Brewery Problem

Brewery brews ale and beer.

- ▶ Production limited by supply of corn, hops and barley malt
- ▶ Recipes for ale and beer require different amounts of resources

	Corn (kg)	Hops (kg)	Malt (kg)	Profit (€)
ale (barrel)	5	4	35	13
beer (barrel)	15	4	20	23
supply	480	160	1190	

Brewery Problem

	Corn (kg)	Hops (kg)	Malt (kg)	Profit (€)
ale (barrel)	5	4	35	13
beer (barrel)	15	4	20	23
supply	480	160	1190	

How can brewer maximize profits?

- ▶ only brew ale: 34 barrels of ale $\implies$ 442 €
- ▶ only brew beer: 32 barrels of beer $\implies$ 736 €
- ▶ 7.5 barrels ale, 29.5 barrels beer $\implies$ 776 €
- ▶ 12 barrels ale, 28 barrels beer $\implies$ 800 €

Brewery Problem

Linear Program

- ▶ Introduce variables a and b that define how much ale and beer to produce.
- ▶ Choose the variables in such a way that the objective function (profit) is maximized.
- ▶ Make sure that no constraints (due to limited supply) are violated.

$$
\begin{aligned}
\max \quad & 13a + 23b \\
\text{s.t.} \quad & 5a + 15b \leq 480 \\
& 4a + 4b \leq 160 \\
& 35a + 20b \leq 1190 \\
& a, b \geq 0
\end{aligned}
$$

Standard Form LPs

LP in standard form:

- ▶ input: numbers a_{ij}, c_j, b_i
- ▶ output: numbers x_j
- ▶ n = #decision variables, m = #constraints
- ▶ maximize linear objective function subject to linear (in)equalities

$$
\begin{aligned}
\max \quad & \sum_{j=1}^{n} c_j x_j \\
\text{s.t.} \quad & \sum_{j=1}^{n} a_{ij} x_j = b_i \quad 1 \leq i \leq m \\
& x_j \geq 0 \quad 1 \leq j \leq n
\end{aligned}
$$

$$
\begin{aligned}
\max \quad & c^T x \\
\text{s.t.} \quad & Ax = b \\
& x \geq 0
\end{aligned}
$$

Standard Form LPs

Original LP

$$\begin{aligned} \max \quad & 13a \;+\; 23b \\ \text{s.t.} \quad 5a \;+\; & 15b \;\le\; 480 \\ 4a \;+\; & 4b \;\le\; 160 \\ 35a \;+\; & 20b \;\le\; 1190 \\ & a, b \;\ge\; 0 \end{aligned}$$

Standard Form

Add a slack variable to every constraint.

$$\begin{aligned} \max \quad 13a \;+\; & 23b \\ \text{s.t.} \quad 5a \;+\; 15b \;+\; & s_c \qquad\qquad\qquad = 480 \\ 4a \;+\; 4b \qquad\;+\; & s_h \qquad\quad = 160 \\ 35a \;+\; 20b \qquad\qquad\;+\; & s_m = 1190 \\ a \;,\; b \;,\; s_c \;,\; s_h \;,\; & s_m \ge 0 \end{aligned}$$

Standard Form LPs

There are different standard forms:

standard form

$$\begin{aligned} \max \quad & c^T x \\ \text{s.t.} \quad Ax \;&=\; b \\ x \;&\ge\; 0 \end{aligned}$$

$$\begin{aligned} \min \quad & c^T x \\ \text{s.t.} \quad Ax \;&=\; b \\ x \;&\ge\; 0 \end{aligned}$$

standard maximization form

$$\begin{aligned} \max \quad & c^T x \\ \text{s.t.} \quad Ax \;&\le\; b \\ x \;&\ge\; 0 \end{aligned}$$

standard minimization form

$$\begin{aligned} \min \quad & c^T x \\ \text{s.t.} \quad Ax \;&\ge\; b \\ x \;&\ge\; 0 \end{aligned}$$

Standard Form LPs

It is easy to transform variants of LPs into (any) standard form:

▶ **less or equal to equality:**

$$a - 3b + 5c \le 12 \implies \begin{aligned} a - 3b + 5c + s &= 12 \\ s &\ge 0 \end{aligned}$$

▶ **greater or equal to equality:**

$$a - 3b + 5c \ge 12 \implies \begin{aligned} a - 3b + 5c - s &= 12 \\ s &\ge 0 \end{aligned}$$

▶ **min to max:**

$$\min a - 3b + 5c \implies \max -a + 3b - 5c$$

Standard Form LPs

It is easy to transform variants of LPs into (any) standard form:

▶ **equality to less or equal:**

$$a - 3b + 5c = 12 \implies \begin{aligned} a - 3b + 5c &\le 12 \\ -a + 3b - 5c &\le -12 \end{aligned}$$

▶ **equality to greater or equal:**

$$a - 3b + 5c = 12 \implies \begin{aligned} a - 3b + 5c &\ge 12 \\ -a + 3b - 5c &\ge -12 \end{aligned}$$

▶ **unrestricted to nonnegative:**

$$x \text{ unrestricted} \implies x = x^+ - x^-, \; x^+ \ge 0, \; x^- \ge 0$$

Standard Form LPs

Observations:

- ▶ a linear program does not contain x^2, $\cos(x)$, etc.
- ▶ transformations between standard forms can be done efficiently and only change the size of the LP by a small constant factor
- ▶ for the standard minimization or maximization LPs we could include the nonnegativity constraints into the set of ordinary constraints; this is of course not possible for the standard form

Fundamental Questions

Definition 1 (Linear Programming Problem (LP))

Let $A \in \mathbb{Q}^{m \times n}$, $b \in \mathbb{Q}^m$, $c \in \mathbb{Q}^n$, $\alpha \in \mathbb{Q}$. Does there exist $x \in \mathbb{Q}^n$ s.t. $Ax = b$, $x \geq 0$, $c^T x \geq \alpha$?
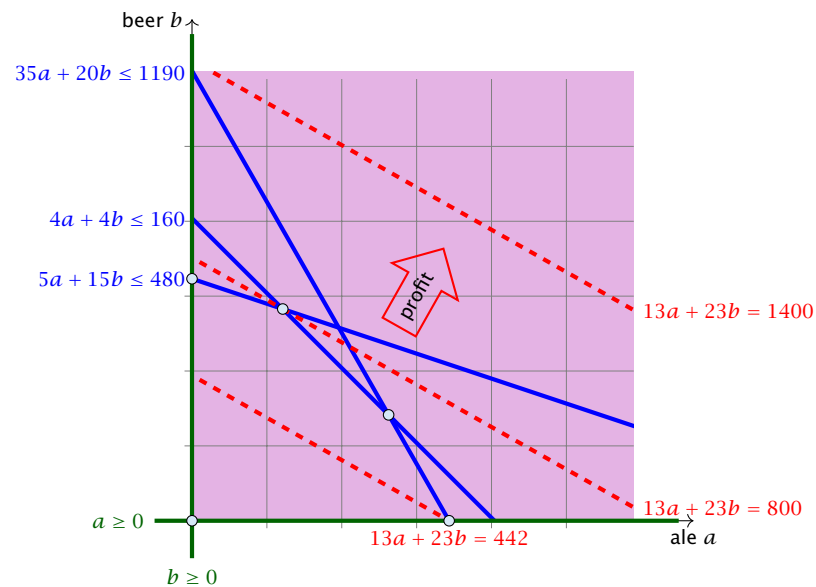
Questions:

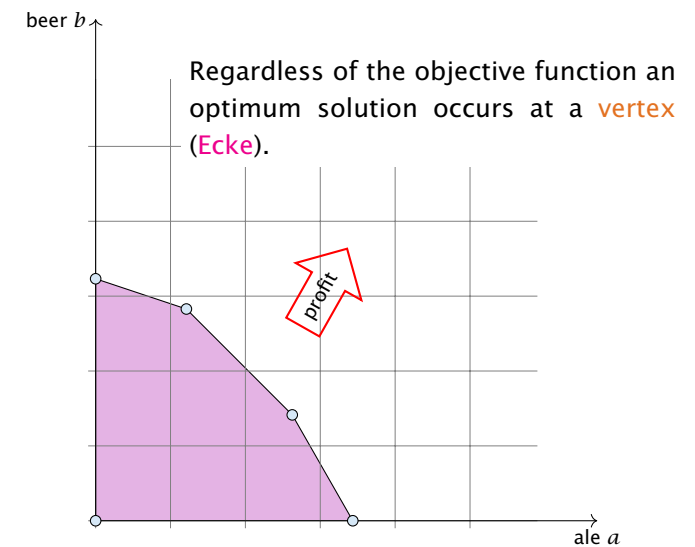- ▶ Is LP in NP?
- ▶ Is LP in co-NP?
- ▶ Is LP in P?

Input size:

- ▶ n number of variables, m constraints, L number of bits to encode the input

Geometry of Linear Programming

Geometry of Linear Programming

Regardless of the objective function an optimum solution occurs at a vertex (Ecke).

Definitions

Let for a Linear Program in standard form
$P = \{x \mid Ax = b, x \geq 0\}$.

- ▶ P is called the feasible region (Lösungsraum) of the LP.
- ▶ A point $x \in P$ is called a feasible point (gültige Lösung).
- ▶ If $P \neq \emptyset$ then the LP is called feasible (erfüllbar). Otherwise, it is called infeasible (unerfüllbar).
- ▶ An LP is bounded (beschränkt) if it is feasible and
  - ▶ $c^T x < \infty$ for all $x \in P$ (for maximization problems)
  - ▶ $c^T x > -\infty$ for all $x \in P$ (for minimization problems)

Definition 2
Given vectors/points $x_1, \ldots, x_k \in \mathbb{R}^n$, $\sum \lambda_i x_i$ is called

- ▶ linear combination if $\lambda_i \in \mathbb{R}$.
- ▶ affine combination if $\lambda_i \in \mathbb{R}$ and $\sum_i \lambda_i = 1$.
- ▶ convex combination if $\lambda_i \in \mathbb{R}$ and $\sum_i \lambda_i = 1$ and $\lambda_i \geq 0$.
- ▶ conic combination if $\lambda_i \in \mathbb{R}$ and $\lambda_i \geq 0$.

Note that a combination involves only finitely many vectors.

Definition 3
A set $X \subseteq \mathbb{R}^n$ is called

- ▶ a linear subspace if it is closed under linear combinations.
- ▶ an affine subspace if it is closed under affine combinations.
- ▶ convex if it is closed under convex combinations.
- ▶ a convex cone if it is closed under conic combinations.

Note that an affine subspace is **not** a vector space

Definition 4
Given a set $X \subseteq \mathbb{R}^n$.

- ▶ $\mathrm{span}(X)$ is the set of all linear combinations of X (linear hull, span)
- ▶ $\mathrm{aff}(X)$ is the set of all affine combinations of X (affine hull)
- ▶ $\mathrm{conv}(X)$ is the set of all convex combinations of X (convex hull)
- ▶ $\mathrm{cone}(X)$ is the set of all conic combinations of X (conic hull)

Definition 5

A function $f : \mathbb{R}^n \to \mathbb{R}$ is convex if for $x, y \in \mathbb{R}^n$ and $\lambda \in [0, 1]$
we have

$$f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda)f(y)$$

Lemma 6

If $P \subseteq \mathbb{R}^n$, and $f : \mathbb{R}^n \to \mathbb{R}$ convex then also

$$Q = \{x \in P \mid f(x) \leq t\}$$

Dimensions

Definition 7

The dimension $\dim(A)$ of an affine subspace $A \subseteq \mathbb{R}^n$ is the
dimension of the vector space $\{x - a \mid x \in A\}$, where $a \in A$.

Definition 8

The dimension $\dim(X)$ of a convex set $X \subseteq \mathbb{R}^n$ is the dimension
of its affine hull $\text{aff}(X)$.

Definition 9

A set $H \subseteq \mathbb{R}^n$ is a hyperplane if $H = \{x \mid a^T x = b\}$, for $a \neq 0$.

Definition 10

A set $H' \subseteq \mathbb{R}^n$ is a (closed) halfspace if $H = \{x \mid a^T x \leq b\}$, for
$a \neq 0$.

Definitions

Definition 11

A polytop is a set $P \subseteq \mathbb{R}^n$ that is the convex hull of a finite set of
points, i.e., $P = \text{conv}(X)$ where $|X| = c$.

Definitions

Definition 12

A polyhedron is a set $P \subseteq \mathbb{R}^n$ that can be represented as the intersection of finitely many half-spaces $\{H(a_1, b_1), \ldots, H(a_m, b_m)\}$, where

$$H(a_i, b_i) = \{x \in \mathbb{R}^n \mid a_i x \leq b_i\} \ .$$

Definition 13

A polyhedron P is bounded if there exists B s.t. $\|x\|_2 \leq B$ for all $x \in P$.

Definitions

Theorem 14

P is a bounded polyhedron iff P is a polytop.

Definition 15

Let $P \subseteq \mathbb{R}^n$, $a \in \mathbb{R}^n$ and $b \in \mathbb{R}$. The hyperplane

$$H(a, b) = \{x \in \mathbb{R}^n \mid a^T x = b\}$$

is a supporting hyperplane of P if $\max\{a^T x \mid x \in P\} = b$.

Definition 16

Let $P \subseteq \mathbb{R}^n$. F is a face of P if $F = P$ or $F = P \cap H$ for some supporting hyperplane H.

Definition 17

Let $P \subseteq \mathbb{R}^n$.

▶ a face v is a vertex of P if $\{v\}$ is a face of P.

▶ a face e is an edge of P if e is a face and $\dim(e) = 1$.

▶ a face F is a facet of P if F is a face and $\dim(F) = \dim(P) - 1$.

Equivalent definition for vertex:

Definition 18

Given polyhedron P. A point $x \in P$ is a vertex if $\exists c \in \mathbb{R}^n$ such that $c^T y < c^T x$, for all $y \in P$, $y \neq x$.

Definition 19

Given polyhedron P. A point $x \in P$ is an extreme point if $\nexists a, b \neq x$, $a, b \in P$, with $\lambda a + (1 - \lambda) b = x$ for $\lambda \in [0, 1]$.

Lemma 20

A vertex is also an extreme point.

Observation

The feasible region of an LP is a Polyhedron.

Convex Sets

Theorem 21

If there exists an optimal solution to an LP (in standard form) then there exists an optimum solution that is an extreme point.

Proof

▶ suppose x is optimal solution that is not extreme point

▶ there exists direction $d \neq 0$ such that $x \pm d \in P$

▶ $Ad = 0$ because $A(x \pm d) = b$

▶ Wlog. assume $c^T d \geq 0$ (by taking either d or $-d$)
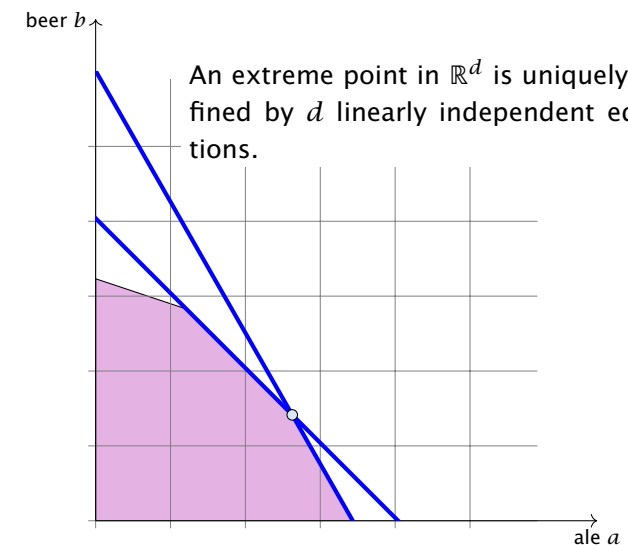
▶ Consider $x + \lambda d$, $\lambda > 0$

Convex Sets

Case 1. [$\exists j$ s.t. $d_j < 0$]

▶ increase λ to λ' until first component of $x + \lambda d$ hits 0

▶ $x + \lambda' d$ is feasible. Since $A(x + \lambda' d) = b$ and $x + \lambda' d \geq 0$

▶ $x + \lambda' d$ has one more zero-component ($d_k = 0$ for $x_k = 0$ as $x \pm d \in P$)

▶ $c^T x' = c^T(x + \lambda' d) = c^T x + \lambda' c^T d \geq c^T x$

Case 2. [$d_j \geq 0$ for all j and $c^T d > 0$]

▶ $x + \lambda d$ is feasible for all $\lambda \geq 0$ since $A(x + \lambda d) = b$ and $x + \lambda d \geq x \geq 0$

▶ as $\lambda \to \infty$, $c^T(x + \lambda d) \to \infty$ as $c^T d > 0$

Algebraic View

An extreme point in $\mathbb{R}^d$ is uniquely defined by d linearly independent equations.

Notation

Suppose $B \subseteq \{1 \ldots n\}$ is a set of column-indices. Define A_B as the subset of columns of A indexed by B.

Theorem 22

Let $P = \{x \mid Ax = b, x \geq 0\}$. For $x \in P$, define $B = \{j \mid x_j > 0\}$. Then x is extreme point **iff** A_B has linearly independent columns.

Theorem 22

Let $P = \{x \mid Ax = b, x \geq 0\}$. For $x \in P$, define $B = \{j \mid x_j > 0\}$. Then x is extreme point **iff** A_B has linearly independent columns.

Proof ($\Leftarrow$)

- ▶ assume x is not extreme point
- ▶ there exists direction d s.t. $x \pm d \in P$
- ▶ $Ad = 0$ because $A(x \pm d) = b$
- ▶ define $B' = \{j \mid d_j \neq 0\}$
- ▶ $A_{B'}$ has linearly dependent columns as $Ad = 0$
- ▶ $d_j = 0$ for all j with $x_j = 0$ as $x \pm d \geq 0$
- ▶ Hence, $B' \subseteq B$, $A_{B'}$ is sub-matrix of A_B

Theorem 22

Let $P = \{x \mid Ax = b, x \geq 0\}$. For $x \in P$, define $B = \{j \mid x_j > 0\}$. Then x is extreme point **iff** A_B has linearly independent columns.

Proof ($\Rightarrow$)

- ▶ assume A_B has linearly dependent columns
- ▶ there exists $d \neq 0$ such that $A_B d = 0$
- ▶ extend d to $\mathbb{R}^n$ by adding 0-components
- ▶ now, $Ad = 0$ and $d_j = 0$ whenever $x_j = 0$
- ▶ for sufficiently small λ we have $x \pm \lambda d \in P$
- ▶ hence, x is not extreme point

Theorem 23

Let $P = \{x \mid Ax = b, x \geq 0\}$. For $x \in P$, define $B = \{j \mid x_j > 0\}$. If A_B has linearly independent columns then x is a vertex of P.

- ▶ define $c_j = \begin{cases} 0 & j \in B \\ -1 & j \notin B \end{cases}$
- ▶ then $c^T x = 0$ and $c^T y \leq 0$ for $y \in P$
- ▶ assume $c^T y = 0$; then $y_j = 0$ for all $j \notin B$
- ▶ $b = Ay = A_B y_B = Ax = A_B x_B$ gives that $A_B(x_B - y_B) = 0$;
- ▶ this means that $x_B = y_B$ since A_B has linearly independent columns
- ▶ we get $y = x$
- ▶ hence, x is a vertex of P

Observation

For an LP we can assume wlog. that the matrix A has full
row-rank. This means $\text{rank}(A) = m$.

▶ assume that $\text{rank}(A) < m$
▶ assume wlog. that the first row A_1 lies in the span of the
  other rows $A_2, \ldots, A_m$; this means

$$A_1 = \sum_{i=2}^{m} \lambda_i \cdot A_i, \text{ for suitable } \lambda_i$$

C1 if now $b_1 = \sum_{i=2}^{m} \lambda_i \cdot b_i$ then for all x with $A_i x = b_i$ we also
  have $A_1 x = b_1$; hence the first constraint is superfluous

C2 if $b_1 \neq \sum_{i=2}^{m} \lambda_i \cdot b_i$ then the LP is infeasible, since for all x
  that fulfill constraints $A_2, \ldots, A_m$ we have

$$A_1 x = \sum_{i=2}^{m} \lambda_i \cdot A_i x = \sum_{i=2}^{m} \lambda_i \cdot b_i \neq b_1$$

From now on we will always assume that the
constraint matrix of a standard form LP has full
row rank.

Theorem 24

*Given $P = \{x \mid Ax = b, x \geq 0\}$. x is extreme point iff there exists
$B \subseteq \{1, \ldots, n\}$ with $|B| = m$ and*

▶ A_B is non-singular
▶ $x_B = A_B^{-1} b \geq 0$
▶ $x_N = 0$

where $N = \{1, \ldots, n\} \setminus B$.

Proof

Take $B = \{j \mid x_j > 0\}$ and augment with linearly independent
columns until $|B| = m$; always possible since $\text{rank}(A) = m$.

Basic Feasible Solutions

$x \in \mathbb{R}^n$ is called basic solution (Basislösung) if $Ax = b$ and
$\text{rank}(A_J) = |J|$ where $J = \{j \mid x_j \neq 0\}$;

x is a basic **feasible** solution (gültige Basislösung) if in addition
$x \geq 0$.

A basis (Basis) is an index set $B \subseteq \{1, \ldots, n\}$ with $\text{rank}(A_B) = m$
and $|B| = m$.

$x \in \mathbb{R}^n$ with $A_B x_B = b$ and $x_j = 0$ for all $j \notin B$ is the basic
solution associated to basis B (die zu B assoziierte Basislösung)

Basic Feasible Solutions

A BFS fulfills the m equality constraints.

In addition, at least $n - m$ of the x_i's are zero. The corresponding non-negativity constraint is fulfilled with equality.
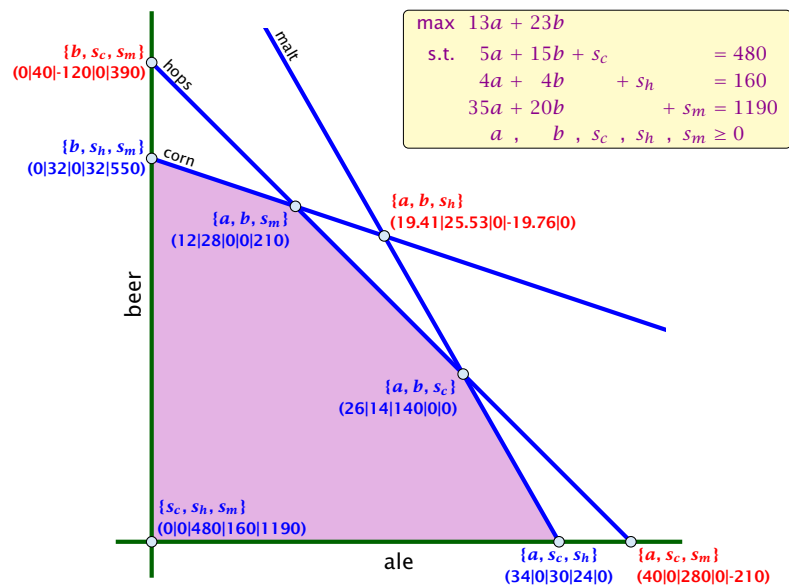
Fact:
In a BFS at least n constraints are fulfilled with equality.

Basic Feasible Solutions

Definition 25
For a general LP ($\max\{c^T x \mid Ax \leq b\}$) with n variables a point x is a basic feasible solution if x is feasible and there exist n (linearly independent) constraints that are tight.

Algebraic View

$$
\begin{aligned}
\max\ & 13a + 23b \\
\text{s.t.}\quad & 5a + 15b + s_c && = 480 \\
& 4a + 4b && + s_h && = 160 \\
& 35a + 20b && && + s_m = 1190 \\
& a,\quad b,\ s_c,\ s_h,\ s_m \geq 0
\end{aligned}
$$

{b, s_c, s_m}
(0|40|-120|0|390)

{b, s_h, s_m}
(0|32|0|32|550)

{a, b, s_m}
(12|28|0|0|210)

{a, b, s_h}
(19.41|25.53|0|-19.76|0)

{a, b, s_c}
(26|14|140|0|0)

{s_c, s_h, s_m}
(0|0|480|160|1190)

{a, s_c, s_h}
(34|0|30|24|0)

{a, s_c, s_m}
(40|0|280|0|-210)

Fundamental Questions

Linear Programming Problem (LP)
Let $A \in \mathbb{Q}^{m \times n}$, $b \in \mathbb{Q}^m$, $c \in \mathbb{Q}^n$, $\alpha \in \mathbb{Q}$. Does there exist $x \in \mathbb{Q}^n$ s.t. $Ax = b$, $x \geq 0$, $c^T x \geq \alpha$?

Questions:
▶ Is LP in NP? yes!
▶ Is LP in co-NP?
▶ Is LP in P?

Proof:
▶ Given a basis B we can compute the associated basis solution by calculating $A_B^{-1} b$ in polynomial time; then we can also compute the profit.

Observation

We can compute an optimal solution to a linear program in time
$\mathcal{O}\left(\binom{n}{m} \cdot \text{poly}(n, m)\right)$.

- there are only $\binom{n}{m}$ different bases.
- compute the profit of each of them and take the maximum

What happens if LP is unbounded?