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Several possibilities:

- Choose path with maximum bottleneck capacity.
- Choose path with sufficiently large bottleneck capacity.
- Choose the shortest augmenting path.

Capacity Scaling

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- Choosing a path with the highest bottleneck increases the flow as much as possible in a single step.

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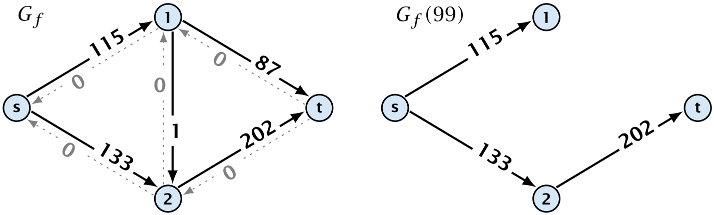

Capacity Scaling

Algorithm 2 maxflow(G, s, t, c)

```
foreach e ∈ E do f_e ← 0;
Δ ← 2^⌈log₂ C⌉
while Δ ≥ 1 do
    G_f(Δ) ← Δ-residual graph
    while there is augmenting path P in G_f(Δ) do
        f ← augment(f, c, P)
        update(G_f(Δ))
    Δ ← Δ/2
return f
```

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- this means we have a maximum flow.

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- In G_f this cut can have capacity at most $m\Delta$.
- This gives me an upper bound on the flow that I can still add.

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Theorem 4

We need $\mathcal{O}(m \log C)$ augmentations. The algorithm can be implemented in time $\mathcal{O}(m^2 \log C)$.