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**Can we obtain a better analysis?**

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## Observation

Simplex visits every **feasible** basis at most once.

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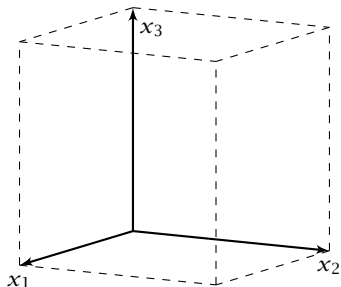
Simplex visits every **feasible** basis at most once.

However, also the number of feasible bases can be very large.



## Example

$$\begin{aligned} \max \quad & c^T x \\ \text{s.t.} \quad & 0 \leq x_1 \leq 1 \\ & 0 \leq x_2 \leq 1 \\ & \vdots \\ & 0 \leq x_n \leq 1 \end{aligned}$$

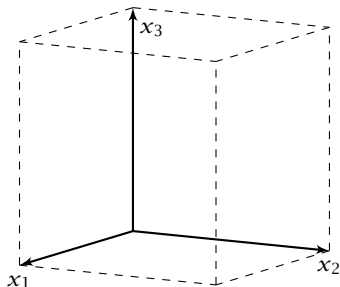


$2n$  constraint on  $n$  variables define an  $n$ -dimensional hypercube as feasible region.

The feasible region has  $2^n$  vertices.

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However, Simplex may still run quickly as it usually does not visit all feasible bases.

In the following we give an example of a feasible region for which there is a bad **Pivoting Rule**.

# Pivoting Rule

A Pivoting Rule defines how to choose the entering and leaving variable for an iteration of Simplex.

In the non-degenerate case after choosing the entering variable the leaving variable is unique.

# Klee Minty Cube

$\max x_n$

s.t.  $0 \leq x_1 \leq 1$

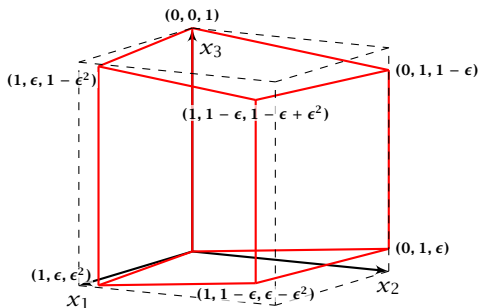
$$\epsilon x_1 \leq x_2 \leq 1 - \epsilon x_1$$

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$\vdots$

$$\epsilon x_{n-1} \leq x_n \leq 1 - \epsilon x_{n-1}$$

$$x_i \geq 0$$



## Observations

- ▶ We have  $2n$  constraints, and  $3n$  variables (after adding slack variables to every constraint).
- ▶ Every basis is defined by  $2n$  variables, and  $n$  non-basic variables.
- ▶ There exist degenerate vertices.
- ▶ The degeneracies come from the non-negativity constraints, which are superfluous.
- ▶ In the following all variables  $x_i$  stay in the basis at all times.
- ▶ Then, we can uniquely specify a basis by choosing for each variable whether it should be equal to its lower bound, or equal to its upper bound (the slack variable corresponding to the non-tight constraint is part of the basis).
- ▶ We can also simply identify each basis/vertex with the corresponding hypercube vertex obtained by letting  $\epsilon \rightarrow 0$ .

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# Analysis

- ▶ In the following we specify a sequence of bases (identified by the corresponding hypercube node) along which the objective function strictly increases.
- ▶ The basis  $(0, \dots, 0, 1)$  is the unique optimal basis.
- ▶ Our sequence  $S_n$  starts at  $(0, \dots, 0)$  ends with  $(0, \dots, 0, 1)$  and visits every node of the hypercube.
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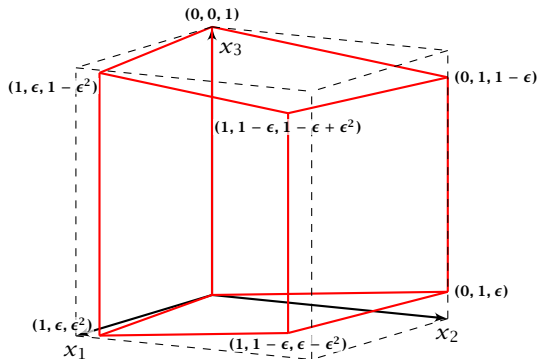
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$$\text{s.t. } 0 \leq x_1 \leq 1$$

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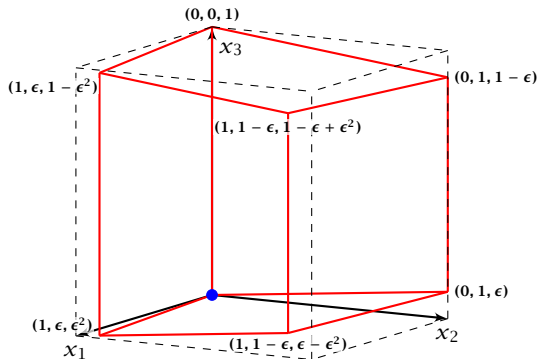
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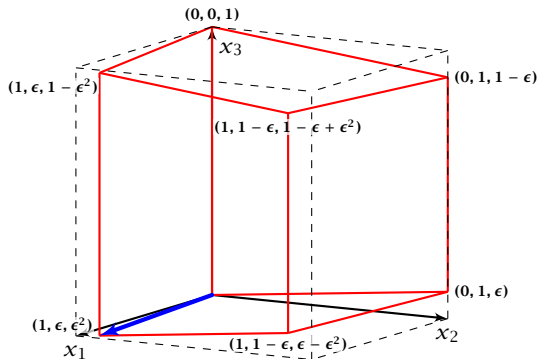
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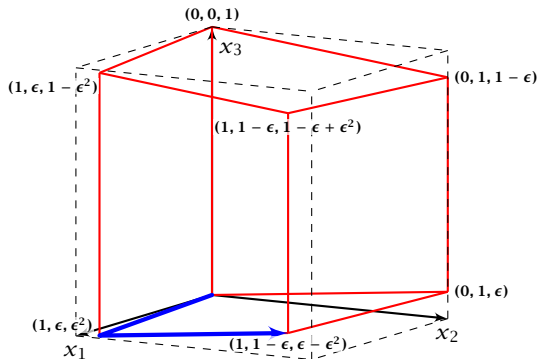
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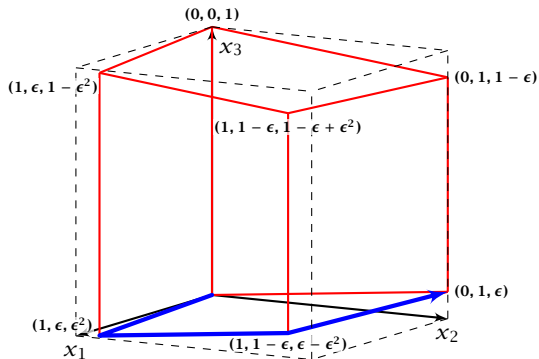
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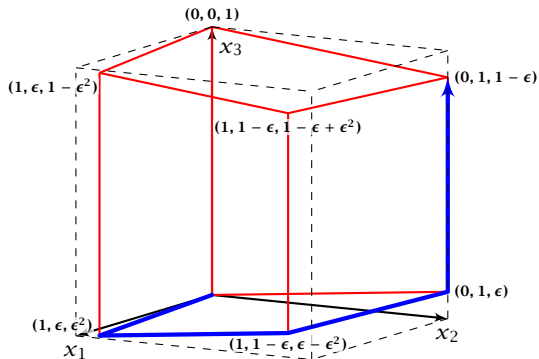
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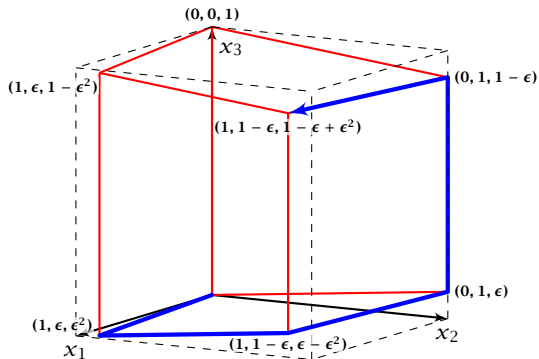
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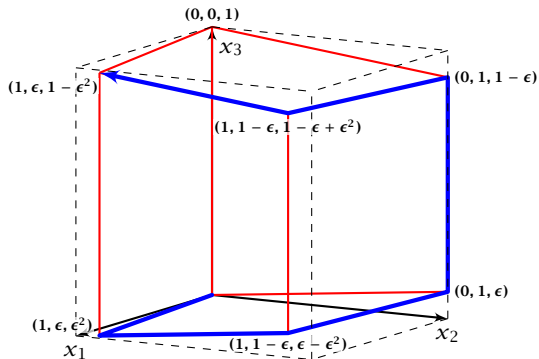
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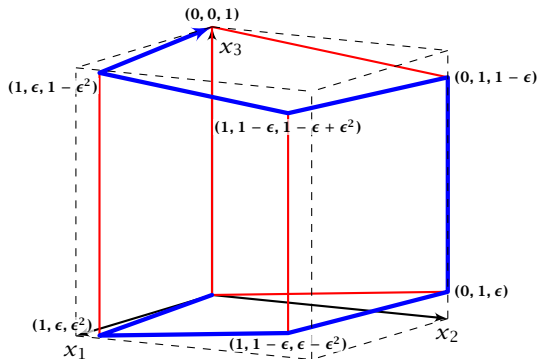
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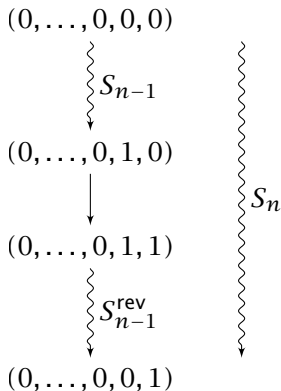
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## Analysis

The sequence  $S_n$  that visits every node of the hypercube is defined recursively



The non-recursive case is  $S_1 = 0 \rightarrow 1$

# Analysis

## Lemma 2

The objective value  $x_n$  is increasing along path  $S_n$ .

Proof by induction:

$n = 1$ : obvious, since  $S_1 = 0 \rightarrow 1$ , and  $1 > 0$ .

$n - 1 \rightarrow n$

By the first part the value of

by induction hypothesis  $x_{n-1}$  is increasing along  $S_{n-1}$ .  
Hence, also

Going from  $n-1$  to  $n$  we have  $x_{n-1} > x_{n-2}$ .  
small enough  $\epsilon$ .

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Going from  $S_{n-1}$  to  $S_n$  we add  $n$  nodes. If  $n$  is small enough,  $x_n$  is increasing.

For the remaining part of the proof we have

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- ▶ Going from  $(0, \dots, 0, 1, 0)$  to  $(0, \dots, 0, 1, 1)$  increases  $x_n$  for small enough  $\epsilon$ .
- ▶ For the remaining path  $S_{n-1}^{\text{rev}}$  we have  $x_n = 1 - \epsilon x_{n-1}$ .
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# Remarks about Simplex

## Observation

The simplex algorithm takes at most  $\binom{n}{m}$  iterations. Each iteration can be implemented in time  $\mathcal{O}(mn)$ .

In practise it usually takes a linear number of iterations.

# Remarks about Simplex

## Theorem

For almost all known **deterministic** pivoting rules (rules for choosing entering and leaving variables) there exist lower bounds that require the algorithm to have exponential running time ( $\Omega(2^{\Omega(n)})$ ) (e.g. Klee Minty 1972).

# Remarks about Simplex

## Theorem

For some standard **randomized** pivoting rules there exist subexponential lower bounds ( $\Omega(2^{\Omega(n^\alpha)})$  for  $\alpha > 0$ ) (Friedmann, Hansen, Zwick 2011).

# Remarks about Simplex

## Conjecture (Hirsch 1957)

The edge-vertex graph of an  $m$ -facet polytope in  $d$ -dimensional Euclidean space has diameter no more than  $m - d$ .

The conjecture has been proven wrong in 2010.

But the question whether the diameter is perhaps of the form  $\mathcal{O}(\text{poly}(m, d))$  is open.