

# Part II

## Foundations

## 3 Goals

- ▶ Gain knowledge about efficient algorithms for important problems, i.e., learn how to solve certain types of problems efficiently.
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## What do you measure?

- ▶ **Memory requirement**
- ▶ Running time
- ▶ Number of comparisons
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- ▶ Number of hard-disc accesses
- ▶ Program size
- ▶ Power consumption
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## How do you measure?

- ▶ Implementing and testing on representative inputs
  - ▶ How do you choose your inputs?
  - ▶ May be very time-consuming.
  - ▶ Very reliable results if done correctly.
  - ▶ Results only hold for a specific machine and for a specific set of inputs.
- ▶ Theoretical analysis in a specific **model of computation**.
  - ▶ Gives a **lower bound** like "this algorithm always runs in  $\Omega(n^2)$  time".
  - ▶ Typically focuses on the **number of comparisons**.
  - ▶ Can this lower bound also be **empirically tested** using sorting algorithms needs at least  $\Omega(n \log n)$  comparisons in the worst case?

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Quick example: How long does the following algorithm always run?

```
int f(int n) {
```

```
    if (n < 1) return 1;
```

```
    return f(n-1) + f(n-2);
```

```
}
```

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Quick question: Does this algorithm always run in  $O(n^2)$  time?

Typical answer: Yes.

Can this lower bound be very computer-architecture dependent?  
Algorithmic complexity is not the same as complexity in the real world.

Why not?

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### Input length

The theoretical bounds are usually given by a function  $f: \mathbb{N} \rightarrow \mathbb{N}$  that maps the **input length** to the running time (or storage space, comparisons, multiplications, program size etc.).

The **input length** may e.g. be

the size of the input (number of bits)

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1. Calculate running time and storage space etc. on a simplified, idealized model of computation, e.g. Random Access Machine (RAM), Turing Machine (TM), . . .
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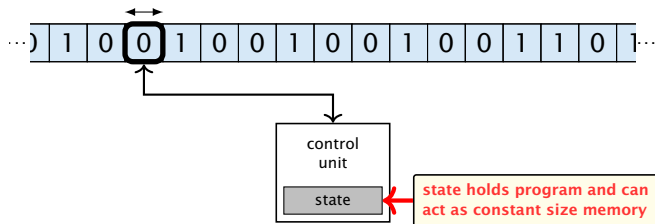
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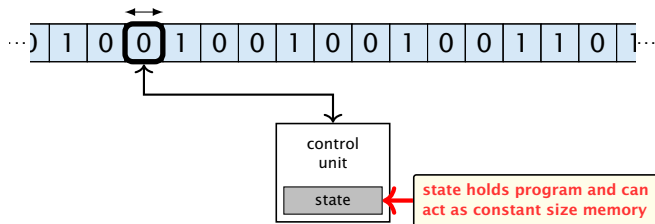
# Turing Machine

- ▶ Very simple model of computation.
  - ▶ Only the “current” memory location can be altered.
  - ▶ Very good model for discussing computability, or polynomial vs. exponential time.
  - ▶ Some simple problems like recognizing whether input is of the form  $x^x$ , where  $x$  is a string, have quadratic lower bound.
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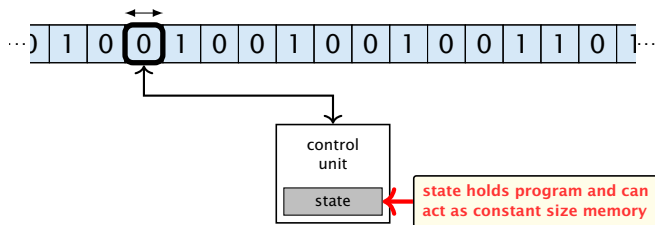
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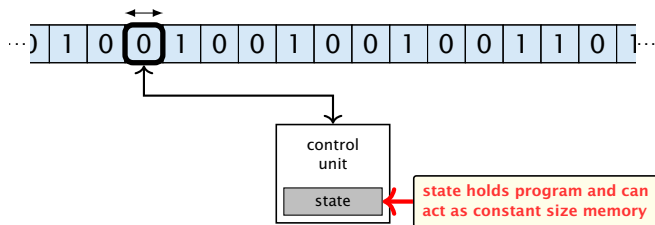
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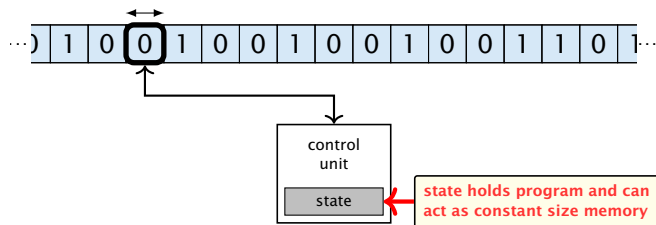
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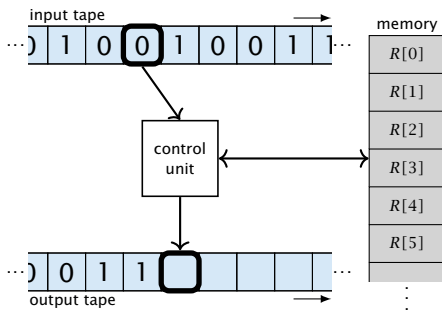
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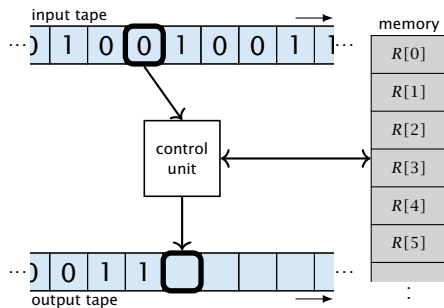
- ▶ Input tape and output tape (sequences of zeros and ones; unbounded length).
- ▶ Memory unit: infinite but countable number of registers  $R[0], R[1], R[2], \dots$
- ▶ Registers hold integers.
- ▶ Indirect addressing.



Note that in the picture on the right the tapes are one-directional, and that a READ- or WRITE-operation always advances its tape.

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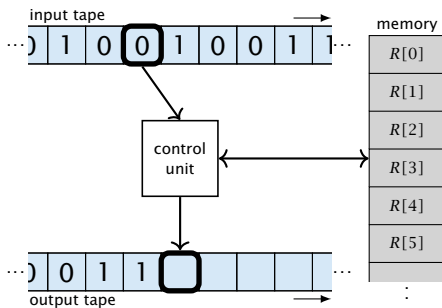
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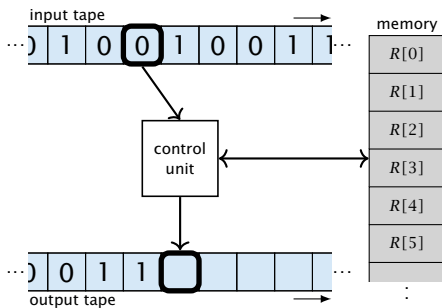
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jumps to position  $x$  in the program;  
sets instruction counter to  $x$ ;  
reads the next operation to perform from register  $R[x]$
  - ▶ `jumpz x R[i]`  
jump to  $x$  if  $R[i] = 0$   
if not the instruction counter is increased by 1;
  - ▶ `jumpi i`  
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# Model of Computation

- ▶ **uniform** cost model

Every operation takes time 1.

- ▶ **logarithmic** cost model

The cost depends on the content of memory cells:

- ▶ The time for a step is equal to the largest operand involved.
- ▶ The amount of space required is equal to the length of the largest operand involved.

**Bounded word RAM model:** cost is uniform but the largest value stored in a register may not exceed  $2^w$ , where usually  $w = \log_2 n$ .

The latter model is quite realistic as the word-size of a standard computer that handles a problem of size  $n$  must be at least  $\log_2 n$  as otherwise the computer could either not store the problem instance or not address all its memory.



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Every operation takes time 1.
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The cost depends on the content of memory cells:
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**Bounded word RAM model:** cost is uniform but the largest value stored in a register may not exceed  $2^w$ , where usually  $w = \log_2 n$ .

The latter model is quite realistic as the word-size of a standard computer that handles a problem of size  $n$  must be at least  $\log_2 n$  as otherwise the computer could either not store the problem instance or not address all its memory.

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- ▶ **best-case** complexity:

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Usually easy to analyze, but not very meaningful.

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Let  $f$  denote functions from  $\mathbb{N}$  to  $\mathbb{R}^+$ .

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# Asymptotic Notation in Equations

How do we interpret an expression like:

$$2n^2 + 3n + 1 = 2n^2 + \Theta(n)$$

Here,  $\Theta(n)$  stands for an anonymous function in the set  $\Theta(n)$  that makes the expression true.

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The  $\Theta(i)$ -symbol on the left represents **one** anonymous function  $f : \mathbb{N} \rightarrow \mathbb{R}^+$ , and then  $\sum_i f(i)$  is computed.

How do we interpret an expression like:

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Careful!

“It is understood” that every occurrence of an  $\Theta$ -symbol (or  $\Theta, \Omega, o, \omega$ ) on the left represents **one** anonymous function.

Hence, the left side is **not** equal to

$$\Theta(1) + \Theta(2) + \dots + \Theta(n-1) + \Theta(n)$$

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# Asymptotic Notation in Equations

We can view an expression containing asymptotic notation as generating a set:

$$n^2 \cdot \mathcal{O}(n) + \mathcal{O}(\log n)$$

represents

$$\{f : \mathbb{N} \rightarrow \mathbb{R}^+ \mid f(n) = n^2 \cdot g(n) + h(n)\}$$

with  $g(n) \in \mathcal{O}(n)$  and  $h(n) \in \mathcal{O}(\log n)$

Recall that according to the previous slide e.g. the expressions  $\sum_{i=1}^n \mathcal{O}(i)$  and  $\sum_{i=1}^{n/2} \mathcal{O}(i) + \sum_{i=n/2+1}^n \mathcal{O}(i)$  generate different sets.

# Asymptotic Notation in Equations

Then an asymptotic equation can be interpreted as containment btw. two sets:

$$n^2 \cdot \mathcal{O}(n) + \mathcal{O}(\log n) = \Theta(n^2)$$

represents

$$n^2 \cdot \mathcal{O}(n) + \mathcal{O}(\log n) \subseteq \Theta(n^2)$$

Note that the equation does not hold.

# Asymptotic Notation

## Lemma 3

Let  $f, g$  be functions with the property

$\exists n_0 > 0 \forall n \geq n_0 : f(n) > 0$  (the same for  $g$ ). Then

- ▶  $c \cdot f(n) \in \Theta(f(n))$  for any constant  $c$
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The expressions also hold for  $\Omega$ . Note that this means that  $f(n) + g(n) \in \Theta(\max\{f(n), g(n)\})$ .

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## Comments

- ▶ Do not use asymptotic notation within induction proofs.
- ▶ For any constants  $a, b$  we have  $\log_a n = \Theta(\log_b n)$ .  
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In general asymptotic classification of running times is a good measure for comparing algorithms:

- ▶ If the running time analysis is tight and actually occurs in practise (i.e., the asymptotic bound is not a purely theoretical worst-case bound), then the algorithm that has better asymptotic running time will always outperform a weaker algorithm for large enough values of  $n$ .
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## 6 Recurrences

### Algorithm 2 mergesort(list $L$ )

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1:  $n \leftarrow \text{size}(L)$ 
2: if  $n \leq 1$  return  $L$ 
3:  $L_1 \leftarrow L[1 \cdots \lfloor \frac{n}{2} \rfloor]$ 
4:  $L_2 \leftarrow L[\lfloor \frac{n}{2} \rfloor + 1 \cdots n]$ 
5: mergesort( $L_1$ )
6: mergesort( $L_2$ )
7:  $L \leftarrow \text{merge}(L_1, L_2)$ 
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This algorithm requires

$$T(n) = T\left(\left\lceil \frac{n}{2} \right\rceil\right) + T\left(\left\lfloor \frac{n}{2} \right\rfloor\right) + \mathcal{O}(n) \leq 2T\left(\left\lceil \frac{n}{2} \right\rceil\right) + \mathcal{O}(n)$$

comparisons when  $n > 1$  and 0 comparisons when  $n \leq 1$ .

# Recurrences

How do we bring the expression for the number of comparisons ( $\approx$  running time) into a **closed form**?

For this we need to **solve** the recurrence.

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# Methods for Solving Recurrences

## 1. Guessing+Induction

Guess the right solution and prove that it is correct via induction. It needs experience to make the right guess.

## 2. Master Theorem

For a lot of recurrences that appear in the analysis of algorithms this theorem can be used to obtain tight asymptotic bounds. It does not provide exact solutions.

## 3. Characteristic Polynomial

Linear homogenous recurrences can be solved via this method.



## 4. Generating Functions

A more general technique that allows to solve certain types of linear inhomogenous relations and also sometimes non-linear recurrence relations.

## 5. Transformation of the Recurrence

Sometimes one can transform the given recurrence relations so that it e.g. becomes linear and can therefore be solved with one of the other techniques.

## 6.1 Guessing+Induction

First we need to get rid of the  $\mathcal{O}$ -notation in our recurrence:

$$T(n) \leq \begin{cases} 2T(\lceil \frac{n}{2} \rceil) + cn & n \geq 2 \\ 0 & \text{otherwise} \end{cases}$$

**Informal way:**

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One way of solving such a recurrence is to **guess** a solution, and check that it is correct by plugging it in.

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Formally, this is not correct if  $n$  is not a power of 2. Also even in this case one would need to do an induction proof.

## 6.1 Guessing+Induction

$$T(n) \leq \begin{cases} 2T(\frac{n}{2}) + cn & n \geq 16 \\ b & \text{otw.} \end{cases}$$

- Note that this proves the statement for  $n \in \mathbb{N}_{\geq 2}$ , as the statement is wrong for  $n = 1$ .
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Suppose statem. is true for  $n' \in \{2, \dots, n-1\}$ , and  $n \geq 16$ .

We prove it for  $n$ :

$$\begin{aligned} T(n) &\leq 2T\left(\frac{n}{2}\right) + cn \\ &\leq 2\left(d\frac{n}{2} \log \frac{n}{2}\right) + cn \\ &= dn(\log n - 1) + cn \\ &= dn \log n + (c - d)n \end{aligned}$$

- Note that this proves the statement for  $n \in \mathbb{N}_{\geq 2}$ , as the statement is wrong for  $n = 1$ .
- The base case is usually omitted, as it is the same for different recurrences.

## 6.1 Guessing+Induction

$$T(n) \leq \begin{cases} 2T\left(\frac{n}{2}\right) + cn & n \geq 16 \\ b & \text{otw.} \end{cases}$$

**Guess:**  $T(n) \leq dn \log n$ .

**Proof.** (by induction)

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Hence, statement is **true** if we choose  $d \geq c$ .



## 6.1 Guessing+Induction

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Note that we can do this as for constant-sized inputs the running time is always some constant ( $b$  in the above case).

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We also make a guess of  $T(n) \leq dn \log n$  and get

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$$\boxed{\frac{n}{2} + 1 \leq \frac{9}{16}n} \leq dn \log\left(\frac{9}{16}n\right) + 2d \log n + cn$$

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$$\leq dn \log n$$

for a suitable choice of  $d$ .



## 6.2 Master Theorem

Note that the cases do not cover all possibilities.

### Lemma 4

Let  $a \geq 1$ ,  $b \geq 1$  and  $\epsilon > 0$  denote constants. Consider the recurrence

$$T(n) = aT\left(\frac{n}{b}\right) + f(n) .$$

#### Case 1.

If  $f(n) = \mathcal{O}(n^{\log_b(a)-\epsilon})$  then  $T(n) = \Theta(n^{\log_b a})$ .

#### Case 2.

If  $f(n) = \Theta(n^{\log_b(a)} \log^k n)$  then  $T(n) = \Theta(n^{\log_b a} \log^{k+1} n)$ ,  
 $k \geq 0$ .

#### Case 3.

If  $f(n) = \Omega(n^{\log_b(a)+\epsilon})$  and for sufficiently large  $n$   
 $af\left(\frac{n}{b}\right) \leq cf(n)$  for some constant  $c < 1$  then  $T(n) = \Theta(f(n))$ .

## 6.2 Master Theorem

We prove the Master Theorem for the case that  $n$  is of the form  $b^{\ell}$ , and we assume that the non-recursive case occurs for problem size  $1$  and incurs cost  $1$ .

# The Recursion Tree

The running time of a recursive algorithm can be visualized by a recursion tree:

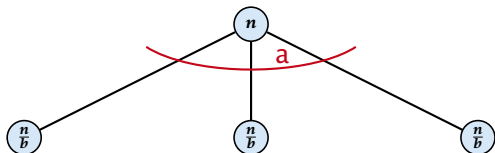
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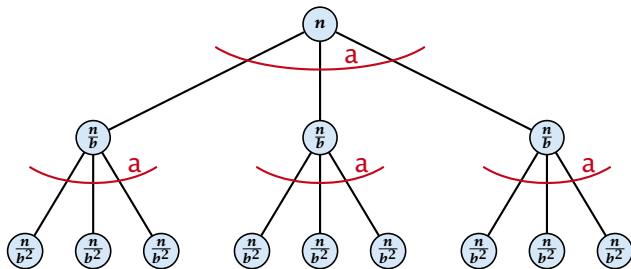
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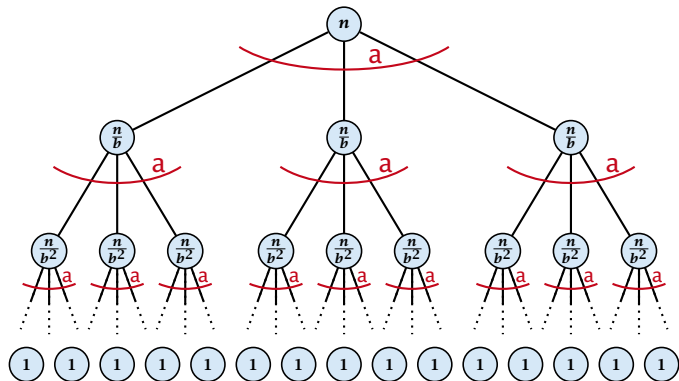
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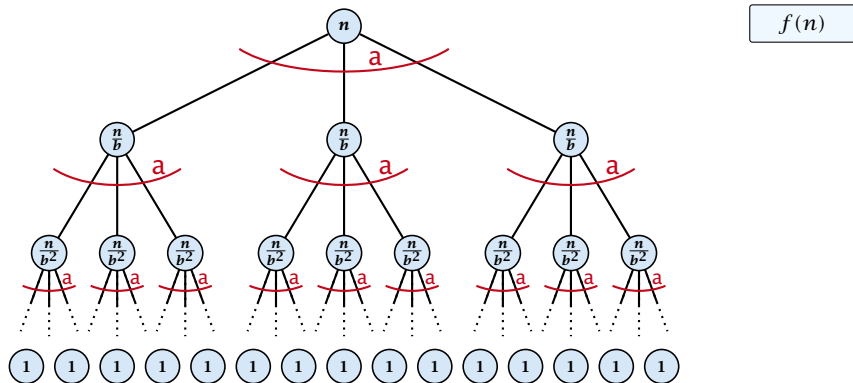
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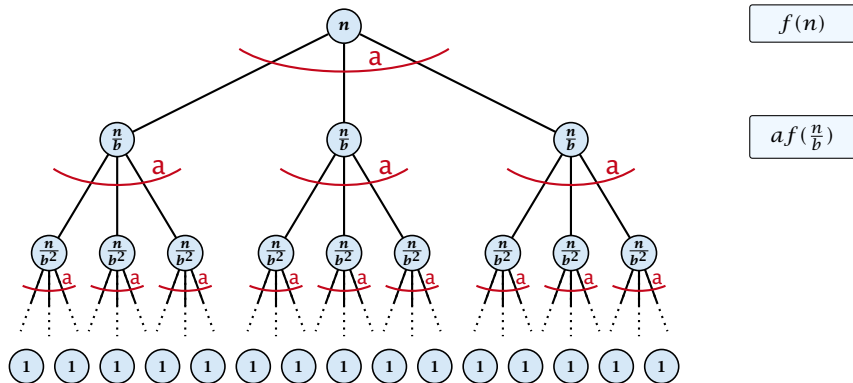
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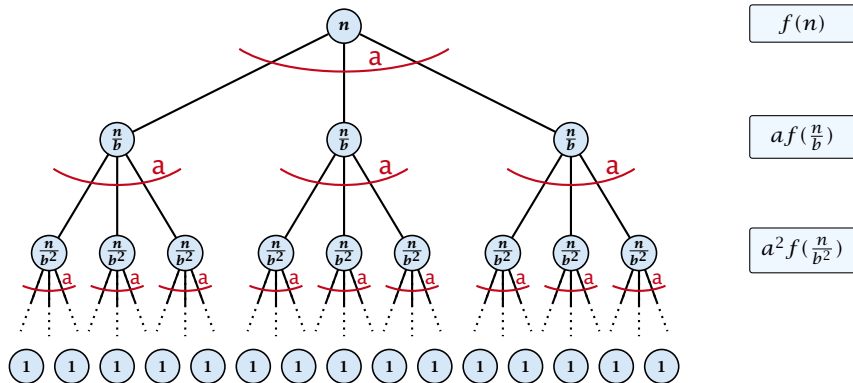
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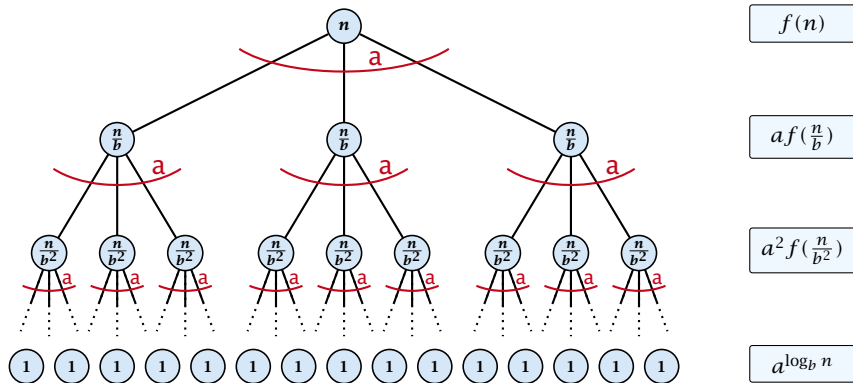
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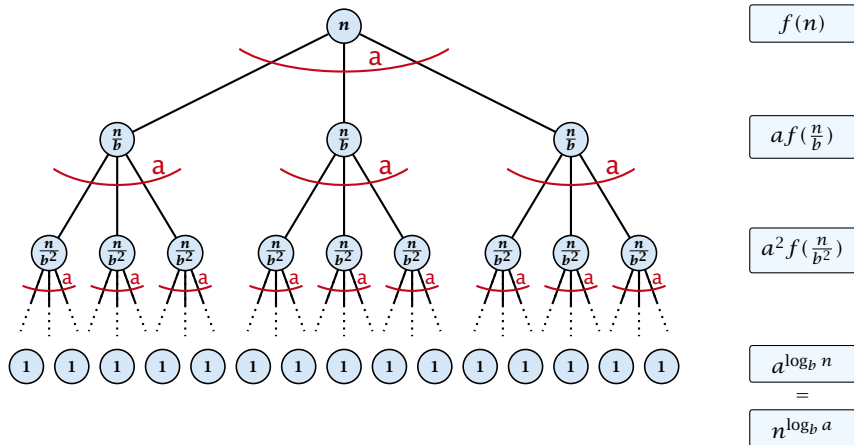
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## 6.2 Master Theorem

This gives

$$T(n) = n^{\log_b a} + \sum_{i=0}^{\log_b n - 1} a^i f\left(\frac{n}{b^i}\right).$$

Case 1. Now suppose that  $f(n) \leq cn^{\log_b a - \epsilon}$ .

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$$\begin{aligned} T(n) - n^{\log_b a} &= \sum_{i=0}^{\log_b n - 1} a^i f\left(\frac{n}{b^i}\right) \\ &\leq c \sum_{i=0}^{\log_b n - 1} a^i \left(\frac{n}{b^i}\right)^{\log_b a - \epsilon} \end{aligned}$$

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$$\boxed{\sum_{i=0}^k q^i = \frac{q^{k+1} - 1}{q - 1}} = cn^{\log_b a - \epsilon} (b^{\epsilon \log_b n} - 1) / (b^{\epsilon} - 1)$$

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Hence,

$$T(n) \leq \left( \frac{c}{b^{\epsilon} - 1} + 1 \right) n^{\log_b(a)}$$



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Hence,

$$T(n) \leq \left( \frac{c}{b^{\epsilon} - 1} + 1 \right) n^{\log_b a} \quad \Rightarrow T(n) = \mathcal{O}(n^{\log_b a}).$$

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$n = b^\ell \Rightarrow \ell = \log_b n$	$= cn^{\log_b a} \sum_{i=0}^{\ell-1} \left(\log_b\left(\frac{b^\ell}{b^i}\right)\right)^k$
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$$\Rightarrow T(n) = \mathcal{O}(n^{\log_b a} \log^{k+1} n).$$

Case 3. Now suppose that  $f(n) \geq dn^{\log_b a + \epsilon}$ , and that for sufficiently large  $n$ :  $af(n/b) \leq cf(n)$ , for  $c < 1$ .

Where did we use  $f(n) \geq \Omega(n^{\log_b a + \epsilon})$ ?

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Hence,

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$$\Rightarrow T(n) = \Theta(f(n)).$$

Where did we use  $f(n) \geq \Omega(n^{\log_b a + \epsilon})$ ?

## Example: Multiplying Two Integers

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1	0	0	0	1	0	0	1	1	$B$
<hr/>						0	0	0	

The diagram illustrates the addition of two 9-bit integers, A and B. The bits of A are 1 1 0 1 1 0 1 0 1 and the bits of B are 1 0 0 0 1 0 0 1 1. A horizontal line is drawn under the bits of B. The result of the addition is shown below the line, with the bits 0 0 0. The carry bits are indicated by small '1's below the bits of B. The bit '1' under the 7th bit of B is highlighted with a light blue box.

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					1	1	1		
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The diagram illustrates the addition of two 9-bit integers, A and B, using a ripple carry adder. The bits of A are 1 1 0 1 1 0 1 0 1 and the bits of B are 1 0 0 0 1 0 0 1 1. A horizontal line separates the inputs from the output. The output bits are 0 1 0 0 0. A vertical box highlights the 5th bit position (from the right), where the carry-in is 1 and the carry-out is 0. Small subscripts '1' and '0' are placed below the 4th and 5th bits of the input rows, respectively, indicating the carry propagation.

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		1	1	0	1	1	1		
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1	1	0	1	1	0	1	0	1	$A$
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<hr/>									
			1	1	0	1	1	1	
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*Note: In the original image, a vertical box highlights the first two bits of the top row (1, 1) and the first bit of the bottom row (1). Below the box, the carry values are shown: 0 under the first bit, and 1 under the second bit.*

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	<small>0</small>	<small>0</small>	<small>1</small>	<small>1</small>	<small>0</small>	<small>1</small>	<small>1</small>	<small>1</small>		
<hr/>										
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$$\begin{array}{r} 1\ 1\ 0\ 1\ 1\ 0\ 1\ 0\ 1\ A \\ 1\ 0\ 0\ 0\ 1\ 0\ 0\ 1\ 1\ B \\ \hline 1\ 0\ 1\ 1\ 0\ 0\ 1\ 0\ 0\ 0 \end{array}$$

This gives that two  $n$ -bit integers can be added in time  $\mathcal{O}(n)$ .

## Example: Multiplying Two Integers

Suppose that we want to multiply an  $n$ -bit integer  $A$  and an  $m$ -bit integer  $B$  ( $m \leq n$ ).

- This is also known as the “school method” for multiplying integers.
- Note that the intermediate numbers that are generated can have at most  $m + n \leq 2n$  bits.

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**Time requirement:**

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**Time requirement:**

- ▶ Computing intermediate results:  $\mathcal{O}(nm)$ .

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### Time requirement:

- ▶ Computing intermediate results:  $\mathcal{O}(nm)$ .
- ▶ Adding  $m$  numbers of length  $\leq 2n$ :  
 $\mathcal{O}((m+n)m) = \mathcal{O}(nm)$ .

## Example: Multiplying Two Integers

**A recursive approach:**

Suppose that integers  $A$  and  $B$  are of length  $n = 2^k$ , for some  $k$ .

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$$\boxed{b_{n-1} \quad \dots \quad b_0} \times \boxed{a_{n-1} \quad \dots \quad a_0}$$

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**A recursive approach:**

Suppose that integers  $A$  and  $B$  are of length  $n = 2^k$ , for some  $k$ .

$$\boxed{b_{n-1} \quad \cdots \quad b_{\frac{n}{2}} \quad b_{\frac{n}{2}-1} \quad \cdots \quad b_0} \times \boxed{a_{n-1} \quad \cdots \quad a_{\frac{n}{2}} \quad a_{\frac{n}{2}-1} \quad \cdots \quad a_0}$$

## Example: Multiplying Two Integers

**A recursive approach:**

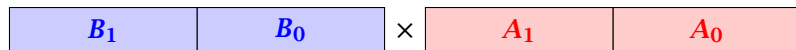
Suppose that integers  $A$  and  $B$  are of length  $n = 2^k$ , for some  $k$ .

$$\begin{array}{|c|c|} \hline B_1 & B_0 \\ \hline \end{array} \times \begin{array}{|c|c|} \hline A_1 & A_0 \\ \hline \end{array}$$

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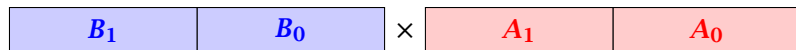
Then it holds that

$$A = A_1 \cdot 2^{\frac{n}{2}} + A_0 \text{ and } B = B_1 \cdot 2^{\frac{n}{2}} + B_0$$

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Hence,

$$A \cdot B = A_1 B_1 \cdot 2^n + (A_1 B_0 + A_0 B_1) \cdot 2^{\frac{n}{2}} + A_0 B_0$$

## Example: Multiplying Two Integers

### Algorithm 3 $\text{mult}(A, B)$

```
1: if  $|A| = |B| = 1$  then  
2:   return  $a_0 \cdot b_0$   
3: split  $A$  into  $A_0$  and  $A_1$   
4: split  $B$  into  $B_0$  and  $B_1$   
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$\mathcal{O}(1)$

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$T\left(\frac{n}{2}\right)$

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8: <b>return</b> $Z_2 \cdot 2^n + Z_1 \cdot 2^{\frac{n}{2}} + Z_0$	$\mathcal{O}(n)$

We get the following recurrence:

$$T(n) = 4T\left(\frac{n}{2}\right) + \mathcal{O}(n) .$$

# Example: Multiplying Two Integers

**Master Theorem:** Recurrence:  $T[n] = aT(\frac{n}{b}) + f(n)$ .

- ▶ Case 1:  $f(n) = \mathcal{O}(n^{\log_b a - \epsilon})$        $T(n) = \Theta(n^{\log_b a})$
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⇒ Not better than the “school method”.

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Again we are in Case 1. We get a running time of  $\Theta(n^{\log_2 3}) \approx \Theta(n^{1.59})$ .

A huge improvement over the "school method".

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## 6.3 The Characteristic Polynomial

Consider the recurrence relation:

$$c_0T(n) + c_1T(n-1) + c_2T(n-2) + \dots + c_kT(n-k) = f(n)$$

This is the general form of a **linear** recurrence relation of **order  $k$**  with constant coefficients ( $c_0, c_k \neq 0$ ).

The order depends on the maximum value of  $T(n-k)$ . The recurrence relation is of order  $k$  if  $c_k \neq 0$ .

The recurrence is **linear** as there are no products of  $T(n-k)$  or  $T(n-k)T(n-k)$ . If  $f(n) = 0$ , then the recurrence relation becomes a **linear homogeneous** recurrence relation of order  $k$ .

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### Observations:

- ▶ The solution  $T[1], T[2], T[3], \dots$  is completely determined by a set of **boundary conditions** that specify values for  $T[1], \dots, T[k]$ .
- ▶ In fact, any  $k$  consecutive values completely determine the solution.
- ▶  $k$  non-consecutive values might not be an appropriate set of boundary conditions (depends on the problem).

### Approach:

- ▶ First determine all solutions that satisfy recurrence relation.
- ▶ Then pick the right one by analyzing boundary conditions.
- ▶ First consider the homogenous case.

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# The Homogenous Case

The solution space

$$S = \left\{ \mathcal{T} = T[1], T[2], T[3], \dots \mid \mathcal{T} \text{ fulfills recurrence relation} \right\}$$

is a **vector space**. This means that if  $\mathcal{T}_1, \mathcal{T}_2 \in S$ , then also  $\alpha\mathcal{T}_1 + \beta\mathcal{T}_2 \in S$ , for arbitrary constants  $\alpha, \beta$ .

How do we find a non-trivial solution?

We guess that the solution is of the form  $\lambda^n$ ,  $\lambda \neq 0$ , and see what happens. In order for this guess to fulfill the recurrence we need

$$c_0\lambda^n + c_1\lambda^{n-1} + c_2 \cdot \lambda^{n-2} + \dots + c_k \cdot \lambda^{n-k} = 0$$

for all  $n \geq k$ .

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Dividing by  $\lambda^{n-k}$  gives that all these constraints are identical to

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This means that if  $\lambda_i$  is a root (Nullstelle) of  $P[\lambda]$  then  $T[n] = \lambda_i^n$  is a solution to the recurrence relation.

Let  $\lambda_1, \dots, \lambda_k$  be the  $k$  (complex) roots of  $P[\lambda]$ . Then, because of the vector space property

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## Lemma 5

Assume that the characteristic polynomial has  $k$  *distinct* roots  $\lambda_1, \dots, \lambda_k$ . Then *all* solutions to the recurrence relation are of the form

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## Proof.

There is one solution for every possible choice of boundary conditions for  $T[1], \dots, T[k]$ .

We show that the above set of solutions contains one solution for every choice of boundary conditions.

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## Proof (cont.).

Suppose I am given boundary conditions  $T[i]$  and I want to see whether I can choose the  $\alpha'_i$ 's such that these conditions are met:



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We show that the column vectors are linearly independent. Then the above equation has a solution.

# Computing the Determinant

$$\begin{vmatrix} \lambda_1 & \lambda_2 & \cdots & \lambda_{k-1} & \lambda_k \\ \lambda_1^2 & \lambda_2^2 & \cdots & \lambda_{k-1}^2 & \lambda_k^2 \\ \vdots & \vdots & & \vdots & \vdots \\ \lambda_1^k & \lambda_2^k & \cdots & \lambda_{k-1}^k & \lambda_k^k \end{vmatrix} =$$

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$$\begin{vmatrix} \lambda_1 & \lambda_2 & \cdots & \lambda_{k-1} & \lambda_k \\ \lambda_1^2 & \lambda_2^2 & \cdots & \lambda_{k-1}^2 & \lambda_k^2 \\ \vdots & \vdots & & \vdots & \vdots \\ \lambda_1^k & \lambda_2^k & \cdots & \lambda_{k-1}^k & \lambda_k^k \end{vmatrix} = \prod_{i=1}^k \lambda_i \cdot \begin{vmatrix} 1 & 1 & \cdots & 1 & 1 \\ \lambda_1 & \lambda_2 & \cdots & \lambda_{k-1} & \lambda_k \\ \vdots & \vdots & & \vdots & \vdots \\ \lambda_1^{k-1} & \lambda_2^{k-1} & \cdots & \lambda_{k-1}^{k-1} & \lambda_k^{k-1} \end{vmatrix}$$
$$= \prod_{i=1}^k \lambda_i \cdot \begin{vmatrix} 1 & \lambda_1 & \cdots & \lambda_1^{k-2} & \lambda_1^{k-1} \\ 1 & \lambda_2 & \cdots & \lambda_2^{k-2} & \lambda_2^{k-1} \\ \vdots & \vdots & & \vdots & \vdots \\ 1 & \lambda_k & \cdots & \lambda_k^{k-2} & \lambda_k^{k-1} \end{vmatrix}$$

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$$\begin{vmatrix} 1 & \lambda_1 - \lambda_1 \cdot 1 & \cdots & \lambda_1^{k-2} - \lambda_1 \cdot \lambda_1^{k-3} & \lambda_1^{k-1} - \lambda_1 \cdot \lambda_1^{k-2} \\ 1 & \lambda_2 - \lambda_1 \cdot 1 & \cdots & \lambda_2^{k-2} - \lambda_1 \cdot \lambda_2^{k-3} & \lambda_2^{k-1} - \lambda_1 \cdot \lambda_2^{k-2} \\ \vdots & \vdots & & \vdots & \vdots \\ 1 & \lambda_k - \lambda_1 \cdot 1 & \cdots & \lambda_k^{k-2} - \lambda_1 \cdot \lambda_k^{k-3} & \lambda_k^{k-1} - \lambda_1 \cdot \lambda_k^{k-2} \end{vmatrix}$$

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$$\begin{vmatrix} 1 & 0 & \cdots & 0 & 0 \\ 1 & (\lambda_2 - \lambda_1) \cdot 1 & \cdots & (\lambda_2 - \lambda_1) \cdot \lambda_2^{k-3} & (\lambda_2 - \lambda_1) \cdot \lambda_2^{k-2} \\ \vdots & \vdots & & \vdots & \vdots \\ 1 & (\lambda_k - \lambda_1) \cdot 1 & \cdots & (\lambda_k - \lambda_1) \cdot \lambda_k^{k-3} & (\lambda_k - \lambda_1) \cdot \lambda_k^{k-2} \end{vmatrix}$$

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$$\prod_{i=2}^k (\lambda_i - \lambda_1) \cdot \begin{vmatrix} 1 & \lambda_2 & \cdots & \lambda_2^{k-3} & \lambda_2^{k-2} \\ \vdots & \vdots & & \vdots & \vdots \\ 1 & \lambda_k & \cdots & \lambda_k^{k-3} & \lambda_k^{k-2} \end{vmatrix}$$

# Computing the Determinant

Repeating the above steps gives:

$$\begin{vmatrix} \lambda_1 & \lambda_2 & \cdots & \lambda_{k-1} & \lambda_k \\ \lambda_1^2 & \lambda_2^2 & \cdots & \lambda_{k-1}^2 & \lambda_k^2 \\ \vdots & \vdots & & \vdots & \vdots \\ \lambda_1^k & \lambda_2^k & \cdots & \lambda_{k-1}^k & \lambda_k^k \end{vmatrix} = \prod_{i=1}^k \lambda_i \cdot \prod_{i>\ell} (\lambda_i - \lambda_\ell)$$

Hence, if all  $\lambda_i$ 's are different, then the determinant is non-zero.



# The Homogeneous Case

## What happens if the roots are not all distinct?

Suppose we have a root  $\lambda_i$  with multiplicity (Vielfachheit) at least 2. Then not only is  $\lambda_i^n$  a solution to the recurrence but also  $n\lambda_i^{n-1}$ .

To see this consider the polynomial

$$P[\lambda] \cdot \lambda^{n-k} = c_0\lambda^n + c_1\lambda^{n-1} + c_2\lambda^{n-2} + \dots + c_k\lambda^{n-k}$$

Since  $\lambda_i$  is a root we can write this as  $Q[\lambda] \cdot (\lambda - \lambda_i)^2$ .

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Calculating the derivative gives a polynomial that still has root  $\lambda_i$ .

# The Homogeneous Case

## What happens if the roots are not all distinct?

Suppose we have a root  $\lambda_i$  with multiplicity (**Vielfachheit**) at least 2. Then not only is  $\lambda_i^n$  a solution to the recurrence but also  $n\lambda_i^{n-1}$ .

To see this consider the polynomial

$$P[\lambda] \cdot \lambda^{n-k} = c_0\lambda^n + c_1\lambda^{n-1} + c_2\lambda^{n-2} + \dots + c_k\lambda^{n-k}$$

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Calculating the derivative gives a polynomial that still has root  $\lambda_i$ .

This means

$$c_0 n \lambda_i^{n-1} + c_1 (n-1) \lambda_i^{n-2} + \dots + c_k (n-k) \lambda_i^{n-k-1} = 0$$

Hence,

$$\underbrace{c_0 n \lambda_i^n}_{T[n]} + \underbrace{c_1 (n-1) \lambda_i^{n-1}}_{T[n-1]} + \dots + \underbrace{c_k (n-k) \lambda_i^{n-k}}_{T[n-k]} = 0$$

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Suppose  $\lambda_i$  has multiplicity  $j$ . We know that

$$c_0 n \lambda_i^n + c_1 (n-1) \lambda_i^{n-1} + \dots + c_k (n-k) \lambda_i^{n-k} = 0$$

(after taking the derivative; multiplying with  $\lambda$ ; plugging in  $\lambda_i$ )

Doing this again gives

$$c_0 n^2 \lambda_i^n + c_1 (n-1)^2 \lambda_i^{n-1} + \dots + c_k (n-k)^2 \lambda_i^{n-k} = 0$$

We can continue  $j-1$  times.

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# The Homogeneous Case

## Lemma 6

Let  $P[\lambda]$  denote the characteristic polynomial to the recurrence

$$c_0T[n] + c_1T[n-1] + \dots + c_kT[n-k] = 0$$

Let  $\lambda_i, i = 1, \dots, m$  be the (complex) roots of  $P[\lambda]$  with multiplicities  $\ell_i$ . Then the general solution to the recurrence is given by

$$T[n] = \sum_{i=1}^m \sum_{j=0}^{\ell_i-1} \alpha_{ij} \cdot (n^j \lambda_i^n) .$$

The full proof is omitted. We have only shown that any choice of  $\alpha_{ij}$ 's is a solution to the recurrence.

## Example: Fibonacci Sequence

$$T[0] = 0$$

$$T[1] = 1$$

$$T[n] = T[n - 1] + T[n - 2] \text{ for } n \geq 2$$

The characteristic polynomial is

$$\lambda^2 - \lambda - 1$$

Finding the roots, gives

$$\lambda_{1/2} = \frac{1}{2} \pm \sqrt{\frac{1}{4} + 1} = \frac{1}{2} (1 \pm \sqrt{5})$$

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Hence, the solution is of the form

$$\alpha \left( \frac{1 + \sqrt{5}}{2} \right)^n + \beta \left( \frac{1 - \sqrt{5}}{2} \right)^n$$

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$$\alpha \left( \frac{1 + \sqrt{5}}{2} \right) + \beta \left( \frac{1 - \sqrt{5}}{2} \right) = 1 \implies \alpha - \beta = \frac{2}{\sqrt{5}}$$

## Example: Fibonacci Sequence

Hence, the solution is

$$\frac{1}{\sqrt{5}} \left[ \left( \frac{1 + \sqrt{5}}{2} \right)^n - \left( \frac{1 - \sqrt{5}}{2} \right)^n \right]$$

# The Inhomogeneous Case

Consider the recurrence relation:

$$c_0T(n) + c_1T(n-1) + c_2T(n-2) + \cdots + c_kT(n-k) = f(n)$$

with  $f(n) \neq 0$ .

While we have a fairly general technique for solving **homogeneous**, linear recurrence relations the inhomogeneous case is different.

# The Inhomogeneous Case

The general solution of the recurrence relation is

$$T(n) = T_h(n) + T_p(n) ,$$

where  $T_h$  is **any** solution to the homogeneous equation, and  $T_p$  is **one** particular solution to the inhomogeneous equation.

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# The Inhomogeneous Case

Example:

$$T[n] = T[n - 1] + 1 \quad T[0] = 1$$

Then,

$$T[n - 1] = T[n - 2] + 1 \quad (n \geq 2)$$

Subtracting the first from the second equation gives,

$$T[n] - T[n - 1] = T[n - 1] - T[n - 2] \quad (n \geq 2)$$

or

$$T[n] = 2T[n - 1] - T[n - 2] \quad (n \geq 2)$$

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$T[0] = 1$  gives  $\alpha = 1$ .

$T[1] = 2$  gives  $1 + \beta = 2 \Rightarrow \beta = 1$ .

## The Inhomogeneous Case

If  $f(n)$  is a polynomial of degree  $r$  this method can be applied  $r + 1$  times to obtain a homogeneous equation:

$$T[n] = T[n - 1] + n^2$$

Shift:

$$T[n - 1] = T[n - 2] + (n - 1)^2$$

Difference:

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Shift:

$$T[n - 1] = 2T[n - 2] - T[n - 3] + 2(n - 1) - 1$$

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Shift:

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and so on...

## 6.4 Generating Functions

### Definition 7 (Generating Function)

Let  $(a_n)_{n \geq 0}$  be a sequence. The corresponding

- ▶ **generating function** (Erzeugendenfunktion) is

$$F(z) := \sum_{n \geq 0} a_n z^n;$$

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## 6.4 Generating Functions

There are two different views:

A generating function is a **formal power series** (formale Potenzreihe).

Then the generating function is an **algebraic object**.

Let  $f = \sum_{n \geq 0} a_n z^n$  and  $g = \sum_{n \geq 0} b_n z^n$ .

- ▶ **Equality:**  $f$  and  $g$  are equal if  $a_n = b_n$  for all  $n$ .
- ▶ **Addition:**  $f + g := \sum_{n \geq 0} (a_n + b_n) z^n$ .
- ▶ **Multiplication:**  $f \cdot g := \sum_{n \geq 0} c_n z^n$  with  $c_n = \sum_{p=0}^n a_p b_{n-p}$ .

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There are no convergence issues here.



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There are two different views:

A generating function is a **formal power series** (**formale Potenzreihe**).

Then the generating function is an **algebraic object**.

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What does  $\sum_{n \geq 0} z^n = \frac{1}{1-z}$  mean in the algebraic view?

It means that the power series  $1 - z$  and the power series  $\sum_{n \geq 0} z^n$  are invers, i.e.,

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Hence, the generating function of the sequence  $a_n = n + 1$  is  $1/(1-z)^2$ .

Formally the derivative of a formal power series  $\sum_{n \geq 0} a_n z^n$  is defined as  $\sum_{n \geq 0} n a_n z^{n-1}$ .

The known rules for differentiation work for this definition. In particular, e.g. the derivative of  $\frac{1}{1-z}$  is  $\frac{1}{(1-z)^2}$ .

Note that this requires a proof if we consider power series as algebraic objects. However, we did not prove this in the lecture.

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We know

$$\sum_{n \geq 0} y^n = \frac{1}{1-y}$$

Hence,

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6. The coefficients of the resulting power series are the  $a_n$ .

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$$\begin{aligned} A(z) &= 1 + \sum_{n \geq 1} (2a_{n-1})z^n \\ &= 1 + 2z \sum_{n \geq 1} a_{n-1}z^{n-1} \end{aligned}$$

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which gives

$$A = \frac{7}{4} \quad B = -\frac{1}{4} \quad C = -\frac{1}{2}$$



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6. This means  $a_n = \frac{7}{4}3^n - \frac{1}{2}n - \frac{3}{4}$ .

## 6.5 Transformation of the Recurrence

### Example 9

$$f_0 = 1$$

$$f_1 = 2$$

$$f_n = f_{n-1} \cdot f_{n-2} \text{ for } n \geq 2 .$$

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Let  $n = 2^k$ :

$$g_k = 3^{k+1} - 2^{k+1}, \text{ hence}$$

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