# 6 Recurrences

**Algorithm 2** mergesort(list $L$)

$n \leftarrow \text{size}(L)$
**if** $n \leq 1$ **return** $L$
$L_1 \leftarrow L[1 \cdots \lfloor \frac{n}{2} \rfloor]$
$L_2 \leftarrow L[\lfloor \frac{n}{2} \rfloor + 1 \cdots n]$
$\text{mergesort}(L_1)$
$\text{mergesort}(L_2)$
$L \leftarrow \text{merge}(L_1, L_2)$
**return** $L$

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**Algorithm 2** mergesort(list $L$)

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n ← size(L)
if n ≤ 1 return L
L_1 ← L[1 ··· ⌊n/2⌋]
L_2 ← L[⌊n/2⌋ + 1 ··· n]
mergesort(L_1)
mergesort(L_2)
L ← merge(L_1, L_2)
return L
```

This algorithm requires

$$T(n) = T\left(\left\lceil \frac{n}{2} \right\rceil\right) + T\left(\left\lfloor \frac{n}{2} \right\rfloor\right) + \mathcal{O}(n) \leq 2T\left(\left\lceil \frac{n}{2} \right\rceil\right) + \mathcal{O}(n)$$

comparisons when $n > 1$ and $0$ comparisons when $n \leq 1$.

# Recurrences

How do we bring the expression for the number of comparisons ($\approx$ running time) into a closed form?

For this we need to solve the recurrence.

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# Methods for Solving Recurrences

1. **Guessing+Induction**
   Guess the right solution and prove that it is correct via induction. It needs experience to make the right guess.
2. **Master Theorem**
   For a lot of recurrences that appear in the analysis of algorithms this theorem can be used to obtain tight asymptotic bounds. It does not provide exact solutions.
3. **Characteristic Polynomial**
   Linear homogenous recurrences can be solved via this method.

# Methods for Solving Recurrences

4. **Generating Functions**
   A more general technique that allows to solve certain types of linear inhomogenous relations and also sometimes non-linear recurrence relations.
5. **Transformation of the Recurrence**
   Sometimes one can transform the given recurrence relations so that it e.g. becomes linear and can therefore be solved with one of the other techniques.

# 6.1 Guessing+Induction

First we need to get rid of the $\mathcal{O}$-notation in our recurrence:

$$T(n) \le \begin{cases} 2T(\lceil \frac{n}{2} \rceil) + cn & n \ge 2 \\ 0 & \text{otherwise} \end{cases}$$

**Informal way:**

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$$T(n) \leq \begin{cases} 2T(\frac{n}{2}) + cn & n \geq 2 \\ 0 & \text{otherwise} \end{cases}$$

One way of solving such a recurrence is to guess a solution, and check that it is correct by plugging it in.

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Suppose we guess $T(n) \leq dn \log n$ for a constant $d$.

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Suppose we guess $T(n) \leq dn\log n$ for a constant $d$. Then

$$\begin{aligned} T(n) &\leq 2T\Big(\frac{n}{2}\Big) + cn \\ &\leq 2\Big(d\frac{n}{2}\log\frac{n}{2}\Big) + cn \\ &= dn(\log n - 1) + cn \end{aligned}$$

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if we choose $d \geq c$.

Formally, this is not correct if $n$ is not a power of 2. Also even in this case one would need to do an induction proof.

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We consider the following recurrence instead of the original one:

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Note that we can do this as for constant-sized inputs the running time is always some constant ($b$ in the above case).

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We also make a guess of $T(n) \leq dn \log n$ and get

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$\log n \le \frac{n}{4}$

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&\le dn\log n - 0.33dn + cn \\
&\le dn\log n
\end{aligned}
$$

for a suitable choice of $d$.

## 6.2 Master Theorem

**Lemma 1**

*Let* $a \geq 1, b \geq 1$ *and* $\epsilon > 0$ *denote constants. Consider the recurrence*

$$T(n) = aT\left(\frac{n}{b}\right) + f(n) \ .$$

*Case 1.*

*If* $f(n) = \mathcal{O}(n^{\log_b(a)-\epsilon})$ *then* $T(n) = \Theta(n^{\log_b a})$.

*Case 2.*

*If* $f(n) = \Theta(n^{\log_b(a)} \log^k n)$ *then* $T(n) = \Theta(n^{\log_b a} \log^{k+1} n)$, $k \geq 0$.

*Case 3.*

*If* $f(n) = \Omega(n^{\log_b(a)+\epsilon})$ *and for sufficiently large* $n$ $af(\frac{n}{b}) \leq cf(n)$ *for some constant* $c < 1$ *then* $T(n) = \Theta(f(n))$.

# 6.2 Master Theorem

We prove the Master Theorem for the case that $n$ is of the form $b^\ell$, and we assume that the non-recursive case occurs for problem size $1$ and incurs cost $1$.

# The Recursion Tree

The running time of a recursive algorithm can be visualized by a recursion tree:

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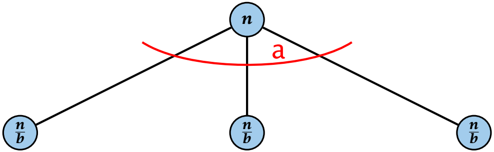

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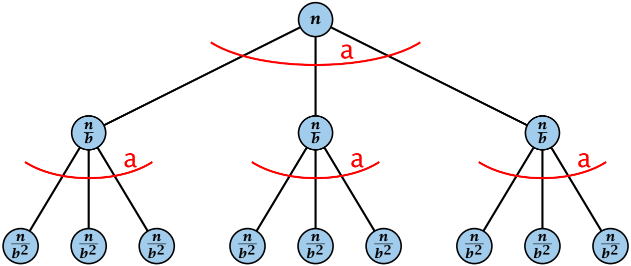

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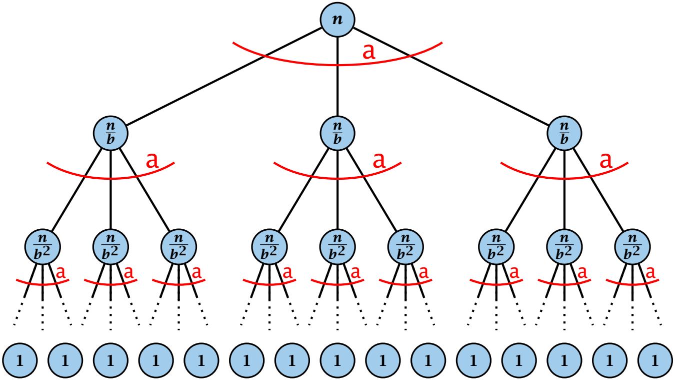

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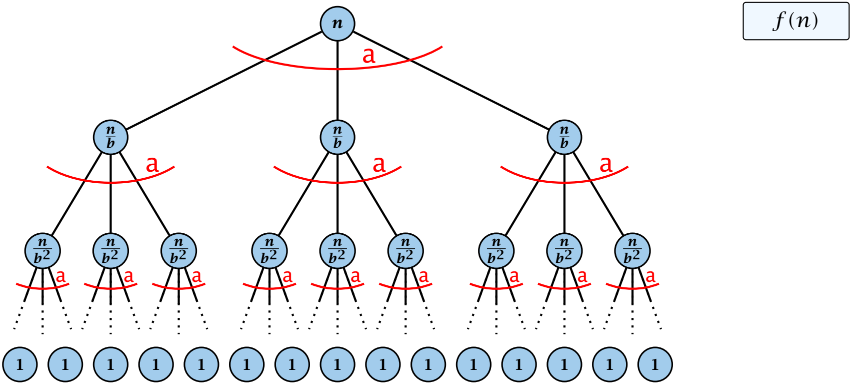

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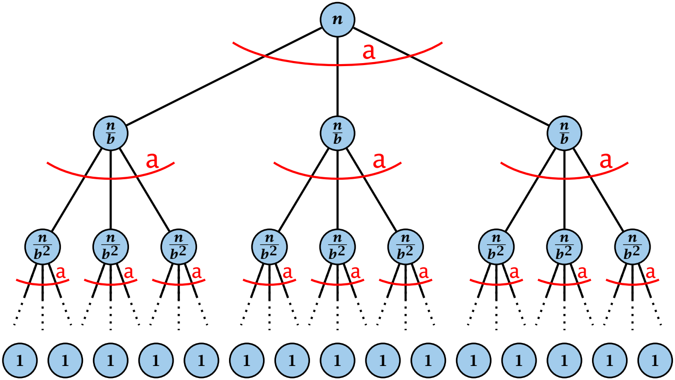


$f(n)$

$af(\frac{n}{b})$

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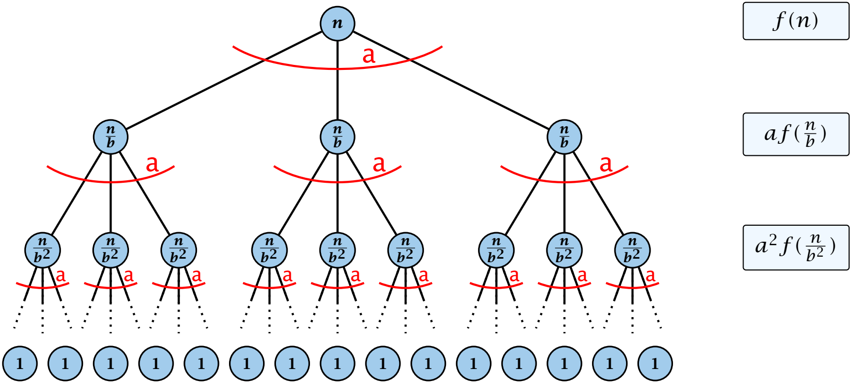

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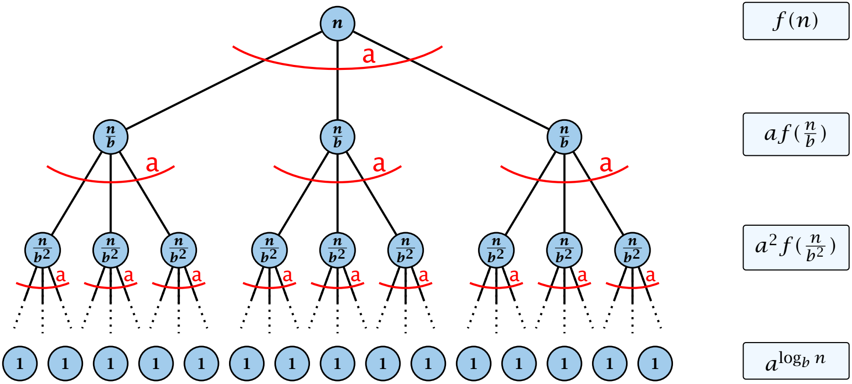

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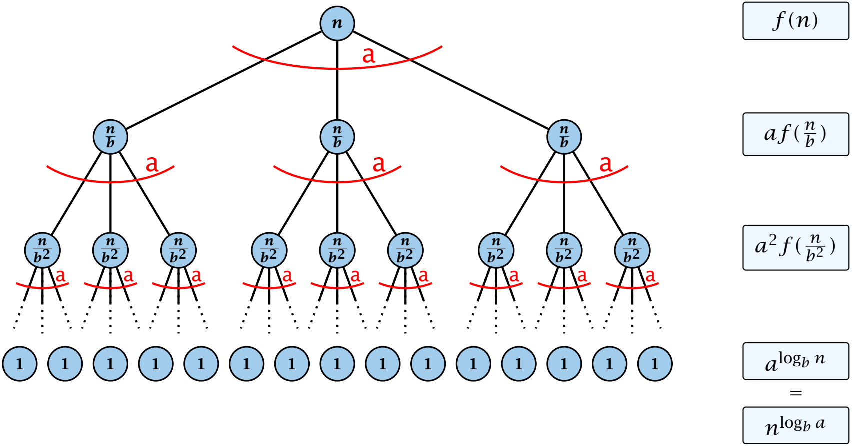

# 6.2 Master Theorem

This gives

$$T(n) = n^{\log_b a} + \sum_{i=0}^{\log_b n - 1} a^i f\left(\frac{n}{b^i}\right) \; .$$

Case 1. Now suppose that $f(n) \leq cn^{\log_b a - \epsilon}$.

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$b^{-i(\log_b a - \epsilon)} = b^{\epsilon i}(b^{\log_b a})^{-i} = b^{\epsilon i} a^{-i}$

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$$= cn^{\log_b a-\epsilon}(b^{\epsilon \log_b n} - 1)/(b^{\epsilon} - 1)$$

$$= cn^{\log_b a-\epsilon}(n^{\epsilon} - 1)/(b^{\epsilon} - 1)$$

$$= \frac{c}{b^{\epsilon} - 1} n^{\log_b a}(n^{\epsilon} - 1)/(n^{\epsilon})$$

**Case 1.** Now suppose that $f(n) \leq cn^{\log_b a - \epsilon}$.

$$
\begin{aligned}
T(n) - n^{\log_b a} &= \sum_{i=0}^{\log_b n - 1} a^i f\left(\frac{n}{b^i}\right) \\
&\leq c \sum_{i=0}^{\log_b n - 1} a^i \left(\frac{n}{b^i}\right)^{\log_b a - \epsilon} \\
&= cn^{\log_b a - \epsilon} \sum_{i=0}^{\log_b n - 1} \left(b^{\epsilon}\right)^i \\
&= cn^{\log_b a - \epsilon}(b^{\epsilon \log_b n} - 1)/(b^{\epsilon} - 1) \\
&= cn^{\log_b a - \epsilon}(n^{\epsilon} - 1)/(b^{\epsilon} - 1) \\
&= \frac{c}{b^{\epsilon} - 1} n^{\log_b a}(n^{\epsilon} - 1)/(n^{\epsilon})
\end{aligned}
$$

$b^{-i(\log_b a - \epsilon)} = b^{\epsilon i}(b^{\log_b a})^{-i} = b^{\epsilon i} a^{-i}$

$\sum_{i=0}^{k} q^i = \frac{q^{k+1} - 1}{q - 1}$

Hence,

$$
T(n) \leq \left(\frac{c}{b^{\epsilon} - 1} + 1\right) n^{\log_b (a)}
$$

Case 1. Now suppose that $f(n) \leq cn^{\log_b a - \epsilon}$.

$$
\begin{aligned}
T(n) - n^{\log_b a} &= \sum_{i=0}^{\log_b n - 1} a^i f\left(\frac{n}{b^i}\right) \\
&\leq c \sum_{i=0}^{\log_b n - 1} a^i \left(\frac{n}{b^i}\right)^{\log_b a - \epsilon}
\end{aligned}
$$

$b^{-i(\log_b a - \epsilon)} = b^{\epsilon i}(b^{\log_b a})^{-i} = b^{\epsilon i} a^{-i}$

$$
= cn^{\log_b a - \epsilon} \sum_{i=0}^{\log_b n - 1} (b^{\epsilon})^i
$$

$\sum_{i=0}^{k} q^i = \frac{q^{k+1} - 1}{q - 1}$

$$
\begin{aligned}
&= cn^{\log_b a - \epsilon}(b^{\epsilon \log_b n} - 1)/(b^{\epsilon} - 1) \\
&= cn^{\log_b a - \epsilon}(n^{\epsilon} - 1)/(b^{\epsilon} - 1) \\
&= \frac{c}{b^{\epsilon} - 1} n^{\log_b a}(n^{\epsilon} - 1)/(n^{\epsilon})
\end{aligned}
$$

Hence,

$$
T(n) \leq \left(\frac{c}{b^{\epsilon} - 1} + 1\right) n^{\log_b (a)}
$$

$\Rightarrow T(n) = \mathcal{O}(n^{\log_b a})$.

Case 2. Now suppose that $f(n) \leq cn^{\log_b a}$.

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$$T(n) - n^{\log_b a}$$

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$$T(n) - n^{\log_b a} = \sum_{i=0}^{\log_b n - 1} a^i f\left(\frac{n}{b^i}\right)$$

Case 2. Now suppose that $f(n) \leq cn^{\log_b a}$.

$$
\begin{aligned}
T(n) - n^{\log_b a} &= \sum_{i=0}^{\log_b n - 1} a^i f\left(\frac{n}{b^i}\right) \\
&\leq c \sum_{i=0}^{\log_b n - 1} a^i \left(\frac{n}{b^i}\right)^{\log_b a}
\end{aligned}
$$

Case 2. Now suppose that $f(n) \le c n^{\log_b a}$.

$$
\begin{aligned}
T(n) - n^{\log_b a} &= \sum_{i=0}^{\log_b n - 1} a^i f\left(\frac{n}{b^i}\right) \\
&\le c \sum_{i=0}^{\log_b n - 1} a^i \left(\frac{n}{b^i}\right)^{\log_b a} \\
&= c n^{\log_b a} \sum_{i=0}^{\log_b n - 1} 1
\end{aligned}
$$

**Case 2.** Now suppose that $f(n) \le cn^{\log_b a}$.

$$
\begin{aligned}
T(n) - n^{\log_b a} &= \sum_{i=0}^{\log_b n - 1} a^i f\left(\frac{n}{b^i}\right) \\
&\le c \sum_{i=0}^{\log_b n - 1} a^i \left(\frac{n}{b^i}\right)^{\log_b a} \\
&= cn^{\log_b a} \sum_{i=0}^{\log_b n - 1} 1 \\
&= cn^{\log_b a} \log_b n
\end{aligned}
$$

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$$
\begin{aligned}
T(n) - n^{\log_b a} &= \sum_{i=0}^{\log_b n - 1} a^i f\left(\frac{n}{b^i}\right) \\
&\leq c \sum_{i=0}^{\log_b n - 1} a^i \left(\frac{n}{b^i}\right)^{\log_b a} \\
&= cn^{\log_b a} \sum_{i=0}^{\log_b n - 1} 1 \\
&= cn^{\log_b a} \log_b n
\end{aligned}
$$

Hence,

$$
T(n) = \mathcal{O}(n^{\log_b a} \log_b n)
$$

Case 2. Now suppose that $f(n) \leq cn^{\log_b a}$.

$$
\begin{aligned}
T(n) - n^{\log_b a} &= \sum_{i=0}^{\log_b n - 1} a^i f\left(\frac{n}{b^i}\right) \\
&\leq c \sum_{i=0}^{\log_b n - 1} a^i \left(\frac{n}{b^i}\right)^{\log_b a} \\
&= cn^{\log_b a} \sum_{i=0}^{\log_b n - 1} 1 \\
&= cn^{\log_b a} \log_b n
\end{aligned}
$$

Hence,

$$T(n) = \mathcal{O}(n^{\log_b a} \log_b n)$$

$\Rightarrow T(n) = \mathcal{O}(n^{\log_b a} \log n)$.

Case 2. Now suppose that $f(n) \geq cn^{\log_b a}$.

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$$T(n) - n^{\log_b a}$$

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$$T(n) - n^{\log_b a} = \sum_{i=0}^{\log_b n-1} a^i f\left(\frac{n}{b^i}\right)$$

$$\geq c \sum_{i=0}^{\log_b n-1} a^i \left(\frac{n}{b^i}\right)^{\log_b a}$$

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$$
\begin{aligned}
T(n) - n^{\log_b a} &= \sum_{i=0}^{\log_b n - 1} a^i f\left(\frac{n}{b^i}\right) \\
&\geq c \sum_{i=0}^{\log_b n - 1} a^i \left(\frac{n}{b^i}\right)^{\log_b a} \\
&= cn^{\log_b a} \sum_{i=0}^{\log_b n - 1} 1
\end{aligned}
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&\geq c \sum_{i=0}^{\log_b n - 1} a^i \left(\frac{n}{b^i}\right)^{\log_b a} \\
&= cn^{\log_b a} \sum_{i=0}^{\log_b n - 1} 1 \\
&= cn^{\log_b a} \log_b n
\end{aligned}
$$

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$$
\begin{aligned}
T(n) - n^{\log_b a} &= \sum_{i=0}^{\log_b n-1} a^i f\left(\frac{n}{b^i}\right) \\
&\geq c \sum_{i=0}^{\log_b n-1} a^i \left(\frac{n}{b^i}\right)^{\log_b a} \\
&= cn^{\log_b a} \sum_{i=0}^{\log_b n-1} 1 \\
&= cn^{\log_b a} \log_b n
\end{aligned}
$$

Hence,

$$T(n) = \Omega(n^{\log_b a} \log_b n)$$

**Case 2.** Now suppose that $f(n) \geq cn^{\log_b a}$.

$$
\begin{aligned}
T(n) - n^{\log_b a} &= \sum_{i=0}^{\log_b n - 1} a^i f\left(\frac{n}{b^i}\right) \\
&\geq c \sum_{i=0}^{\log_b n - 1} a^i \left(\frac{n}{b^i}\right)^{\log_b a} \\
&= cn^{\log_b a} \sum_{i=0}^{\log_b n - 1} 1 \\
&= cn^{\log_b a} \log_b n
\end{aligned}
$$

Hence,

$$T(n) = \Omega(n^{\log_b a} \log_b n) \qquad \Rightarrow T(n) = \Omega(n^{\log_b a} \log n).$$

Case 2. Now suppose that $f(n) \leq cn^{\log_b a}(\log_b(n))^k$.

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$$T(n) - n^{\log_b a} = \sum_{i=0}^{\log_b n - 1} a^i f\left(\frac{n}{b^i}\right)$$
$$\leq c \sum_{i=0}^{\log_b n - 1} a^i \left(\frac{n}{b^i}\right)^{\log_b a} \cdot \left(\log_b\left(\frac{n}{b^i}\right)\right)^k$$

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$$= cn^{\log_b a} \sum_{i=0}^{\ell-1} \left(\log_b\left(\frac{b^\ell}{b^i}\right)\right)^k$$

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$$= cn^{\log_b a} \sum_{i=0}^{\ell-1} (\ell - i)^k$$

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$$
\begin{aligned}
T(n) - n^{\log_b a} &= \sum_{i=0}^{\log_b n-1} a^i f\left(\frac{n}{b^i}\right) \\
&\leq c \sum_{i=0}^{\log_b n-1} a^i \left(\frac{n}{b^i}\right)^{\log_b a} \cdot \left(\log_b\left(\frac{n}{b^i}\right)\right)^k \\
&= cn^{\log_b a} \sum_{i=0}^{\ell-1} \left(\log_b\left(\frac{b^\ell}{b^i}\right)\right)^k \\
&= cn^{\log_b a} \sum_{i=0}^{\ell-1} (\ell - i)^k \\
&= cn^{\log_b a} \sum_{i=1}^{\ell} i^k
\end{aligned}
$$

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&\leq c \sum_{i=0}^{\log_b n-1} a^i \left(\frac{n}{b^i}\right)^{\log_b a} \cdot \left(\log_b\left(\frac{n}{b^i}\right)\right)^k \\
&= cn^{\log_b a} \sum_{i=0}^{\ell-1} \left(\log_b\left(\frac{b^\ell}{b^i}\right)\right)^k \\
&= cn^{\log_b a} \sum_{i=0}^{\ell-1} (\ell - i)^k \\
&= cn^{\log_b a} \sum_{i=1}^{\ell} i^k \approx \frac{1}{k}\ell^{k+1}
\end{aligned}
$$

$n = b^\ell \Rightarrow \ell = \log_b n$

**Case 2.** Now suppose that $f(n) \le cn^{\log_b a}(\log_b(n))^k$.

$$
\begin{aligned}
T(n) - n^{\log_b a} &= \sum_{i=0}^{\log_b n - 1} a^i f\left(\frac{n}{b^i}\right) \\
&\le c \sum_{i=0}^{\log_b n - 1} a^i \left(\frac{n}{b^i}\right)^{\log_b a} \cdot \left(\log_b\left(\frac{n}{b^i}\right)\right)^k \\
&= cn^{\log_b a} \sum_{i=0}^{\ell-1} \left(\log_b\left(\frac{b^\ell}{b^i}\right)\right)^k \\
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&\approx \frac{c}{k} n^{\log_b a} \ell^{k+1}
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$$
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&\leq c \sum_{i=0}^{\log_b n - 1} a^i \left(\frac{n}{b^i}\right)^{\log_b a} \cdot \left(\log_b\left(\frac{n}{b^i}\right)\right)^k \\
&= cn^{\log_b a} \sum_{i=0}^{\ell-1} \left(\log_b\left(\frac{b^\ell}{b^i}\right)\right)^k \\
&= cn^{\log_b a} \sum_{i=0}^{\ell-1} (\ell - i)^k \\
&= cn^{\log_b a} \sum_{i=1}^{\ell} i^k \\
&\approx \frac{c}{k} n^{\log_b a} \ell^{k+1}
\end{aligned}
$$

$n = b^\ell \Rightarrow \ell = \log_b n$

$\Rightarrow T(n) = \mathcal{O}(n^{\log_b a} \log^{k+1} n)$.

Case 3. Now suppose that $f(n) \geq dn^{\log_b a+\epsilon}$, and that for sufficiently large $n$: $af(n/b) \leq cf(n)$, for $c < 1$.

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$$T(n) - n^{\log_b a} = \sum_{i=0}^{\log_b n-1} a^i f\left(\frac{n}{b^i}\right)$$

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$$
\begin{aligned}
T(n) - n^{\log_b a} &= \sum_{i=0}^{\log_b n-1} a^i f\left(\frac{n}{b^i}\right) \\
&\leq \sum_{i=0}^{\log_b n-1} c^i f(n) + \mathcal{O}(n^{\log_b a})
\end{aligned}
$$

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\end{aligned}
$$

$q < 1 : \sum_{i=0}^{n} q^i = \frac{1-q^{n+1}}{1-q} \leq \frac{1}{1-q}$

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&\leq \frac{1}{1-c} f(n) + \mathcal{O}(n^{\log_b a})
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$$
\begin{aligned}
T(n) - n^{\log_b a} &= \sum_{i=0}^{\log_b n-1} a^i f\left(\frac{n}{b^i}\right) \\
&\leq \sum_{i=0}^{\log_b n-1} c^i f(n) + \mathcal{O}(n^{\log_b a}) \\
&\leq \frac{1}{1-c} f(n) + \mathcal{O}(n^{\log_b a})
\end{aligned}
$$

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Hence,

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$$T(n) - n^{\log_b a} = \sum_{i=0}^{\log_b n-1} a^i f\left(\frac{n}{b^i}\right)$$

$$\leq \sum_{i=0}^{\log_b n-1} c^i f(n) + \mathcal{O}(n^{\log_b a})$$

$$\leq \frac{1}{1-c} f(n) + \mathcal{O}(n^{\log_b a})$$

$q < 1 : \sum_{i=0}^{n} q^i = \frac{1-q^{n+1}}{1-q} \leq \frac{1}{1-q}$

Hence,

$$T(n) \leq \mathcal{O}(f(n))$$

$\Rightarrow T(n) = \Theta(f(n))$.

# Example: Multiplying Two Integers

Suppose we want to multiply two $n$-bit Integers, but our registers can only perform operations on integers of constant size.

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$$
\begin{array}{ccccccccccc}
1 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & & A \\
1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & & B \\
\hline
\end{array}
$$

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Suppose we want to multiply two $n$-bit Integers, but our registers can only perform operations on integers of constant size.

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| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 1 | 1 | 0 | 1 | 1 | 0 | 1 | 0 | 1 | $A$ |
| 1 | 0 | 0 | 0 | 1 | 0 | 0 | 1 | 1 | $B$ |

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| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 1 | 1 | 0 | 1 | 1 | 0 | 1 | 0 | 1 | $A$ |
| 1 | 0 | 0 | 0 | 1 | 0 | 0 | 1 | 1 | $B$ |
| | | | | | | | $_1$ | | |
| | | | | | | | | 0 | |

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| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 1 | 1 | 0 | 1 | 1 | 0 | 1 | 0 | 1 | $A$ |
| 1 | 0 | 0 | 0 | 1 | 0 | 0 | 1 | 1 | $B$ |
| | | | | | | | 1 | | |
| | | | | | | | | 0 | |

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| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 1 | 1 | 0 | 1 | 1 | 0 | 1 | 0 | 1 | $A$ |
| 1 | 0 | 0 | 0 | 1 | 0 | $0_1$ | $1_1$ | 1 | $B$ |
| | | | | | | | 0 | 0 | |

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| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 1 | 1 | 0 | 1 | 1 | 0 | 1 | 0 | 1 | $A$ |
| 1 | 0 | 0 | 0 | 1 | 0 | 0 | 1 | 1 | $B$ |
| | | | | | | 1 | 1 | | |
| | | | | | | | 0 | 0 | |

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For this we first need to be able to add two integers $A$ and $B$:

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 1 | 1 | 0 | 1 | 1 | 0 | 1 | 0 | 1 | $A$ |
| 1 | 0 | 0 | 0 | 1 | 0 | 0 | 1 | 1 | $B$ |
| | | | | | 1 | 1 | 1 | | |
| | | | | | | 0 | 0 | 0 | |

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| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 1 | 1 | 0 | 1 | 1 | 0 | 1 | 0 | 1 | $A$ |
| 1 | 0 | 0 | 0 | 1 | 0 | 0 | 1 | 1 | $B$ |
| | | | | | 1 | 1 | 1 | | |
| | | | | | | 0 | 0 | 0 | |

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|---|---|---|---|---|---|---|---|---|---|
| 1 | 1 | 0 | 1 | 1 | 0 | 1 | 0 | 1 | $A$ |
| 1 | 0 | 0 | 0 | 1 | 0 | 0 | 1 | 1 | $B$ |
| | | | | 0 | 1 | 1 | 1 | | |
| | | | | | 1 | 0 | 0 | 0 | |

# Example: Multiplying Two Integers

Suppose we want to multiply two $n$-bit Integers, but our registers can only perform operations on integers of constant size.

For this we first need to be able to add two integers $A$ and $B$:

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 1 | 1 | 0 | 1 | 1 | 0 | 1 | 0 | 1 | $A$ |
| 1 | 0 | 0 | 0 | 1 | 0 | 0 | 1 | 1 | $B$ |
| | | | | 0 | 1 | 1 | 1 | | |
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| | | | 1 | 0 | 1 | 1 | 1 | | |
| | | | | 0 | 1 | 0 | 0 | 0 | |

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| | | | 1 | 0 | 1 | 1 | 1 | | |
| | | | | 0 | 1 | 0 | 0 | 0 | |

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| | | 1 | 1 | 0 | 1 | 1 | 1 | | |
| | | | 0 | 0 | 1 | 0 | 0 | 0 | |

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| | 0 | 1 | 1 | 0 | 1 | 1 | 1 | | |
| | | 1 | 0 | 0 | 1 | 0 | 0 | 0 | |

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| | 0 | 1 | 1 | 0 | 1 | 1 | 1 | | |
| | | 1 | 0 | 0 | 1 | 0 | 0 | 0 | |

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| 0 | 0 | 1 | 1 | 0 | 1 | 1 | 1 | | |
| | 1 | 1 | 0 | 0 | 1 | 0 | 0 | 0 | |

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|---|---|---|---|---|---|---|---|---|---|
| 1 | 1 | 0 | 1 | 1 | 0 | 1 | 0 | 1 | $A$ |
| 1 | 0 | 0 | 0 | 1 | 0 | 0 | 1 | 1 | $B$ |
| 0 | 0 | 1 | 1 | 0 | 1 | 1 | 1 | | |
| | 1 | 1 | 0 | 0 | 1 | 0 | 0 | 0 | |

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|---|---|---|---|---|---|---|---|---|---|---|
| | 1 | 1 | 0 | 1 | 1 | 0 | 1 | 0 | 1 | $A$ |
| | 1 | 0 | 0 | 0 | 1 | 0 | 0 | 1 | 1 | $B$ |
| 1 | 0 | 0 | 1 | 1 | 0 | 1 | 1 | 1 | | |
| | 0 | 1 | 1 | 0 | 0 | 1 | 0 | 0 | 0 | |

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|---|---|---|---|---|---|---|---|---|---|---|
| | 1 | 1 | 0 | 1 | 1 | 0 | 1 | 0 | 1 | $A$ |
| | 1 | 0 | 0 | 0 | 1 | 0 | 0 | 1 | 1 | $B$ |
| 1 | 0 | 0 | 1 | 1 | 0 | 1 | 1 | 1 | | |
| | 0 | 1 | 1 | 0 | 0 | 1 | 0 | 0 | 0 | |

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|---|---|---|---|---|---|---|---|---|---|---|
| | 1 | 1 | 0 | 1 | 1 | 0 | 1 | 0 | 1 | $A$ |
| | 1 | 0 | 0 | 0 | 1 | 0 | 0 | 1 | 1 | $B$ |
| 1 | 0 | 0 | 1 | 1 | 0 | 1 | 1 | 1 | | |
| 1 | 0 | 1 | 1 | 0 | 0 | 1 | 0 | 0 | 0 | |

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| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| | 1 | 1 | 0 | 1 | 1 | 0 | 1 | 0 | 1 | $A$ |
| | 1 | 0 | 0 | 0 | 1 | 0 | 0 | 1 | 1 | $B$ |
| 1 | 0 | 0 | 1 | 1 | 0 | 1 | 1 | 1 | | |
| 1 | 0 | 1 | 1 | 0 | 0 | 1 | 0 | 0 | 0 | |

This gives that two $n$-bit integers can be added in time $\mathcal{O}(n)$.

# Example: Multiplying Two Integers

Suppose that we want to multiply an $n$-bit integer $A$ and an $m$-bit integer $B$ ($m \le n$).

# Example: Multiplying Two Integers

Suppose that we want to multiply an $n$-bit integer $A$ and an $m$-bit integer $B$ ($m \le n$).

$$\begin{array}{cccccccccc} 1 & 0 & 0 & 0 & 1 & \times & 1 & 0 & 1 & 1 \\ \hline \end{array}$$

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Suppose that we want to multiply an $n$-bit integer $A$ and an $m$-bit integer $B$ ($m \leq n$).

$$\begin{array}{ccccccccccc} 1 & 0 & 0 & 0 & 1 & \times & 1 & 0 & 1 & \boxed{1} \\ \hline \end{array}$$

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Suppose that we want to multiply an $n$-bit integer $A$ and an $m$-bit integer $B$ ($m \leq n$).

$$
\begin{array}{ccccccccccc}
1 & 0 & 0 & 0 & 1 & \times & 1 & 0 & 1 & \boxed{1} \\
\hline
 & & & & & 1 & 0 & 0 & 0 & 1
\end{array}
$$

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Suppose that we want to multiply an $n$-bit integer $A$ and an $m$-bit integer $B$ ($m \le n$).

$$
\begin{array}{ccccccccccc}
1 & 0 & 0 & 0 & 1 & \times & 1 & 0 & \boxed{1} & 1 \\
\hline
 & & & & & 1 & 0 & 0 & 0 & 1 \\
 & & & & & & & & & 0
\end{array}
$$

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Suppose that we want to multiply an $n$-bit integer $A$ and an $m$-bit integer $B$ $(m \le n)$.

$$
\begin{array}{rrrrrrrrrr}
1 & 0 & 0 & 0 & 1 & \times & 1 & 0 & \boxed{1} & 1 \\
\hline
 & & & & & 1 & 0 & 0 & 0 & 1 \\
 & & & & 1 & 0 & 0 & 0 & 1 & 0
\end{array}
$$

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$$
\begin{array}{cccccccccc}
1 & 0 & 0 & 0 & 1 & \times & 1 & \boxed{0} & 1 & 1 \\
\hline
 & & & & & 1 & 0 & 0 & 0 & 1 \\
 & & & & 1 & 0 & 0 & 0 & 1 & 0
\end{array}
$$

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\begin{array}{rrrrrrrrrr}
1 & 0 & 0 & 0 & 1 & \times & 1 & \boxed{0} & 1 & 1 \\
\hline
 & & & & & 1 & 0 & 0 & 0 & 1 \\
 & & & & 1 & 0 & 0 & 0 & 1 & 0 \\
 & & & & & & & & 0 & 0
\end{array}
$$

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$$
\begin{array}{ccccccccccc}
1 & 0 & 0 & 0 & 1 & \times & 1 & \boxed{0} & 1 & 1 \\
\hline
 & & & & & 1 & 0 & 0 & 0 & 1 \\
 & & & & 1 & 0 & 0 & 0 & 1 & 0 \\
 & & & 0 & 0 & 0 & 0 & 0 & 0 & 0
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| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 0 | 0 | 0 | 1 | × | 1 | 0 | 1 | 1 |
| | | | | | | 1 | 0 | 0 | 0 | 1 |
| | | | | | 1 | 0 | 0 | 0 | 1 | 0 |
| | | | 0 | 0 | 0 | 0 | 0 | 0 | 0 |

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| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 1 | 0 | 0 | 0 | 1 | $\times$ | 1 | 0 | 1 | 1 |
| | | | | | 1 | 0 | 0 | 0 | 1 |
| | | | | 1 | 0 | 0 | 0 | 1 | 0 |
| | | | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| | | | | | | | 0 | 0 | 0 |

# Example: Multiplying Two Integers

Suppose that we want to multiply an $n$-bit integer $A$ and an $m$-bit integer $B$ ($m \leq n$).

$$
\begin{array}{ccccccccccc}
 & & & 1 & 0 & 0 & 0 & 1 & \times & \boxed{1} & 0 & 1 & 1 \\
\hline
 & & & & & & & & & 1 & 0 & 0 & 0 & 1 \\
\end{array}
$$

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| | | | 1 | 0 | 0 | 0 | 1 |
| | | 1 | 0 | 0 | 0 | 1 | 0 |
| | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 1 | 0 | 0 | 0 | 1 | 0 | 0 | 0 |

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Suppose that we want to multiply an $n$-bit integer $A$ and an $m$-bit integer $B$ $(m \le n)$.

$$
\begin{array}{rrrrrrrrrr}
1 & 0 & 0 & 0 & 1 & \times & 1 & 0 & 1 & 1 \\
\hline
 & & & & & 1 & 0 & 0 & 0 & 1 \\
 & & & & 1 & 0 & 0 & 0 & 1 & 0 \\
 & & & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 & & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
\hline
\end{array}
$$

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Suppose that we want to multiply an $n$-bit integer $A$ and an $m$-bit integer $B$ ($m \le n$).

$$
\begin{array}{cccccccccc}
1 & 0 & 0 & 0 & 1 & \times & 1 & 0 & 1 & 1 \\
\hline
 & & & & & 1 & 0 & 0 & 0 & 1 \\
 & & & & 1 & 0 & 0 & 0 & 1 & 0 \\
 & & & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 & & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
\hline
 & & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 1
\end{array}
$$

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Suppose that we want to multiply an $n$-bit integer $A$ and an $m$-bit integer $B$ ($m \le n$).

$$
\begin{array}{rrrrrrrrrr}
1 & 0 & 0 & 0 & 1 & \times & 1 & 0 & 1 & 1 \\
\hline
 & & & & & 1 & 0 & 0 & 0 & 1 \\
 & & & & 1 & 0 & 0 & 0 & 1 & 0 \\
 & & & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 & & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
\hline
 & & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 1
\end{array}
$$

**Time requirement:**

# Example: Multiplying Two Integers

Suppose that we want to multiply an $n$-bit integer $A$ and an $m$-bit integer $B$ ($m \leq n$).

$$
\begin{array}{rrrrrrrrrr}
 & 1 & 0 & 0 & 0 & 1 & \times & 1 & 0 & 1 & 1 \\
\hline
 & & & & & & 1 & 0 & 0 & 0 & 1 \\
 & & & & & 1 & 0 & 0 & 0 & 1 & 0 \\
 & & & & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 & & & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
\hline
 & & & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 1
\end{array}
$$

**Time requirement:**

- Computing intermediate results: $\mathcal{O}(nm)$.

# Example: Multiplying Two Integers

Suppose that we want to multiply an $n$-bit integer $A$ and an $m$-bit integer $B$ ($m \le n$).

$$
\begin{array}{r}
10001 \times 1011 \\
\hline
10001 \\
100010 \\
0000000 \\
10001000 \\
\hline
10111011
\end{array}
$$

**Time requirement:**

- Computing intermediate results: $\mathcal{O}(nm)$.
- Adding $m$ numbers of length $\le 2n$: $\mathcal{O}((m+n)m) = \mathcal{O}(nm)$.

# Example: Multiplying Two Integers

**A recursive approach:**

Suppose that integers $A$ and $B$ are of length $n = 2^k$, for some $k$.

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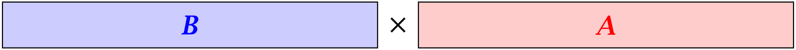
Suppose that integers $A$ and $B$ are of length $n = 2^k$, for some $k$.



# Example: Multiplying Two Integers

**A recursive approach:**

Suppose that integers $A$ and $B$ are of length $n = 2^k$, for some $k$.

$$b_{n-1} \quad \cdots \quad b_0 \times a_{n-1} \quad \cdots \quad a_0$$

# Example: Multiplying Two Integers

**A recursive approach:**

Suppose that integers $A$ and $B$ are of length $n = 2^k$, for some $k$.

$$\boxed{b_{n-1} \quad \cdots \quad b_{\frac{n}{2}} \; b_{\frac{n}{2}-1} \quad \cdots \quad b_0} \times \boxed{a_{n-1} \quad \cdots \quad a_{\frac{n}{2}} \; a_{\frac{n}{2}-1} \quad \cdots \quad a_0}$$

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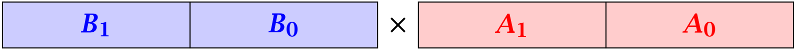

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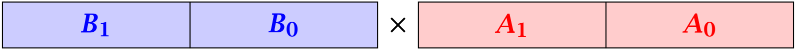


Then it holds that

$$A = A_1 \cdot 2^{\frac{n}{2}} + A_0 \text{ and } B = B_1 \cdot 2^{\frac{n}{2}} + B_0$$

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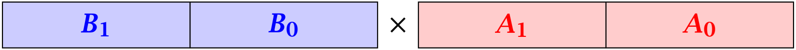
**A recursive approach:**

Suppose that integers $A$ and $B$ are of length $n = 2^k$, for some $k$.




Then it holds that

$$A = A_1 \cdot 2^{\frac{n}{2}} + A_0 \text{ and } B = B_1 \cdot 2^{\frac{n}{2}} + B_0$$

Hence,

$$A \cdot B = A_1 B_1 \cdot 2^n + (A_1 B_0 + A_0 B_1) \cdot 2^{\frac{n}{2}} + A_0 B_0$$

# Example: Multiplying Two Integers

**Algorithm 3** mult$(A, B)$

```
if |A| = |B| = 1 then
        return a_0 · b_0
split A into A_0 and A_1
split B into B_0 and B_1
Z_2 ← mult(A_1, B_1)
Z_1 ← mult(A_1, B_0) + mult(A_0, B_1)
Z_0 ← mult(A_0, B_0)
return Z_2 · 2^n + Z_1 · 2^(n/2) + Z_0
```

# Example: Multiplying Two Integers

**Algorithm 3** mult$(A, B)$

```
1: if |A| = |B| = 1 then                      O(1)
2:     return a_0 · b_0
3: split A into A_0 and A_1
4: split B into B_0 and B_1
5: Z_2 ← mult(A_1, B_1)
6: Z_1 ← mult(A_1, B_0) + mult(A_0, B_1)
7: Z_0 ← mult(A_0, B_0)
8: return Z_2 · 2^n + Z_1 · 2^(n/2) + Z_0
```

1: **if** $|A| = |B| = 1$ **then** — $\mathcal{O}(1)$
2: **return** $a_0 \cdot b_0$
3: split $A$ into $A_0$ and $A_1$
4: split $B$ into $B_0$ and $B_1$
5: $Z_2 \leftarrow \mathrm{mult}(A_1, B_1)$
6: $Z_1 \leftarrow \mathrm{mult}(A_1, B_0) + \mathrm{mult}(A_0, B_1)$
7: $Z_0 \leftarrow \mathrm{mult}(A_0, B_0)$
8: **return** $Z_2 \cdot 2^n + Z_1 \cdot 2^{\frac{n}{2}} + Z_0$

# Example: Multiplying Two Integers

**Algorithm 3** mult$(A, B)$

**if** $|A| = |B| = 1$ **then** $\mathcal{O}(1)$
  **return** $a_0 \cdot b_0$ $\mathcal{O}(1)$
split $A$ into $A_0$ and $A_1$
split $B$ into $B_0$ and $B_1$
$Z_2 \leftarrow \mathrm{mult}(A_1, B_1)$
$Z_1 \leftarrow \mathrm{mult}(A_1, B_0) + \mathrm{mult}(A_0, B_1)$
$Z_0 \leftarrow \mathrm{mult}(A_0, B_0)$
**return** $Z_2 \cdot 2^n + Z_1 \cdot 2^{\frac{n}{2}} + Z_0$

# Example: Multiplying Two Integers

**Algorithm 3** mult$(A, B)$

```
if |A| = |B| = 1 then                      O(1)
        return a_0 · b_0                   O(1)
split A into A_0 and A_1                   O(n)
split B into B_0 and B_1
Z_2 ← mult(A_1, B_1)
Z_1 ← mult(A_1, B_0) + mult(A_0, B_1)
Z_0 ← mult(A_0, B_0)
return Z_2 · 2^n + Z_1 · 2^(n/2) + Z_0
```

- **if** $|A| = |B| = 1$ **then** — $\mathcal{O}(1)$
  - **return** $a_0 \cdot b_0$ — $\mathcal{O}(1)$
- split $A$ into $A_0$ and $A_1$ — $\mathcal{O}(n)$
- split $B$ into $B_0$ and $B_1$
- $Z_2 \leftarrow \mathrm{mult}(A_1, B_1)$
- $Z_1 \leftarrow \mathrm{mult}(A_1, B_0) + \mathrm{mult}(A_0, B_1)$
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# Example: Multiplying Two Integers

**Algorithm 3** mult$(A, B)$

| | |
|---|---|
| **if** $|A| = |B| = 1$ **then** | $\mathcal{O}(1)$ |
|     **return** $a_0 \cdot b_0$ | $\mathcal{O}(1)$ |
| split $A$ into $A_0$ and $A_1$ | $\mathcal{O}(n)$ |
| split $B$ into $B_0$ and $B_1$ | $\mathcal{O}(n)$ |
| $Z_2 \leftarrow \text{mult}(A_1, B_1)$ | |
| $Z_1 \leftarrow \text{mult}(A_1, B_0) + \text{mult}(A_0, B_1)$ | |
| $Z_0 \leftarrow \text{mult}(A_0, B_0)$ | |
| **return** $Z_2 \cdot 2^n + Z_1 \cdot 2^{\frac{n}{2}} + Z_0$ | |

# Example: Multiplying Two Integers

**Algorithm 3** mult$(A, B)$

| | |
|---|---|
| **if** $\|A\| = \|B\| = 1$ **then** | $\mathcal{O}(1)$ |
|     **return** $a_0 \cdot b_0$ | $\mathcal{O}(1)$ |
| split $A$ into $A_0$ and $A_1$ | $\mathcal{O}(n)$ |
| split $B$ into $B_0$ and $B_1$ | $\mathcal{O}(n)$ |
| $Z_2 \leftarrow \mathrm{mult}(A_1, B_1)$ | $T(\frac{n}{2})$ |
| $Z_1 \leftarrow \mathrm{mult}(A_1, B_0) + \mathrm{mult}(A_0, B_1)$ | |
| $Z_0 \leftarrow \mathrm{mult}(A_0, B_0)$ | |
| **return** $Z_2 \cdot 2^n + Z_1 \cdot 2^{\frac{n}{2}} + Z_0$ | |

# Example: Multiplying Two Integers

**Algorithm 3** mult$(A, B)$

| | |
|---|---|
| **if** $\|A\| = \|B\| = 1$ **then** | $\mathcal{O}(1)$ |
|     **return** $a_0 \cdot b_0$ | $\mathcal{O}(1)$ |
| split $A$ into $A_0$ and $A_1$ | $\mathcal{O}(n)$ |
| split $B$ into $B_0$ and $B_1$ | $\mathcal{O}(n)$ |
| $Z_2 \leftarrow \text{mult}(A_1, B_1)$ | $T(\frac{n}{2})$ |
| $Z_1 \leftarrow \text{mult}(A_1, B_0) + \text{mult}(A_0, B_1)$ | $2T(\frac{n}{2}) + \mathcal{O}(n)$ |
| $Z_0 \leftarrow \text{mult}(A_0, B_0)$ | |
| **return** $Z_2 \cdot 2^n + Z_1 \cdot 2^{\frac{n}{2}} + Z_0$ | |

# Example: Multiplying Two Integers

**Algorithm 3** mult$(A, B)$

| | |
|---|---|
| **if** $\|A\| = \|B\| = 1$ **then** | $\mathcal{O}(1)$ |
|     **return** $a_0 \cdot b_0$ | $\mathcal{O}(1)$ |
| split $A$ into $A_0$ and $A_1$ | $\mathcal{O}(n)$ |
| split $B$ into $B_0$ and $B_1$ | $\mathcal{O}(n)$ |
| $Z_2 \leftarrow \text{mult}(A_1, B_1)$ | $T(\frac{n}{2})$ |
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| $Z_0 \leftarrow \text{mult}(A_0, B_0)$ | $T(\frac{n}{2})$ |
| **return** $Z_2 \cdot 2^n + Z_1 \cdot 2^{\frac{n}{2}} + Z_0$ | |

# Example: Multiplying Two Integers

| **Algorithm 3** mult$(A, B)$ | |
|---|---|
| **if** $\|A\| = \|B\| = 1$ **then** | $\mathcal{O}(1)$ |
|     **return** $a_0 \cdot b_0$ | $\mathcal{O}(1)$ |
| split $A$ into $A_0$ and $A_1$ | $\mathcal{O}(n)$ |
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# Example: Multiplying Two Integers

**Algorithm 3** mult$(A, B)$

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| 2: | **return** $a_0 \cdot b_0$ | $\mathcal{O}(1)$ |
| 3: | split $A$ into $A_0$ and $A_1$ | $\mathcal{O}(n)$ |
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| 8: | **return** $Z_2 \cdot 2^n + Z_1 \cdot 2^{\frac{n}{2}} + Z_0$ | $\mathcal{O}(n)$ |

We get the following recurrence:

$$T(n) = 4T\left(\frac{n}{2}\right) + \mathcal{O}(n) \ .$$

# Example: Multiplying Two Integers

**Master Theorem:** Recurrence: $T[n] = aT(\frac{n}{b}) + f(n)$.

- Case 1: $f(n) = \mathcal{O}(n^{\log_b a-\epsilon})$ $\quad T(n) = \Theta(n^{\log_b a})$
- Case 2: $f(n) = \Theta(n^{\log_b a}\log^k n)$ $\quad T(n) = \Theta(n^{\log_b a}\log^{k+1} n)$
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In our case $a = 4$, $b = 2$, and $f(n) = \Theta(n)$. Hence, we are in Case 1, since $n = \mathcal{O}(n^{2-\epsilon}) = \mathcal{O}(n^{\log_b a - \epsilon})$.

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$\Longrightarrow$ Not better then the “school method”.

# Example: Multiplying Two Integers

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Hence,

**Algorithm 4** mult$(A, B)$

**if** $|A| = |B| = 1$ **then**
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split $A$ into $A_0$ and $A_1$
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$Z_2 \leftarrow \text{mult}(A_1, B_1)$
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```
if |A| = |B| = 1 then                                  O(1)
      return a_0 · b_0                                 O(1)
split A into A_0 and A_1                               O(n)
split B into B_0 and B_1                               O(n)
Z_2 ← mult(A_1, B_1)
Z_0 ← mult(A_0, B_0)
Z_1 ← mult(A_0 + A_1, B_0 + B_1) − Z_2 − Z_0
return Z_2 · 2^n + Z_1 · 2^(n/2) + Z_0
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We get the following recurrence:

$$T(n) = 3T\left(\frac{n}{2}\right) + \mathcal{O}(n) \ .$$

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Again we are in Case 1. We get a running time of $\Theta(n^{\log_2 3}) \approx \Theta(n^{1.59})$.

A huge improvement over the “school method”.

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# 6.3 The Characteristic Polynomial

Consider the recurrence relation:

$$c_0T(n) + c_1T(n-1) + c_2T(n-2) + \cdots + c_kT(n-k) = f(n)$$

This is the general form of a linear recurrence relation of order $k$ with constant coefficients ($c_0, c_k \neq 0$).

- $T(n)$ only depends on the $k$ preceding values. This means the recurrence relation is of order $k$.
- The recurrence is linear as there are no products of $T[n]$'s.
- If $f(n) = 0$ then the recurrence relation becomes a linear, [illegible] recurrence relation of order $k$.

Note that we ignore boundary conditions for the moment.

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**Observations:**

- The solution $T[1], T[2], T[3], \ldots$ is completely determined by a set of boundary conditions that specify values for $T[1], \ldots, T[k]$.
- In fact, any $k$ consecutive values completely determine the solution.
- $k$ non-concecutive values might not be an appropriate set of boundary conditions (depends on the problem).

**Approach:**

- First determine all solutions that satisfy recurrence relation.
- Then pick the right one by analyzing boundary conditions.
- First consider the homogenous case.

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- $k$ non-concecutive values might not be an appropriate set of boundary conditions (depends on the problem).

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- First determine all solutions that satisfy recurrence relation.
- Then pick the right one by analyzing boundary conditions.
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# The Homogenous Case

The solution space

$$S = \Big\{ \mathcal{T} = T[1], T[2], T[3], \ldots \;\Big|\; \mathcal{T} \text{ fulfills recurrence relation} \Big\}$$

is a vector space. This means that if $\mathcal{T}_1, \mathcal{T}_2 \in S$, then also $\alpha\mathcal{T}_1 + \beta\mathcal{T}_2 \in S$, for arbitrary constants $\alpha, \beta$.

**How do we find a non-trivial solution?**

We guess that the solution is of the form $\lambda^n$, $\lambda \neq 0$, and see what happens. In order for this guess to fulfill the recurrence we need

$$c_0\lambda^n + c_1\lambda^{n-1} + c_2 \cdot \lambda^{n-2} + \cdots + c_k \cdot \lambda^{n-k} = 0$$

for all $n \geq k$.

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Dividing by $\lambda^{n-k}$ gives that all these constraints are identical to

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This means that if $\lambda_i$ is a root (Nullstelle) of $P[\lambda]$ then $T[n] = \lambda_i^n$ is a solution to the recurrence relation.

Let $\lambda_1, \ldots, \lambda_k$ be the $k$ (complex) roots of $P[\lambda]$. Then, because of the vector space property

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**Lemma 2**
*Assume that the characteristic polynomial has $k$ distinct roots $\lambda_1, \ldots, \lambda_k$. Then all solutions to the recurrence relation are of the form*

$$\alpha_1\lambda_1^n + \alpha_2\lambda_2^n + \cdots + \alpha_k\lambda_k^n \enspace .$$

**Proof.**
There is one solution for every possible choice of boundary conditions for $T[1], \ldots, T[k]$.

We show that the above set of solutions contains one solution for every choice of boundary conditions.

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**Proof (cont.).**

Suppose I am given boundary conditions $T[i]$ and I want to see whether I can choose the $\alpha_i's$ such that these conditions are met:

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\end{array}
$$

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Suppose I am given boundary conditions $T[i]$ and I want to see whether I can choose the $\alpha_i' s$ such that these conditions are met:

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We show that the column vectors are linearly independent. Then the above equation has a solution.

# Computing the Determinant

$$\begin{vmatrix} \lambda_1 & \lambda_2 & \cdots & \lambda_{k-1} & \lambda_k \\ \lambda_1^2 & \lambda_2^2 & \cdots & \lambda_{k-1}^2 & \lambda_k^2 \\ \vdots & \vdots & & \vdots & \vdots \\ \lambda_1^k & \lambda_2^k & \cdots & \lambda_{k-1}^k & \lambda_k^k \end{vmatrix} =$$

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$$= \prod_{i=1}^{k} \lambda_i \cdot \begin{vmatrix} 1 & \lambda_1 & \cdots & \lambda_1^{k-2} & \lambda_1^{k-1} \\ 1 & \lambda_2 & \cdots & \lambda_2^{k-2} & \lambda_2^{k-1} \\ \vdots & \vdots & & \vdots & \vdots \\ 1 & \lambda_k & \cdots & \lambda_k^{k-2} & \lambda_k^{k-1} \end{vmatrix}$$

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$$
\begin{vmatrix}
1 & \lambda_1 - \lambda_1 \cdot 1 & \cdots & \lambda_1^{k-2} - \lambda_1 \cdot \lambda_1^{k-3} & \lambda_1^{k-1} - \lambda_1 \cdot \lambda_1^{k-2} \\
1 & \lambda_2 - \lambda_1 \cdot 1 & \cdots & \lambda_2^{k-2} - \lambda_1 \cdot \lambda_2^{k-3} & \lambda_2^{k-1} - \lambda_1 \cdot \lambda_2^{k-2} \\
\vdots & \vdots & & \vdots & \vdots \\
1 & \lambda_k - \lambda_1 \cdot 1 & \cdots & \lambda_k^{k-2} - \lambda_1 \cdot \lambda_k^{k-3} & \lambda_k^{k-1} - \lambda_1 \cdot \lambda_k^{k-2}
\end{vmatrix} =
$$

$$
\begin{vmatrix}
1 & 0 & \cdots & 0 & 0 \\
1 & (\lambda_2 - \lambda_1) \cdot 1 & \cdots & (\lambda_2 - \lambda_1) \cdot \lambda_2^{k-3} & (\lambda_2 - \lambda_1) \cdot \lambda_2^{k-2} \\
\vdots & \vdots & & \vdots & \vdots \\
1 & (\lambda_k - \lambda_1) \cdot 1 & \cdots & (\lambda_k - \lambda_1) \cdot \lambda_k^{k-3} & (\lambda_k - \lambda_1) \cdot \lambda_k^{k-2}
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$$\prod_{i=2}^{k}(\lambda_i - \lambda_1)\cdot \begin{vmatrix} 1 & \lambda_2 & \cdots & \lambda_2^{k-3} & \lambda_2^{k-2} \\ \vdots & \vdots & & \vdots & \vdots \\ 1 & \lambda_k & \cdots & \lambda_k^{k-3} & \lambda_k^{k-2} \end{vmatrix}$$

# Computing the Determinant

Repeating the above steps gives:

$$\begin{vmatrix} \lambda_1 & \lambda_2 & \cdots & \lambda_{k-1} & \lambda_k \\ \lambda_1^2 & \lambda_2^2 & \cdots & \lambda_{k-1}^2 & \lambda_k^2 \\ \vdots & \vdots & & \vdots & \vdots \\ \lambda_1^k & \lambda_2^k & \cdots & \lambda_{k-1}^k & \lambda_k^k \end{vmatrix} = \prod_{i=1}^{k} \lambda_i \cdot \prod_{i>\ell} (\lambda_i - \lambda_\ell)$$

Hence, if all $\lambda_i$'s are different, then the determinant is non-zero.

# The Homogeneous Case

**What happens if the roots are not all distinct?**

Suppose we have a root $\lambda_i$ with multiplicity (Vielfachheit) at least 2. Then not only is $\lambda_i^n$ a solution to the recurrence but also $n\lambda_i^n$.

To see this consider the polynomial

$$P[\lambda] \cdot \lambda^{n-k} = c_0\lambda^n + c_1\lambda^{n-1} + c_2\lambda^{n-2} + \cdots + c_k\lambda^{n-k}$$

Since $\lambda_i$ is a root we can write this as $Q[\lambda] \cdot (\lambda - \lambda_i)^2$. Calculating the derivative gives a polynomial that still has root $\lambda_i$.

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This means

$$c_0 n\lambda_i^{n-1} + c_1(n-1)\lambda_i^{n-2} + \cdots + c_k(n-k)\lambda_i^{n-k-1} = 0$$

Hence,

$$\underbrace{c_0 n\lambda_i^{n}}_{T[n]} + c_1\underbrace{(n-1)\lambda_i^{n-1}}_{T[n-1]} + \cdots + c_k\underbrace{(n-k)\lambda_i^{n-k}}_{T[n-k]} = 0$$

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$$c_0 n\lambda_i^{n-1} + c_1(n-1)\lambda_i^{n-2} + \cdots + c_k(n-k)\lambda_i^{n-k-1} = 0$$

Hence,

$$\underbrace{c_0 n\lambda_i^{n}}_{T[n]} + c_1\underbrace{(n-1)\lambda_i^{n-1}}_{T[n-1]} + \cdots + c_k\underbrace{(n-k)\lambda_i^{n-k}}_{T[n-k]} = 0$$

# The Homogeneous Case

Suppose $\lambda_i$ has multiplicity $j$. We know that

$$c_0 n\lambda_i^n + c_1(n-1)\lambda_i^{n-1} + \cdots + c_k(n-k)\lambda_i^{n-k} = 0$$

(after taking the derivative; multiplying with $\lambda$; plugging in $\lambda_i$)

Doing this again gives

$$c_0 n^2\lambda_i^n + c_1(n-1)^2\lambda_i^{n-1} + \cdots + c_k(n-k)^2\lambda_i^{n-k} = 0$$

We can continue $j-1$ times.

Hence, $n^\ell\lambda_i^n$ is a solution for $\ell \in 0,\ldots,j-1$.

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# The Homogeneous Case

**Lemma 3**

*Let $P[\lambda]$ denote the characteristic polynomial to the recurrence*

$$c_0 T[n] + c_1 T[n-1] + \cdots + c_k T[n-k] = 0$$

*Let $\lambda_i$, $i = 1, \ldots, m$ be the (complex) roots of $P[\lambda]$ with multiplicities $\ell_i$. Then the general solution to the recurrence is given by*

$$T[n] = \sum_{i=1}^{m} \sum_{j=0}^{\ell_i - 1} \alpha_{ij} \cdot (n^j \lambda_i^n) \ .$$

The full proof is omitted. We have only shown that any choice of $\alpha_{ij}$'s is a solution to the recurrence.

# Example: Fibonacci Sequence

$$T[0] = 0$$
$$T[1] = 1$$
$$T[n] = T[n-1] + T[n-2] \text{ for } n \geq 2$$

The characteristic polynomial is

$$\lambda^2 - \lambda - 1$$

Finding the roots, gives

$$\lambda_{1/2} = \frac{1}{2} \pm \sqrt{\frac{1}{4} + 1} = \frac{1}{2}\left(1 \pm \sqrt{5}\right)$$

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Hence, the solution is of the form

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$$\alpha\left(\frac{1+\sqrt{5}}{2}\right) + \beta\left(\frac{1-\sqrt{5}}{2}\right) = 1 \Longrightarrow \alpha - \beta = \frac{2}{\sqrt{5}}$$

# Example: Fibonacci Sequence

Hence, the solution is

$$\frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right]$$

# The Inhomogeneous Case

Consider the recurrence relation:

$$c_0T(n) + c_1T(n-1) + c_2T(n-2) + \cdots + c_kT(n-k) = f(n)$$

with $f(n) \neq 0$.

While we have a fairly general technique for solving homogeneous, linear recurrence relations the inhomogeneous case is different.

# The Inhomogeneous Case

The general solution of the recurrence relation is

$$T(n) = T_h(n) + T_p(n) \ ,$$

where $T_h$ is any solution to the homogeneous equation, and $T_p$ is one particular solution to the inhomogeneous equation.

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Example:

$$T[n] = T[n-1] + 1 \qquad T[0] = 1$$

Then,

$$T[n-1] = T[n-2] + 1 \qquad (n \geq 2)$$

Subtracting the first from the second equation gives,

$$T[n] - T[n-1] = T[n-1] - T[n-2] \qquad (n \geq 2)$$

or

$$T[n] = 2T[n-1] - T[n-2] \qquad (n \geq 2)$$

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$T[1] = 2$ gives $1 + \beta = 2 \Longrightarrow \beta = 1$.

## The Inhomogeneous Case

If $f(n)$ is a polynomial of degree $r$ this method can be applied $r+1$ times to obtain a homogeneous equation:

$$T[n] = T[n-1] + n^2$$

Shift:

$$T[n-1] = T[n-2] + (n-1)^2 = T[n-2] + n^2 - 2n + 1$$

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Shift:

$$\begin{aligned} T[n-1] &= 2T[n-2] - T[n-3] + 2(n-1) - 1 \\ &= 2T[n-2] - T[n-3] + 2n - 3 \end{aligned}$$

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and so on...

# 6.4 Generating Functions

**Definition 4 (Generating Function)**

Let $(a_n)_{n\geq 0}$ be a sequence. The corresponding

- generating function (Erzeugendenfunktion) is

$$F(z) := \sum_{n\geq 0} a_n z^n;$$

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**Example 5**

1. The generating function of the sequence $(1, 0, 0, \ldots)$ is

$$F(z) = 1 \,.$$

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# 6.4 Generating Functions

There are two different views:

A generating function is a formal power series (formale Potenzreihe).

Then the generating function is an algebraic object.

Let $f = \sum_{n\geq 0} a_n z^n$ and $g = \sum_{n\geq 0} b_n z^n$.

- **Equality:** $f$ and $g$ are equal if $a_n = b_n$ for all $n$.
- **Addition:** $f + g := \sum_{n\geq 0}(a_n + b_n) z^n$.
- **Multiplication:** $f \cdot g := \sum_{n\geq 0} c_n z^n$ with $c_n = \sum_{p=0}^{n} a_p b_{n-p}$.

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We view a power series as a function $f : \mathbb{C} \to \mathbb{C}$.

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It means that the power series $1-z$ and the power series $\sum_{n\geq 0} z^n$ are invers, i.e.,

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It means that the power series $1 - z$ and the power series $\sum_{n\geq 0} z^n$ are invers, i.e.,

$$\big(1 - z\big) \cdot \Big( \sum_{n\geq 0}^{\infty} z^n \Big) = 1 \ .$$

This is well-defined.

# 6.4 Generating Functions

Suppose we are given the generating function

$$\sum_{n\geq 0} z^n = \frac{1}{1-z} \ .$$

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Hence, the generating function of the sequence $a_n = n + 1$ is $1/(1-z)^2$.

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$$\sum_{n\geq 1} n(n+1)z^{n-1} = \frac{2}{(1-z)^3}$$

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Hence, the generating function of the sequence $a_n = (n+1)(n+2)$ is $\frac{2}{(1-z)^3}$.

# 6.4 Generating Functions

Computing the $k$-th derivative of $\sum z^n$.

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$$\sum_{n\geq k} n(n-1)\cdot\ldots\cdot(n-k+1)z^{n-k} = \sum_{n\geq 0}(n+k)\cdot\ldots\cdot(n+1)z^n$$
$$= \frac{k!}{(1-z)^{k+1}} \; .$$

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Hence:

$$\sum_{n\geq 0}\binom{n+k}{k}z^n = \frac{1}{(1-z)^{k+1}} \ .$$

The generating function of the sequence $a_n = \binom{n+k}{k}$ is $\frac{1}{(1-z)^{k+1}}$.

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$$\sum_{n\geq 0} nz^n = \sum_{n\geq 0}(n+1)z^n - \sum_{n\geq 0} z^n$$

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$$\begin{aligned}\sum_{n\geq 0} nz^n &= \sum_{n\geq 0}(n+1)z^n - \sum_{n\geq 0} z^n \\ &= \frac{1}{(1-z)^2} - \frac{1}{1-z}\end{aligned}$$

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$$= \frac{z}{(1-z)^2}$$

The generating function of the sequence $a_n = n$ is $\frac{z}{(1-z)^2}$.

# 6.4 Generating Functions

We know

$$\sum_{n\geq 0} y^n = \frac{1}{1-y}$$

Hence,

$$\sum_{n\geq 0} a^n z^n = \frac{1}{1-az}$$

The generating function of the sequence $f_n = a^n$ is $\frac{1}{1-az}$.

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## Example: $a_n = a_{n-1} + 1$, $a_0 = 1$

Suppose we have the recurrence $a_n = a_{n-1} + 1$ for $n \geq 1$ and $a_0 = 1$.

$$A(z)$$

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$$\begin{aligned} A(z) &= \sum_{n\geq 0} a_n z^n \\ &= a_0 + \sum_{n\geq 1} (a_{n-1} + 1) z^n \end{aligned}$$

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$$\begin{aligned} A(z) &= \sum_{n\geq 0} a_n z^n \\ &= a_0 + \sum_{n\geq 1} (a_{n-1} + 1) z^n \\ &= 1 + z \sum_{n\geq 1} a_{n-1} z^{n-1} + \sum_{n\geq 1} z^n \end{aligned}$$

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&= 1 + z \sum_{n\geq 1} a_{n-1} z^{n-1} + \sum_{n\geq 1} z^n \\
&= z \sum_{n\geq 0} a_n z^n + \sum_{n\geq 0} z^n
\end{aligned}$$

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&= 1 + z \sum_{n\geq 1} a_{n-1} z^{n-1} + \sum_{n\geq 1} z^n \\
&= z \sum_{n\geq 0} a_n z^n + \sum_{n\geq 0} z^n \\
&= zA(z) + \sum_{n\geq 0} z^n \\
&= zA(z) + \frac{1}{1-z}
\end{aligned}
$$

## Example: $a_n = a_{n-1} + 1$, $a_0 = 1$

Solving for $A(z)$ gives

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$$\sum_{n\geq 0} a_n z^n = A(z) = \frac{1}{(1-z)^2}$$

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$$\sum_{n\geq 0} a_n z^n = A(z) = \frac{1}{(1-z)^2} = \sum_{n\geq 0}(n+1)z^n$$

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$$\sum_{n\geq 0} a_n z^n = A(z) = \frac{1}{(1-z)^2} = \sum_{n\geq 0}(n+1)z^n$$

Hence, $a_n = n + 1$.

# Some Generating Functions

| *n*-th sequence element | generating function |
|---|---|
| | |
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| | |
| | |
| | |

# Some Generating Functions

| *n-th sequence element* | *generating function* |
|---|---|
| $1$ | $\frac{1}{1-z}$ |
| | |
| | |
| | |
| | |
| | |
| | |

# Some Generating Functions

| *n*-th sequence element | generating function |
|---|---|
| $1$ | $\frac{1}{1-z}$ |
| $n+1$ | $\frac{1}{(1-z)^2}$ |
| | |
| | |
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| | |

# Some Generating Functions

| *n-th sequence element* | *generating function* |
|---|---|
| $1$ | $\frac{1}{1-z}$ |
| $n+1$ | $\frac{1}{(1-z)^2}$ |
| $\binom{n+k}{k}$ | $\frac{1}{(1-z)^{k+1}}$ |
| | |
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| | |

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|---|---|
| $1$ | $\frac{1}{1-z}$ |
| $n+1$ | $\frac{1}{(1-z)^2}$ |
| $\binom{n+k}{k}$ | $\frac{1}{(1-z)^{k+1}}$ |
| $n$ | $\frac{z}{(1-z)^2}$ |
| | |
| | |
| | |

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| $1$ | $\frac{1}{1-z}$ |
| $n+1$ | $\frac{1}{(1-z)^2}$ |
| $\binom{n+k}{k}$ | $\frac{1}{(1-z)^{k+1}}$ |
| $n$ | $\frac{z}{(1-z)^2}$ |
| $a^n$ | $\frac{1}{1-az}$ |
| | |
| | |

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| $n$ | $\frac{z}{(1-z)^2}$ |
| $a^n$ | $\frac{1}{1-az}$ |
| $n^2$ | $\frac{z(1+z)}{(1-z)^3}$ |
| | |

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| $a^n$ | $\frac{1}{1-az}$ |
| $n^2$ | $\frac{z(1+z)}{(1-z)^3}$ |
| $\frac{1}{n!}$ | $e^z$ |

# Some Generating Functions

| *n-th sequence element* | *generating function* |
|---|---|
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# Some Generating Functions

| *n-th sequence element* | *generating function* |
|---|---|
| $cf_n$ | $cF$ |
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# Some Generating Functions

| *n-th sequence element* | *generating function* |
|---|---|
| $cf_n$ | $cF$ |
| $f_n + g_n$ | $F + G$ |
| | |
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| | |

# Some Generating Functions

| *n-th sequence element* | *generating function* |
|---|---|
| $cf_n$ | $cF$ |
| $f_n + g_n$ | $F + G$ |
| $\sum_{i=0}^{n} f_i g_{n-i}$ | $F \cdot G$ |
| | |
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| $\sum_{i=0}^{n} f_i g_{n-i}$ | $F \cdot G$ |
| $f_{n-k} \quad (n \geq k);\quad 0$ otw. | $z^k F$ |
| | |
| | |
| | |

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| $\sum_{i=0}^{n} f_i g_{n-i}$ | $F \cdot G$ |
| $f_{n-k} \quad (n \geq k);\quad 0$ otw. | $z^k F$ |
| $\sum_{i=0}^{n} f_i$ | $\dfrac{F(z)}{1-z}$ |
| | |
| | |

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| $\sum_{i=0}^{n} f_i$ | $\frac{F(z)}{1-z}$ |
| $nf_n$ | $z\frac{\mathrm{d}F(z)}{\mathrm{d}z}$ |
| | |

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| $nf_n$ | $z\frac{\mathrm{d}F(z)}{\mathrm{d}z}$ |
| $c^n f_n$ | $F(cz)$ |

# Solving Recursions with Generating Functions

1. Set $A(z) = \sum_{n\geq 0} a_n z^n$.

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   Techniques:

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5. Write $f(z)$ as a formal power series.
   Techniques:
   - partial fraction decomposition (Partialbruchzerlegung)
   - lookup in tables
6. The coefficients of the resulting power series are the $a_n$.

## Example: $a_n = 2a_{n-1}, a_0 = 1$

1. Set up generating function:

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1. Set up generating function:

$$A(z) = \sum_{n\geq 0} a_n z^n$$

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$$A(z) = a_0 + \sum_{n\geq 1} a_n z^n$$

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2. Plug in:

$$A(z) = 1 + \sum_{n\geq 1} (2a_{n-1}) z^n$$

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3. Transform right hand side so that infinite sums can be replaced by $A(z)$ or by simple function.

$$\begin{aligned} A(z) &= 1 + \sum_{n\geq 1} (2a_{n-1})z^n \\ &= 1 + 2z \sum_{n\geq 1} a_{n-1}z^{n-1} \end{aligned}$$

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&= 1 + 2z\sum_{n\geq 0} a_n z^n
\end{aligned}
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&= 1 + 2z \sum_{n\geq 1} a_{n-1}z^{n-1} \\
&= 1 + 2z \sum_{n\geq 0} a_n z^n \\
&= 1 + 2z \cdot A(z)
\end{aligned}
$$

4. Solve for $A(z)$.

## Example: $a_n = 2a_{n-1}$, $a_0 = 1$

3. Transform right hand side so that infinite sums can be replaced by $A(z)$ or by simple function.

$$\begin{aligned} A(z) &= 1 + \sum_{n\geq 1}(2a_{n-1})z^n \\ &= 1 + 2z\sum_{n\geq 1}a_{n-1}z^{n-1} \\ &= 1 + 2z\sum_{n\geq 0}a_n z^n \\ &= 1 + 2z\cdot A(z) \end{aligned}$$

4. Solve for $A(z)$.

$$A(z) = \frac{1}{1-2z}$$

# Example: $a_n = 2a_{n-1}, a_0 = 1$

5. Rewrite $f(z)$ as a power series:

$$A(z) = \frac{1}{1 - 2z}$$

## Example: $a_n = 2a_{n-1}, a_0 = 1$

**5.** Rewrite $f(z)$ as a power series:

$$\sum_{n\geq 0} a_n z^n = A(z) = \frac{1}{1-2z}$$

# Example: $a_n = 2a_{n-1}$, $a_0 = 1$

**5.** Rewrite $f(z)$ as a power series:

$$\sum_{n\geq 0} a_n z^n = A(z) = \frac{1}{1-2z} = \sum_{n\geq 0} 2^n z^n$$

## Example: $a_n = 3a_{n-1} + n$, $a_0 = 1$

1. Set up generating function:

$$A(z) = \sum_{n\geq 0} a_n z^n$$

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1. Set up generating function:

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**2./3.** Transform right hand side:

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&= 1 + 3z \sum_{n\geq 1} a_{n-1} z^{n-1} + \sum_{n\geq 1} n z^n
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gives

$$A(z) = \frac{(1-z)^2 + z}{(1-3z)(1-z)^2} = \frac{z^2 - z + 1}{(1-3z)(1-z)^2}$$

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**5.** Write $f(z)$ as a formal power series:

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$$\begin{aligned} z^2 - z + 1 &= A(1-z)^2 + B(1-3z)(1-z) + C(1-3z) \\ &= A(1-2z+z^2) + B(1-4z+3z^2) + C(1-3z) \end{aligned}$$

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## Example: $a_n = 3a_{n-1} + n$, $a_0 = 1$

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   This leads to the following conditions:

$$\begin{aligned} A + B + C &= 1 \\ 2A + 4B + 3C &= 1 \\ A + 3B &= 1 \end{aligned}$$

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**5.** Write $f(z)$ as a formal power series:

This leads to the following conditions:

$$A + B + C = 1$$
$$2A + 4B + 3C = 1$$
$$A + 3B = 1$$

which gives

$$A = \frac{7}{4} \quad B = -\frac{1}{4} \quad C = -\frac{1}{2}$$

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$$= \frac{7}{4} \cdot \sum_{n \geq 0} 3^n z^n - \frac{1}{4} \cdot \sum_{n \geq 0} z^n - \frac{1}{2} \cdot \sum_{n \geq 0} (n+1) z^n$$

$$= \sum_{n \geq 0} \Big( \frac{7}{4} \cdot 3^n - \frac{1}{4} - \frac{1}{2}(n+1) \Big) z^n$$

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&= \sum_{n\ge 0}\Big(\frac{7}{4}\cdot 3^n - \frac{1}{4} - \frac{1}{2}(n+1)\Big)z^n \\
&= \sum_{n\ge 0}\Big(\frac{7}{4}\cdot 3^n - \frac{1}{2}n - \frac{3}{4}\Big)z^n
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\end{aligned}
$$

**6.** This means $a_n = \frac{7}{4}3^n - \frac{1}{2}n - \frac{3}{4}$.

# 6.5 Transformation of the Recurrence

**Example 6**

$$f_0 = 1$$
$$f_1 = 2$$
$$f_n = f_{n-1} \cdot f_{n-2} \text{ for } n \geq 2 \,.$$

# 6.5 Transformation of the Recurrence

**Example 6**

$$f_0 = 1$$
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Define

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$$g_n = F_n \; (n\text{-th Fibonacci number})$$
$$f_n = 2^{F_n}$$

# 6.5 Transformation of the Recurrence

**Example 7**

$$f_1 = 1$$
$$f_n = 3f_{\frac{n}{2}} + n; \text{ for } n = 2^k,\ k \geq 1\ ;$$

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Define

$$g_k := f_{2^k} \;.$$

Then:

$$g_0 = 1$$
$$g_k = 3g_{k-1} + 2^k,\; k \geq 1$$

# 6 Recurrences

We get

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$$\begin{aligned} g_k &= 3\left[g_{k-1}\right] + 2^k \\ &= 3\left[3g_{k-2} + 2^{k-1}\right] + 2^k \end{aligned}$$

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$$\begin{aligned} g_k &= 3\left[g_{k-1}\right] + 2^k \\ &= 3\left[3g_{k-2} + 2^{k-1}\right] + 2^k \\ &= 3^2\left[g_{k-2}\right] + 32^{k-1} + 2^k \end{aligned}$$

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\begin{aligned}
g_k &= 3\left[g_{k-1}\right] + 2^k \\
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&= 3^2\left[3g_{k-3} + 2^{k-2}\right] + 32^{k-1} + 2^k \\
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\end{aligned}
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&= 2^k \cdot \sum_{i=0}^{k} \left(\frac{3}{2}\right)^i \\
&= 2^k \cdot \frac{(\frac{3}{2})^{k+1} - 1}{1/2}
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&= 3^3 g_{k-3} + 3^2 2^{k-2} + 32^{k-1} + 2^k \\
&= 2^k \cdot \sum_{i=0}^{k} \left(\frac{3}{2}\right)^i \\
&= 2^k \cdot \frac{(\frac{3}{2})^{k+1} - 1}{1/2} = 3^{k+1} - 2^{k+1}
\end{aligned}
$$

# 6 Recurrences

Let $n = 2^k$:

$$g_k = 3^{k+1} - 2^{k+1}, \text{ hence}$$
$$f_n = 3 \cdot 3^k - 2 \cdot 2^k$$

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