

# Flows

## Definition 2

An  $(s, t)$ -flow in a (complete) directed graph  $G = (V, V \times V, c)$  is a function  $f : V \times V \rightarrow \mathbb{R}_0^+$  that satisfies

1. For each edge  $(x, y)$

$$0 \leq f_{xy} \leq c_{xy} .$$

(capacity constraints)

2. For each  $v \in V \setminus \{s, t\}$

$$\sum_x f_{vx} = \sum_x f_{xv} .$$

(flow conservation constraints)

## Definition 3

The **value of an  $(s, t)$ -flow  $f$**  is defined as

$$\text{val}(f) = \sum_x f_{sx} - \sum_x f_{xs} .$$

## Maximum Flow Problem:

Find an  $(s, t)$ -flow with maximum value.

# LP-Formulation of Maxflow

$$\begin{array}{ll} \max & \sum_z f_{sz} - \sum_z f_{zs} \\ \text{s.t.} & \forall (z, w) \in V \times V \quad f_{zw} \leq c_{zw} \quad \ell_{zw} \\ & \forall w \neq s, t \quad \sum_z f_{zw} - \sum_z f_{wz} = 0 \quad p_w \\ & f_{zw} \geq 0 \end{array}$$

$$\begin{array}{ll} \min & \sum_{(x,y)} c_{xy} \ell_{xy} \\ \text{s.t.} & f_{xy} \ (x, y \neq s, t): \quad 1\ell_{xy} - 1p_x + 1p_y \geq 0 \\ & f_{sy} \ (y \neq s, t): \quad 1\ell_{sy} \quad + 1p_y \geq 1 \\ & f_{xs} \ (x \neq s, t): \quad 1\ell_{xs} - 1p_x \quad \geq -1 \\ & f_{ty} \ (y \neq s, t): \quad 1\ell_{ty} \quad + 1p_y \geq 0 \\ & f_{xt} \ (x \neq s, t): \quad 1\ell_{xt} - 1p_x \quad \geq 0 \\ & f_{st}: \quad 1\ell_{st} \quad \geq 1 \\ & f_{ts}: \quad 1\ell_{ts} \quad \geq -1 \\ & \ell_{xy} \quad \geq 0 \end{array}$$

# LP-Formulation of Maxflow

$$\begin{array}{ll} \min & \sum_{(x,y)} c_{xy} l_{xy} \\ \text{s.t.} & f_{xy} (x, y \neq s, t) : 1l_{xy} - 1p_x + 1p_y \geq 0 \\ & f_{sy} (y \neq s, t) : 1l_{sy} - 1 + 1p_y \geq 0 \\ & f_{xs} (x \neq s, t) : 1l_{xs} - 1p_x + 1 \geq 0 \\ & f_{ty} (y \neq s, t) : 1l_{ty} - 0 + 1p_y \geq 0 \\ & f_{xt} (x \neq s, t) : 1l_{xt} - 1p_x + 0 \geq 0 \\ & f_{st} : 1l_{st} - 1 + 0 \geq 0 \\ & f_{ts} : 1l_{ts} - 0 + 1 \geq 0 \\ & l_{xy} \geq 0 \end{array}$$

# LP-Formulation of Maxflow

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with  $p_t = 0$  and  $p_s = 1$ .

# LP-Formulation of Maxflow

$$\begin{array}{ll} \min & \sum_{(xy)} c_{xy} \ell_{xy} \\ \text{s.t. } & f_{xy} : \quad 1\ell_{xy} - 1p_x + 1p_y \geq 0 \\ & \ell_{xy} \geq 0 \\ & p_s = 1 \\ & p_t = 0 \end{array}$$

We can interpret the  $\ell_{xy}$  value as assigning a length to every edge.

The value  $p_x$  for a variable, then can be seen as the distance of  $x$  to  $t$  (where the distance from  $s$  to  $t$  is required to be 1 since  $p_s = 1$ ).

The constraint  $p_x \leq \ell_{xy} + p_y$  then simply follows from triangle inequality ( $d(x, t) \leq d(x, y) + d(y, t) \Rightarrow d(x, t) \leq \ell_{xy} + d(y, t)$ ).

One can show that there is an optimum LP-solution for the dual problem that gives an integral assignment of variables.

This means  $p_x = 1$  or  $p_x = 0$  for our case. This gives rise to a cut in the graph with vertices having value 1 on one side and the other vertices on the other side. The objective function then evaluates the capacity of this cut.

This shows that the Maxflow/Mincut theorem follows from linear programming duality.