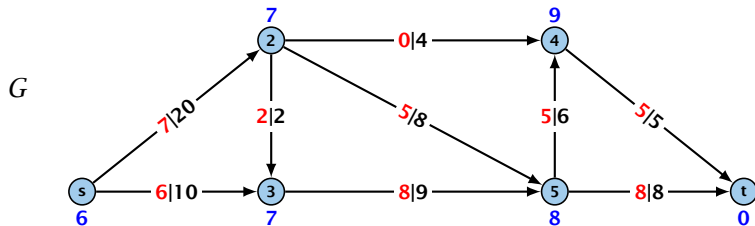
G



G_f



Preflow Push Algorithm



Analysis

Lemma 5

An active node has a path to s in the residual graph.

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- ▶ Let $f(B) = \sum_{v \in B} f(v)$ be the excess flow of all nodes in B .

Let $f : E \rightarrow \mathbb{R}_0^+$ be a preflow. We introduce the notation

$$f(x, y) = \begin{cases} 0 & (x, y) \notin E \\ f((x, y)) & (x, y) \in E \end{cases}$$

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Hence, the excess flow $f(b)$ must be 0 for every node $b \in B$.

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There are only $\mathcal{O}(n^2)$ relabel operations.

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- ▶ Since the label of v is at most $2n - 1$, there are at most n pushes along (u, v) .

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- ▶ A non-saturating push decreases Φ by at least 1 as the node that is pushed from becomes inactive and has a label that is strictly larger than the target.
- ▶ Hence,

$$\begin{aligned} \# \text{non-saturating_pushes} &\leq \# \text{relabels} + 2n \cdot \# \text{saturating_pushes} \\ &\leq \mathcal{O}(n^2m) . \end{aligned}$$

Theorem 10

There is an implementation of the generic push relabel algorithm with running time $\mathcal{O}(n^2m)$.

Analysis

Proof:

For every node maintain a list of admissible edges starting at that node. Further maintain a list of active nodes.

A push along an edge (u, v) can be performed in constant time

- check whether edge (u, v) needs to be added to the list
- check whether v needs to be added (starting push)
- check whether u becomes inactive and has to be deleted from the set of active nodes

A relabel at a node u can be performed in time $\mathcal{O}(n)$

- check for all outgoing edges if they become inadmissible
- check for all incoming edges if they become inadmissible

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A push along an edge (u, v) can be performed in constant time
- check whether v is an active node (edge (u, v) is admissible)
- decrease $d[u]$ by $d[v]$ (push) (if $d[u] > d[v]$)
- check whether u becomes inactive and has to be deleted from the set of active nodes

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Proof:

For every node maintain a list of admissible edges starting at that node. Further maintain a list of active nodes.

A push along an edge (u, v) can be performed in constant time

- ▶ check whether edge (v, u) needs to be added to G_f
- ▶ check whether (u, v) needs to be deleted (saturating push)
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Analysis

For special variants of push relabel algorithms we organize the neighbours of a node into a linked list (possible neighbours in the residual graph G_f). Then we use the discharge-operation:

Algorithm 4 discharge(u)

```
1: while  $u$  is active do  
2:    $v \leftarrow u.current\text{-neighbour}$   
3:   if  $v = \text{null}$  then  
4:     relabel( $u$ )  
5:      $u.current\text{-neighbour} \leftarrow u.neighbour\text{-list-head}$   
6:   else  
7:     if  $(u, v)$  admissible then push( $u, v$ )  
8:     else  $u.current\text{-neighbour} \leftarrow v.next\text{-in-list}$ 
```

Note that $u.current\text{-neighbour}$ is a global variable. It is only changed within the discharge routine, but keeps its value between consecutive calls to discharge.

Lemma 11

If $v = \text{null}$ in Line 3, then there is no outgoing admissible edge from u .

Proof.

- ▶ While pushing from u the current-neighbour pointer is only advanced if the current edge is not admissible.
- ▶ The only thing that could make the edge admissible again would be a relabel at u .
- ▶ If we reach the end of the list ($v = \text{null}$) all edges are not admissible. □

This shows that $\text{discharge}(u)$ is correct, and that we can perform a relabel in Line 4.