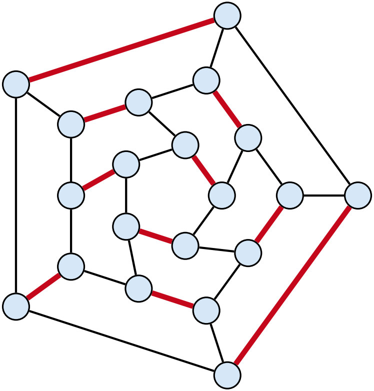
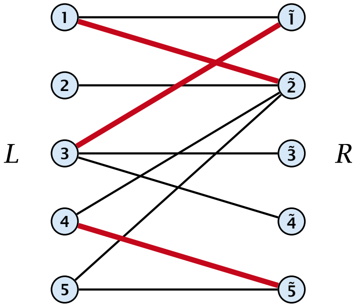
Matching

- Input: undirected graph $G = (V, E)$.
- $M \subseteq E$ is a matching if each node appears in at most one edge in M.
- Maximum Matching: find a matching of maximum cardinality



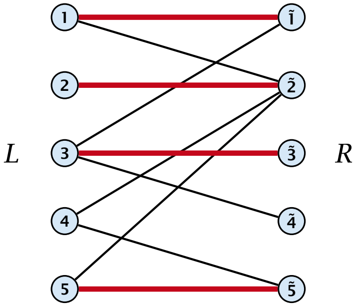
Bipartite Matching

- Input: undirected, bipartite graph $G = (L \uplus R, E)$.
- $M \subseteq E$ is a matching if each node appears in at most one edge in M.
- Maximum Matching: find a matching of maximum cardinality



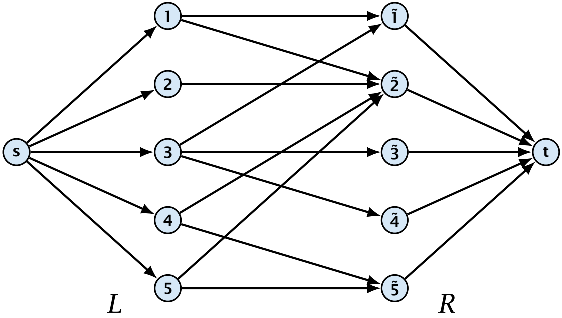
Bipartite Matching

- Input: undirected, bipartite graph $G = (L \uplus R, E)$.
- $M \subseteq E$ is a matching if each node appears in at most one edge in M.
- Maximum Matching: find a matching of maximum cardinality



Maxflow Formulation

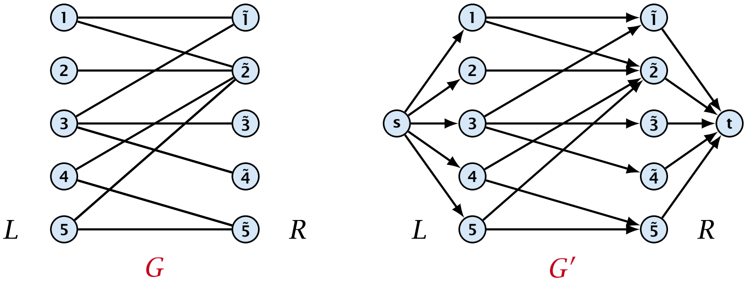
- Input: undirected, bipartite graph $G = (L \uplus R \uplus \{s,t\}, E')$.
- Direct all edges from L to R.
- Add source s and connect it to all nodes on the left.
- Add t and connect all nodes on the right to t.
- All edges have unit capacity.



Proof

Max cardinality matching in G ≤ value of maxflow in G'

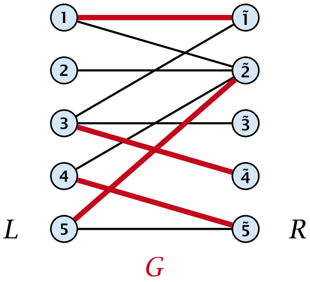
- Given a maximum matching M of cardinality k.
- Consider flow f that sends one unit along each of k paths.
- f is a flow and has cardinality k.

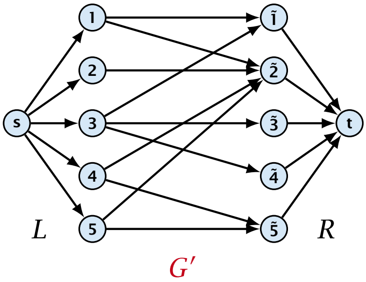


Proof

Max cardinality matching in $G \leq$ value of maxflow in G'

- Given a maximum matching M of cardinality k.
- Consider flow f that sends one unit along each of k paths.
- f is a flow and has cardinality k.

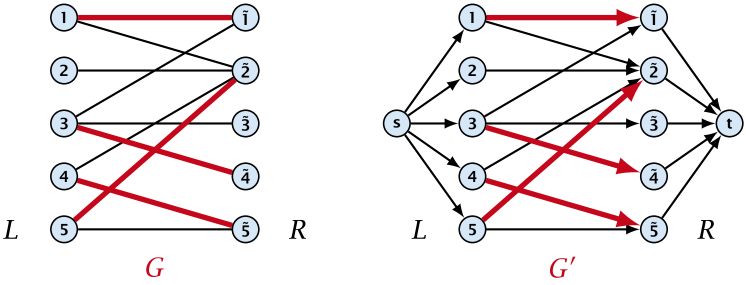




Proof

Max cardinality matching in $G \leq$ value of maxflow in G'

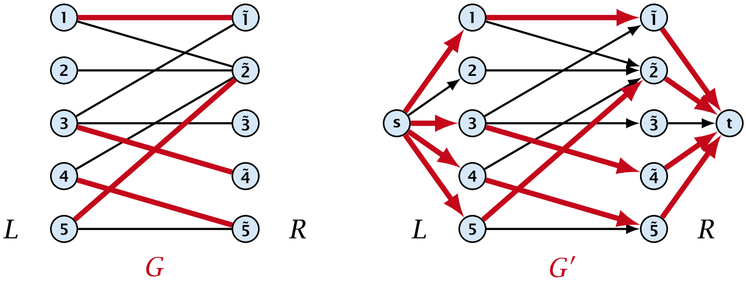
- Given a maximum matching M of cardinality k.
- Consider flow f that sends one unit along each of k paths.
- f is a flow and has cardinality k.



Proof

Max cardinality matching in $G \leq$ value of maxflow in G'

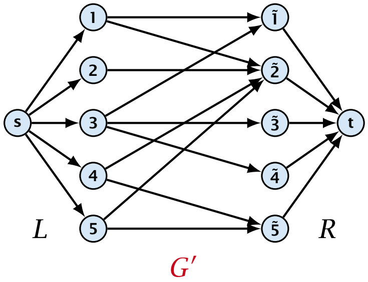
- Given a maximum matching M of cardinality k.
- Consider flow f that sends one unit along each of k paths.
- f is a flow and has cardinality k.

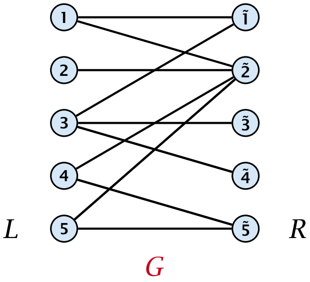


Proof

Max cardinality matching in $G \geq$ value of maxflow in G'

- Let f be a maxflow in G' of value k
- Integrality theorem $\Rightarrow$ k integral; we can assume f is 0/1.
- Consider M= set of edges from L to R with $f(e) = 1$.
- Each node in L and R participates in at most one edge in M.
- $|M| = k$, as the flow must use at least k middle edges.

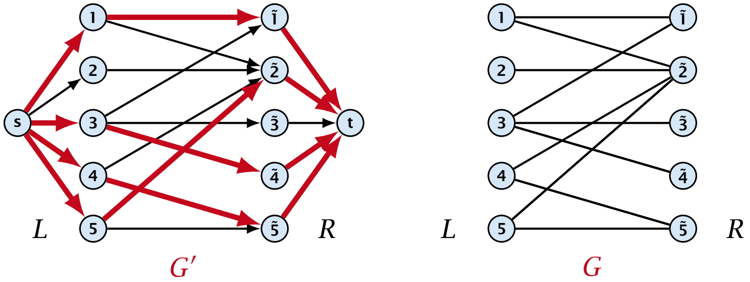




Proof

Max cardinality matching in $G \geq$ value of maxflow in G'

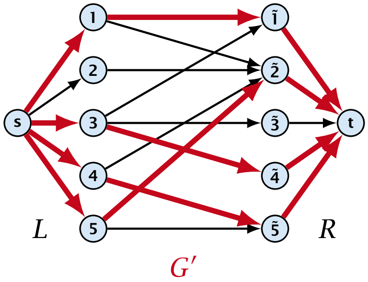
- Let f be a maxflow in G' of value k
- Integrality theorem $\Rightarrow$ k integral; we can assume f is 0/1.
- Consider M= set of edges from L to R with $f(e) = 1$.
- Each node in L and R participates in at most one edge in M.
- $|M| = k$, as the flow must use at least k middle edges.

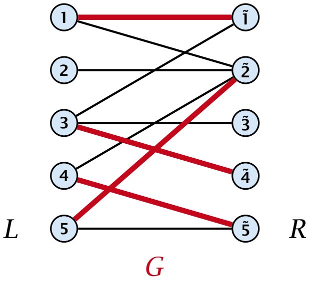


Proof

Max cardinality matching in $G \geq$ value of maxflow in G'

- Let f be a maxflow in G' of value k
- Integrality theorem $\Rightarrow$ k integral; we can assume f is 0/1.
- Consider M= set of edges from L to R with $f(e) = 1$.
- Each node in L and R participates in at most one edge in M.
- $|M| = k$, as the flow must use at least k middle edges.





12.1 Matching

Which flow algorithm to use?

- Generic augmenting path: $\mathcal{O}(m\,\mathrm{val}(f^*)) = \mathcal{O}(mn)$.
- Capacity scaling: $\mathcal{O}(m^2 \log C) = \mathcal{O}(m^2)$.
- Shortest augmenting path: $\mathcal{O}(mn^2)$.

For unit capacity simple graphs shortest augmenting path can be implemented in time $\mathcal{O}(m\sqrt{n})$.

A graph is a unit capacity simple graph if

- every edge has capacity 1
- a node has either at most one leaving edge **or** at most one entering edge

Baseball Elimination

team i	wins w_i	losses ℓ_i	remaining games Atl	Phi	NY	Mon
Atlanta	83	71	–	1	6	1
Philadelphia	80	79	1	–	0	2
New York	78	78	6	0	–	0
Montreal	77	82	1	2	0	–

Which team can end the season with most wins?

- Montreal is eliminated, since even after winning all remaining games there are only 80 wins.
- But also Philadelphia is eliminated. Why?

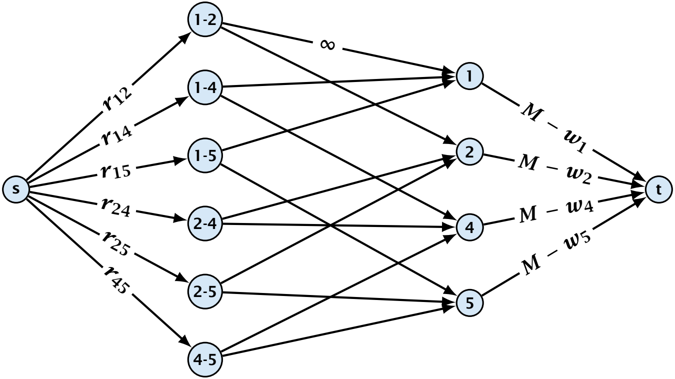
Baseball Elimination

Formal definition of the problem:

- Given a set S of teams, and one specific team $z \in S$.
- Team x has already won w_x games.
- Team x still has to play team y, r_{xy} times.
- Does team z still have a chance to finish with the most number of wins.

Baseball Elimination

Flow network for $z = 3$. M is number of wins Team 3 can still obtain.



Idea. Distribute the results of remaining games in such a way that no team gets too many wins.

Certificate of Elimination

Let $T \subseteq S$ be a subset of teams. Define

$$w(T) := \sum_{i \in T} w_i, \qquad r(T) := \sum_{i,j \in T, i<j} r_{ij}$$

wins of teams in T

remaining games among teams in T

If $\frac{w(T)+r(T)}{|T|} > M$ then one of the teams in T will have more than M wins in the end. A team that can win at most M games is therefore eliminated.

Theorem 1

A team z is eliminated if and only if the flow network for z does not allow a flow of value $\sum_{ij \in S\setminus\{z\}, i<j} r_{ij}$.

Theorem 1

A team z is eliminated if and only if the flow network for z does not allow a flow of value $\sum_{ij\in S\setminus\{z\},i<j} r_{ij}$.

Proof (⇐)

- Consider the mincut A in the flow network. Let T be the set of team-nodes in A.

Theorem 1

A team z is eliminated if and only if the flow network for z does not allow a flow of value $\sum_{ij\in S\setminus\{z\},i<j} r_{ij}$.

Proof (⇐)

- Consider the mincut A in the flow network. Let T be the set of team-nodes in A.
- If for node x-y not both team-nodes x and y are in T, then x-$y \notin A$ as otw. the cut would cut an infinite capacity edge.

Theorem 1

A team z is eliminated if and only if the flow network for z does not allow a flow of value $\sum_{ij\in S\setminus\{z\},i<j} r_{ij}$.

Proof (⇐)

- Consider the mincut A in the flow network. Let T be the set of team-nodes in A.
- If for node x-y not both team-nodes x and y are in T, then x-$y \notin A$ as otw. the cut would cut an infinite capacity edge.
- We don't find a flow that saturates all source edges:

$$r(S \setminus \{z\})$$

Theorem 1

A team z is eliminated if and only if the flow network for z does not allow a flow of value $\sum_{ij\in S\setminus\{z\},i<j} r_{ij}$.

Proof (⇐)

- Consider the mincut A in the flow network. Let T be the set of team-nodes in A.
- If for node x-y not both team-nodes x and y are in T, then x-$y \notin A$ as otw. the cut would cut an infinite capacity edge.
- We don't find a flow that saturates all source edges:

$$r(S \setminus \{z\}) > \text{cap}(A, V \setminus A)$$

Theorem 1

A team z is eliminated if and only if the flow network for z does not allow a flow of value $\sum_{ij \in S \setminus \{z\}, i<j} r_{ij}$.

Proof (⇐)

- Consider the mincut A in the flow network. Let T be the set of team-nodes in A.
- If for node x-y not both team-nodes x and y are in T, then x-$y \notin A$ as otw. the cut would cut an infinite capacity edge.
- We don't find a flow that saturates all source edges:

$$
\begin{aligned}
r(S \setminus \{z\}) &> \mathrm{cap}(A, V \setminus A) \\
&\geq \sum_{i<j:\, i \notin T \vee j \notin T} r_{ij} + \sum_{i \in T} (M - w_i)
\end{aligned}
$$

Theorem 1

A team z is eliminated if and only if the flow network for z does not allow a flow of value $\sum_{ij\in S\setminus\{z\},i<j} r_{ij}$.

Proof (⇐)

- Consider the mincut A in the flow network. Let T be the set of team-nodes in A.
- If for node x-y not both team-nodes x and y are in T, then x-$y \notin A$ as otw. the cut would cut an infinite capacity edge.
- We don't find a flow that saturates all source edges:

$$\begin{aligned} r(S\setminus\{z\}) &> \mathrm{cap}(A, V\setminus A) \\ &\geq \sum_{i<j:\, i\notin T \vee j\notin T} r_{ij} + \sum_{i\in T}(M - w_i) \\ &\geq r(S\setminus\{z\}) - r(T) + |T|M - w(T) \end{aligned}$$

Theorem 1

A team z is eliminated if and only if the flow network for z does not allow a flow of value $\sum_{ij\in S\setminus\{z\}, i<j} r_{ij}$.

Proof (⇐)

- Consider the mincut A in the flow network. Let T be the set of team-nodes in A.
- If for node x-y not both team-nodes x and y are in T, then x-$y \notin A$ as otw. the cut would cut an infinite capacity edge.
- We don't find a flow that saturates all source edges:

$$
\begin{aligned}
r(S \setminus \{z\}) &> \mathrm{cap}(A, V \setminus A) \\
&\geq \sum_{i<j:\, i\notin T \vee j\notin T} r_{ij} + \sum_{i\in T} (M - w_i) \\
&\geq r(S \setminus \{z\}) - r(T) + |T|M - w(T)
\end{aligned}
$$

- This gives $M < (w(T) + r(T))/|T|$, i.e., z is eliminated.

Baseball Elimination

Proof (⇒)

- Suppose we have a flow that saturates all source edges.
- We can assume that this flow is integral.
- For every pairing x-y it defines how many games team x and team y should win.
- The flow leaving the team-node x can be interpreted as the additional number of wins that team x will obtain.
- This is less than $M - w_x$ because of capacity constraints.
- Hence, we found a set of results for the remaining games, such that no team obtains more than M wins in total.
- Hence, team z is not eliminated.

Baseball Elimination

Proof (⇒)

- Suppose we have a flow that saturates all source edges.
- We can assume that this flow is integral.
- For every pairing x-y it defines how many games team x and team y should win.
- The flow leaving the team-node x can be interpreted as the additional number of wins that team x will obtain.
- This is less than $M - w_x$ because of capacity constraints.
- Hence, we found a set of results for the remaining games, such that no team obtains more than M wins in total.
- Hence, team z is not eliminated.

Baseball Elimination

Proof (⇒)

- Suppose we have a flow that saturates all source edges.
- We can assume that this flow is integral.
- For every pairing x-y it defines how many games team x and team y should win.
- The flow leaving the team-node x can be interpreted as the additional number of wins that team x will obtain.
- This is less than $M - w_x$ because of capacity constraints.
- Hence, we found a set of results for the remaining games, such that no team obtains more than M wins in total.
- Hence, team z is not eliminated.

Baseball Elimination

Proof (⇒)

- Suppose we have a flow that saturates all source edges.
- We can assume that this flow is integral.
- For every pairing x-y it defines how many games team x and team y should win.
- The flow leaving the team-node x can be interpreted as the additional number of wins that team x will obtain.
- This is less than $M - w_x$ because of capacity constraints.
- Hence, we found a set of results for the remaining games, such that no team obtains more than M wins in total.
- Hence, team z is not eliminated.

Baseball Elimination

Proof (⇒)

- Suppose we have a flow that saturates all source edges.
- We can assume that this flow is integral.
- For every pairing x-y it defines how many games team x and team y should win.
- The flow leaving the team-node x can be interpreted as the additional number of wins that team x will obtain.
- This is less than $M - w_x$ because of capacity constraints.
- Hence, we found a set of results for the remaining games, such that no team obtains more than M wins in total.
- Hence, team z is not eliminated.

Baseball Elimination

Proof (⇒)

- Suppose we have a flow that saturates all source edges.
- We can assume that this flow is integral.
- For every pairing x-y it defines how many games team x and team y should win.
- The flow leaving the team-node x can be interpreted as the additional number of wins that team x will obtain.
- This is less than $M - w_x$ because of capacity constraints.
- Hence, we found a set of results for the remaining games, such that no team obtains more than M wins in total.
- Hence, team z is not eliminated.

Baseball Elimination

Proof (⇒)

- Suppose we have a flow that saturates all source edges.
- We can assume that this flow is integral.
- For every pairing x-y it defines how many games team x and team y should win.
- The flow leaving the team-node x can be interpreted as the additional number of wins that team x will obtain.
- This is less than $M - w_x$ because of capacity constraints.
- Hence, we found a set of results for the remaining games, such that no team obtains more than M wins in total.
- Hence, team z is not eliminated.

Project Selection

Project selection problem:

- Set P of possible projects. Project v has an associated profit p_v (can be positive or negative).
- Some projects have requirements (taking course EA2 requires course EA1).
- Dependencies are modelled in a graph. Edge (u, v) means "can't do project u without also doing project v."
- A subset A of projects is feasible if the prerequisites of every project in A also belong to A.

Goal: Find a feasible set of projects that maximizes the profit.

Project Selection

Project selection problem:

- Set P of possible projects. Project v has an associated profit p_v (can be positive or negative).
- Some projects have requirements (taking course EA2 requires course EA1).
- Dependencies are modelled in a graph. Edge (u,v) means "can't do project u without also doing project v."
- A subset A of projects is feasible if the prerequisites of every project in A also belong to A.

Goal: Find a feasible set of projects that maximizes the profit.

Project Selection

Project selection problem:

- Set P of possible projects. Project v has an associated profit p_v (can be positive or negative).
- Some projects have requirements (taking course EA2 requires course EA1).
- Dependencies are modelled in a graph. Edge (u,v) means "can't do project u without also doing project v."
- A subset A of projects is feasible if the prerequisites of every project in A also belong to A.

Goal: Find a feasible set of projects that maximizes the profit.

Project Selection

Project selection problem:

- Set P of possible projects. Project v has an associated profit p_v (can be positive or negative).
- Some projects have requirements (taking course EA2 requires course EA1).
- Dependencies are modelled in a graph. Edge (u, v) means "can't do project u without also doing project v."
- A subset A of projects is feasible if the prerequisites of every project in A also belong to A.

Goal: Find a feasible set of projects that maximizes the profit.

Project Selection

Project selection problem:

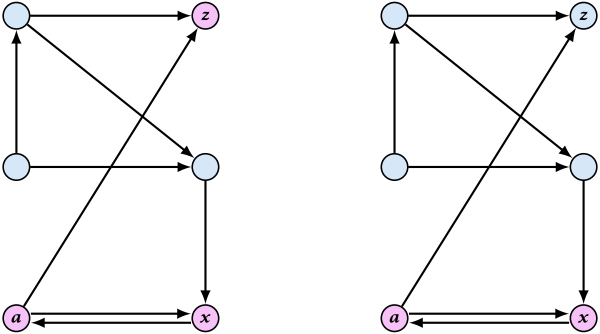
- Set P of possible projects. Project v has an associated profit p_v (can be positive or negative).
- Some projects have requirements (taking course EA2 requires course EA1).
- Dependencies are modelled in a graph. Edge (u, v) means "can't do project u without also doing project v."
- A subset A of projects is feasible if the prerequisites of every project in A also belong to A.

Goal: Find a feasible set of projects that maximizes the profit.

Project Selection

The prerequisite graph:

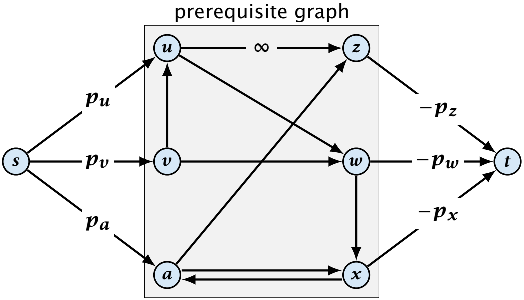
- $\{x, a, z\}$ is a feasible subset.
- $\{x, a\}$ is infeasible.



Project Selection

Mincut formulation:

- Edges in the prerequisite graph get infinite capacity.
- Add edge (s, v) with capacity p_v for nodes v with positive profit.
- Create edge (v, t) with capacity $-p_v$ for nodes v with negative profit.



Theorem 2

A is a mincut if $A \setminus \{s\}$ is the optimal set of projects.

Theorem 2

A is a mincut if $A \setminus \{s\}$ is the optimal set of projects.

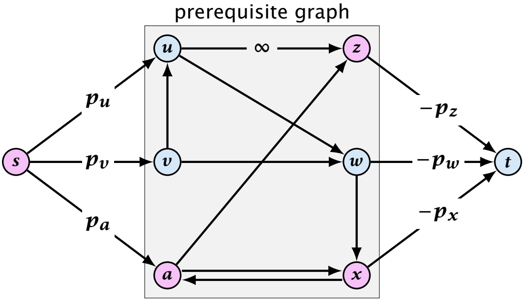
Proof.

- A is feasible because of capacity infinity edges.

For the formula we define $p_s := 0$.

The step follows by adding $\sum_{v \in A: p_v > 0} p_v - \sum_{v \in A: p_v > 0} p_v = 0$.

Note that minimizing the capacity of the cut $(A, V \setminus A)$ corresponds to maximizing profits of projects in A.



Theorem 2

A is a mincut if $A \setminus \{s\}$ is the optimal set of projects.

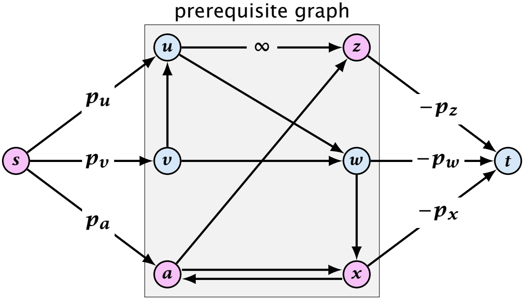
Proof.

- A is feasible because of capacity infinity edges.
- $\text{cap}(A, V \setminus A)$

For the formula we define $p_s := 0$.

The step follows by adding $\sum_{v \in A: p_v > 0} p_v - \sum_{v \in A: p_v > 0} p_v = 0$.

Note that minimizing the capacity of the cut $(A, V \setminus A)$ corresponds to maximizing profits of projects in A.



Theorem 2

A is a mincut if $A \setminus \{s\}$ is the optimal set of projects.

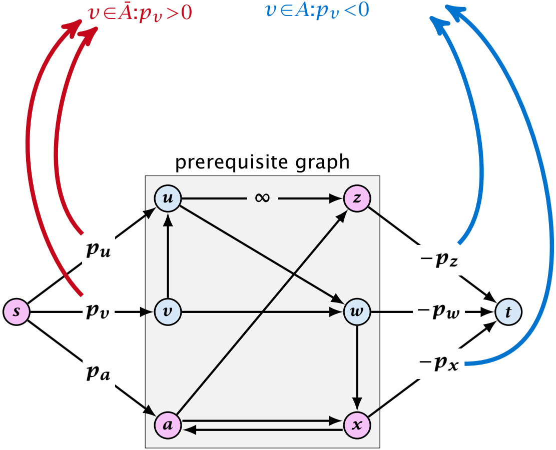
Proof.

- A is feasible because of capacity infinity edges.
- $\mathrm{cap}(A, V \setminus A) = \sum_{v \in \bar{A}: p_v > 0} p_v + \sum_{v \in A: p_v < 0} (-p_v)$

For the formula we define $p_s := 0$.

The step follows by adding $\sum_{v \in A: p_v > 0} p_v - \sum_{v \in A: p_v > 0} p_v = 0$.

Note that minimizing the capacity of the cut $(A, V \setminus A)$ corresponds to maximizing profits of projects in A.



Theorem 2

A is a mincut if $A \setminus \{s\}$ is the optimal set of projects.

Proof.

- A is feasible because of capacity infinity edges.
- $\text{cap}(A, V \setminus A) = \sum_{v \in \bar{A}: p_v > 0} p_v + \sum_{v \in A: p_v < 0} (-p_v)$

$$= \sum_{v: p_v > 0} p_v - \sum_{v \in A} p_v$$

For the formula we define $p_s := 0$.

The step follows by adding $\sum_{v \in A: p_v > 0} p_v - \sum_{v \in A: p_v > 0} p_v = 0$.

Note that minimizing the capacity of the cut $(A, V \setminus A)$ corresponds to maximizing profits of projects in A.