5.3 Strong Duality
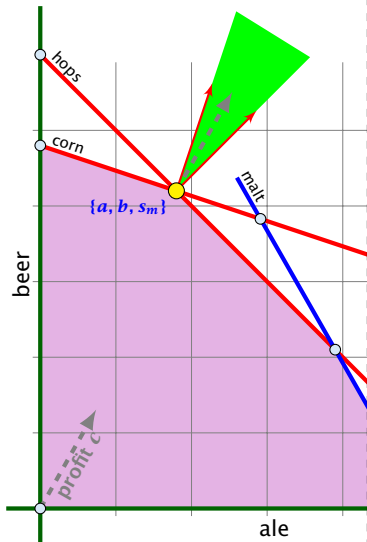
$P = \max\{c^T x \mid Ax \le b, x \ge 0\}$

n_A: number of variables, m_A: number of constraints

We can put the non-negativity constraints into A (which gives us unrestricted variables): $\bar{P} = \max\{c^T x \mid \bar{A}x \le \bar{b}\}$

$n_{\bar{A}} = n_A$, $m_{\bar{A}} = m_A + n_A$

Dual $D = \min\{\bar{b}^T y \mid \bar{A}^T y = c, y \ge 0\}$.

5.3 Strong Duality

If we have a conic combination y of c then $b^T y$ is an upper bound of the profit we can obtain (weak duality):

$$c^T x = (\bar{A}^T y)^T x = y^T \bar{A} x \leq y^T \bar{b}$$

If x and y are optimal then the duality gap is 0 (strong duality). This means

$$
\begin{aligned}
0 &= c^T x - y^T \bar{b} \\
&= (\bar{A}^T y)^T x - y^T \bar{b} \\
&= y^T (\bar{A} x - \bar{b})
\end{aligned}
$$

The last term can only be 0 if y_i is 0 whenever the i-th constraint is not tight. This means we have a conic combination of c by normals (columns of $\bar{A}^T$) of *tight* constraints.

Conversely, if we have x such that the normals of tight constraint (at x) give rise to a conic combination of c, we know that x is optimal.

The profit vector c lies in the cone generated by the normals for the hops and the corn constraint (the tight constraints).

Strong Duality

Theorem 2 (Strong Duality)

Let P and D be a primal dual pair of linear programs, and let z^* and w^* denote the optimal solution to P and D, respectively. Then
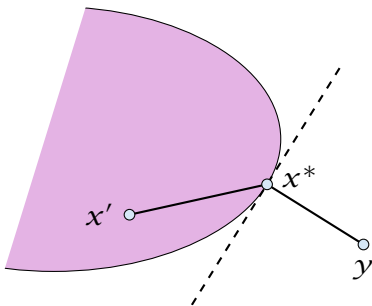
$$z^* = w^*$$

Lemma 3 (Weierstrass)

Let X be a compact set and let $f(x)$ be a continuous function on X. Then $\min\{f(x) : x \in X\}$ exists.
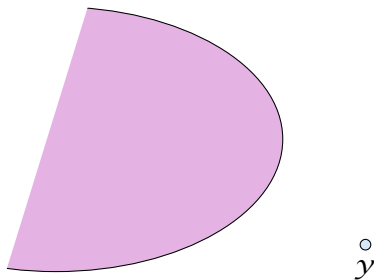
(without proof)

Lemma 4 (Projection Lemma)

Let $X \subseteq \mathbb{R}^m$ be a non-empty convex set, and let $y \notin X$. Then there exist $x^* \in X$ with minimum distance from y. Moreover for all $x \in X$ we have $(y - x^*)^T(x - x^*) \leq 0$.
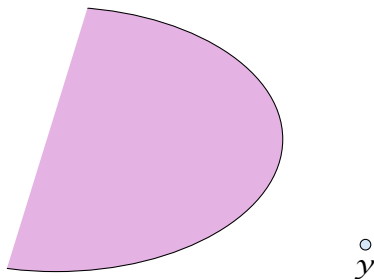
Proof of the Projection Lemma

▶ Define $f(x) = \|y - x\|$.

▶ We want to apply Weierstrass but X may not be bounded.

▶ $X \neq \emptyset$. Hence, there exists $x' \in X$.

▶ Define $X' = \{x \in X \mid \|y - x\| \leq \|y - x'\|\}$. This set is closed and bounded.

▶ Applying Weierstrass gives the existence.

Proof of the Projection Lemma

- Define $f(x) = \|y - x\|$.
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$\circ$
y

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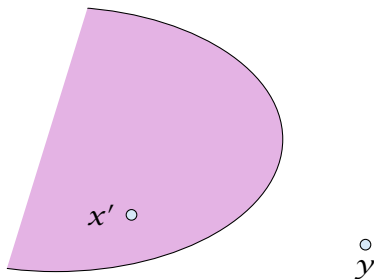▶ Applying Weierstrass gives the existence.

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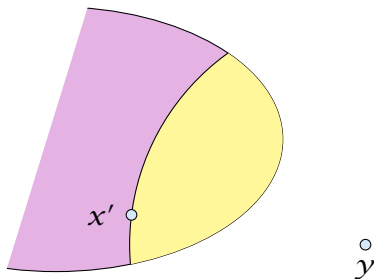
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Proof of the Projection Lemma

- ▶ Define $f(x) = \|y - x\|$.
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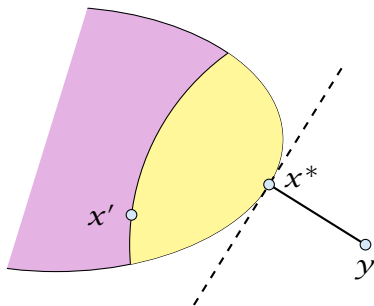- ▶ Applying Weierstrass gives the existence.

Proof of the Projection Lemma (continued)

x^* is minimum. Hence $\|y - x^*\|^2 \le \|y - x\|^2$ for all $x \in X$.

Proof of the Projection Lemma (continued)

x^* is minimum. Hence $\|y - x^*\|^2 \leq \|y - x\|^2$ for all $x \in X$.

By convexity: $x \in X$ then $x^* + \epsilon(x - x^*) \in X$ for all $0 \leq \epsilon \leq 1$.

Proof of the Projection Lemma (continued)

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$\|y - x^*\|^2$

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$$\|y - x^*\|^2 \le \|y - x^* - \epsilon(x - x^*)\|^2$$

Proof of the Projection Lemma (continued)

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By convexity: $x \in X$ then $x^* + \epsilon(x - x^*) \in X$ for all $0 \leq \epsilon \leq 1$.

$$
\begin{aligned}
\|y - x^*\|^2 &\leq \|y - x^* - \epsilon(x - x^*)\|^2 \\
&= \|y - x^*\|^2 + \epsilon^2 \|x - x^*\|^2 - 2\epsilon(y - x^*)^T(x - x^*)
\end{aligned}
$$

Proof of the Projection Lemma (continued)

x^* is minimum. Hence $\|y - x^*\|^2 \leq \|y - x\|^2$ for all $x \in X$.

By convexity: $x \in X$ then $x^* + \epsilon(x - x^*) \in X$ for all $0 \leq \epsilon \leq 1$.

$$\|y - x^*\|^2 \leq \|y - x^* - \epsilon(x - x^*)\|^2$$
$$= \|y - x^*\|^2 + \epsilon^2 \|x - x^*\|^2 - 2\epsilon(y - x^*)^T(x - x^*)$$

Hence, $(y - x^*)^T(x - x^*) \leq \frac{1}{2}\epsilon \|x - x^*\|^2$.

Proof of the Projection Lemma (continued)

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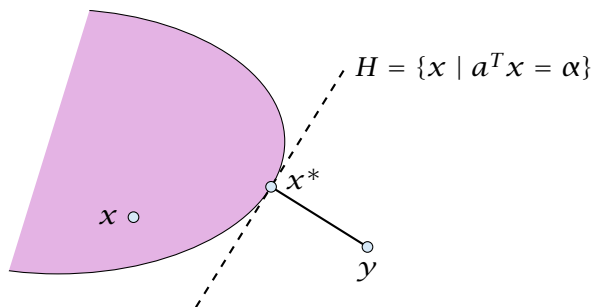
Letting $\epsilon \to 0$ gives the result.

Theorem 5 (Separating Hyperplane)

*Let $X \subseteq \mathbb{R}^m$ be a non-empty closed convex set, and let $y \notin X$.
Then there exists a separating hyperplane $\{x \in \mathbb{R} : a^T x = \alpha\}$
where $a \in \mathbb{R}^m$, $\alpha \in \mathbb{R}$ that separates y from X. ($a^T y < \alpha$;
$a^T x \geq \alpha$ for all $x \in X$)*

Proof of the Hyperplane Lemma

▸ Let $x^* \in X$ be closest point to y in X.

▸ By previous lemma $(y - x^*)^T(x - x^*) \leq 0$ for all $x \in X$.

▸ Choose $a = (x^* - y)$ and $\alpha = a^T x^*$.

▸ For $x \in X : a^T(x - x^*) \geq 0$, and, hence, $a^T x \geq \alpha$.

▸ Also, $a^T y = a^T(x^* - a) = \alpha - \|a\|^2 < \alpha$

$H = \{x \mid a^T x = \alpha\}$

x^*

$x \circ$

y

Proof of the Hyperplane Lemma

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- Choose $a = (x^* - y)$ and $\alpha = a^T x^*$.
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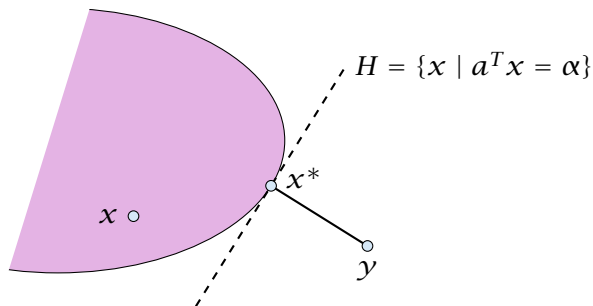
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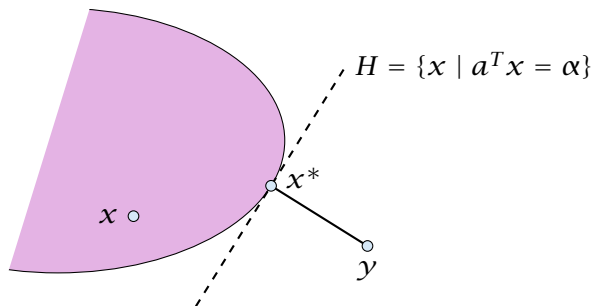- Also, $a^T y = a^T (x^* - a) = \alpha - \|a\|^2 < \alpha$

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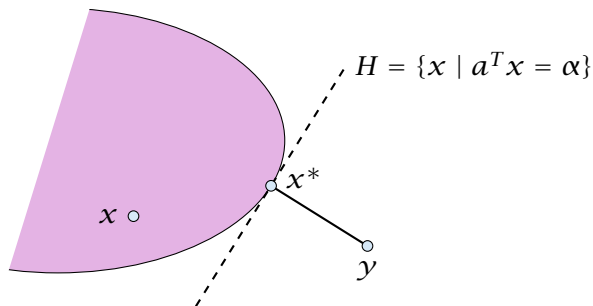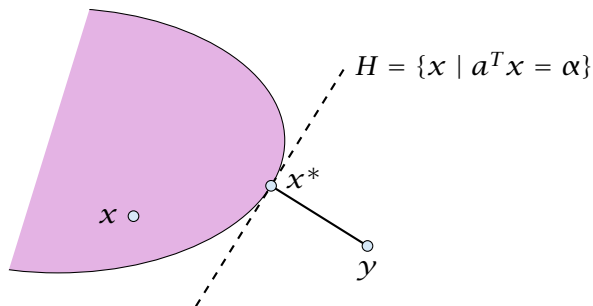- ▶ Also, $a^T y = a^T(x^* - a) = \alpha - \|a\|^2 < \alpha$

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y

Lemma 6 (Farkas Lemma)

Let A be an $m \times n$ matrix, $b \in \mathbb{R}^m$. Then *exactly one* of the following statements holds.

1. $\exists x \in \mathbb{R}^n$ with $Ax = b$, $x \geq 0$
2. $\exists y \in \mathbb{R}^m$ with $A^T y \geq 0$, $b^T y < 0$

Assume $\hat{x}$ satisfies 1. and $\hat{y}$ satisfies 2. Then

$$0 > y^T b = y^T A x \geq 0$$

Hence, at most one of the statements can hold.

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Farkas Lemma
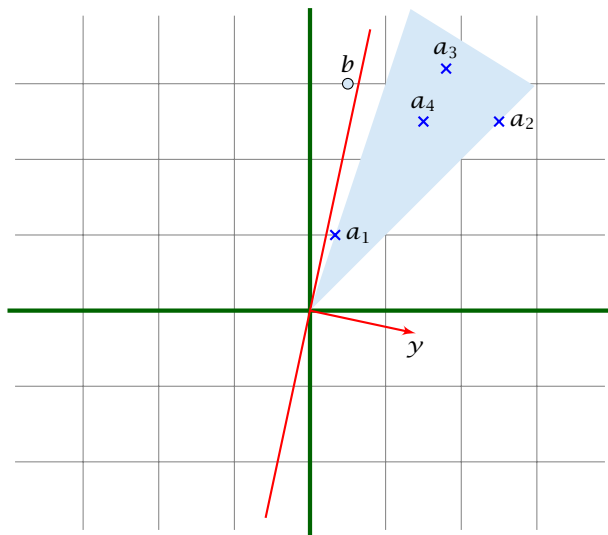
If b is not in the cone generated by the columns of A, there exists a hyperplane y that separates b from the cone.

Proof of Farkas Lemma

Now, assume that 1. does not hold.

Consider $S = \{Ax : x \geq 0\}$ so that S closed, convex, $b \notin S$.

We want to show that there is y with $A^T y \geq 0$, $b^T y < 0$.

Let y be a hyperplane that separates b from S. Hence, $y^T b < \alpha$ and $y^T s \geq \alpha$ for all $s \in S$.

$0 \in S \Rightarrow \alpha \leq 0 \Rightarrow y^T b < 0$

$y^T Ax \geq \alpha$ for all $x \geq 0$. Hence, $y^T A \geq 0$ as we can choose x arbitrarily large.

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Lemma 7 (Farkas Lemma; different version)

Let A be an $m \times n$ matrix, $b \in \mathbb{R}^m$. Then exactly one of the following statements holds.

1. $\exists x \in \mathbb{R}^n$ with $Ax \leq b$, $x \geq 0$
2. $\exists y \in \mathbb{R}^m$ with $A^T y \geq 0$, $b^T y < 0$, $y \geq 0$

Rewrite the conditions:

1. $\exists x \in \mathbb{R}^n$ with $\begin{bmatrix} A\ I \end{bmatrix} \cdot \begin{bmatrix} x \\ s \end{bmatrix} = b$, $x \geq 0$, $s \geq 0$

2. $\exists y \in \mathbb{R}^m$ with $\begin{bmatrix} A^T \\ I \end{bmatrix} y \geq 0$, $b^T y < 0$

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Let A be an $m \times n$ matrix, $b \in \mathbb{R}^m$. Then exactly one of the following statements holds.

   1. $\exists x \in \mathbb{R}^n$ *with* $Ax \le b$, $x \ge 0$

   2. $\exists y \in \mathbb{R}^m$ *with* $A^T y \ge 0$, $b^T y < 0$, $y \ge 0$

Rewrite the conditions:

   1. $\exists x \in \mathbb{R}^n$ with $\begin{bmatrix} A\ I \end{bmatrix} \cdot \begin{bmatrix} x \\ s \end{bmatrix} = b$, $x \ge 0$, $s \ge 0$

   2. $\exists y \in \mathbb{R}^m$ with $\begin{bmatrix} A^T \\ I \end{bmatrix} y \ge 0$, $b^T y < 0$

Proof of Strong Duality

P: $z = \max\{c^T x \mid Ax \leq b, x \geq 0\}$

D: $w = \min\{b^T y \mid A^T y \geq c, y \geq 0\}$

Theorem 8 (Strong Duality)
*Let P and D be a primal dual pair of linear programs, and let z
and w denote the optimal solution to P and D, respectively (i.e.,
P and D are non-empty). Then*

$$z = w \ .$$

Proof of Strong Duality

Proof of Strong Duality

$z \le w$: follows from weak duality

Proof of Strong Duality

$z \leq w$: follows from weak duality

$z \geq w$:

Proof of Strong Duality

$z \leq w$: follows from weak duality

$z \geq w$:
We show $z < \alpha$ implies $w < \alpha$.

Proof of Strong Duality

$z \le w$: follows from weak duality

$z \ge w$:

We show $z < \alpha$ implies $w < \alpha$.

$$
\begin{aligned}
\exists x \in \mathbb{R}^n & \\
\text{s.t.} \quad Ax & \le b \\
-c^T x & \le -\alpha \\
x & \ge 0
\end{aligned}
$$

Proof of Strong Duality

$z \leq w$: follows from weak duality

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\text{s.t.} \quad Ax &\leq b \\
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x &\geq 0
\end{aligned}
$$

$$
\begin{aligned}
\exists y \in \mathbb{R}^m; v \in \mathbb{R} \\
\text{s.t.} \quad A^T y - cv &\geq 0 \\
b^T y - \alpha v &< 0 \\
y, v &\geq 0
\end{aligned}
$$

Proof of Strong Duality

$z \leq w$: follows from weak duality

$z \geq w$:

We show $z < \alpha$ implies $w < \alpha$.

$$
\begin{array}{l}
\exists x \in \mathbb{R}^n \\
\quad \text{s.t.} \quad
\begin{array}{rcl}
Ax & \leq & b \\
-c^T x & \leq & -\alpha \\
x & \geq & 0
\end{array}
\end{array}
$$

$$
\begin{array}{l}
\exists y \in \mathbb{R}^m; v \in \mathbb{R} \\
\quad \text{s.t.} \quad
\begin{array}{rcl}
A^T y - c v & \geq & 0 \\
b^T y - \alpha v & < & 0 \\
y, v & \geq & 0
\end{array}
\end{array}
$$

From the definition of α we know that the first system is infeasible; hence the second must be feasible.

Proof of Strong Duality

$$\exists y \in \mathbb{R}^m; v \in \mathbb{R}$$
$$\text{s.t.} \quad A^T y - c v \geq 0$$
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Proof of Strong Duality

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b^T y - \alpha v & < 0 \\
y, v & \geq 0
\end{aligned}
$$

If the solution y, v has $v = 0$ we have that

$$
\begin{aligned}
\exists y \in \mathbb{R}^m & \\
\text{s.t.} \quad A^T y & \geq 0 \\
b^T y & < 0 \\
y & \geq 0
\end{aligned}
$$

is feasible.

Proof of Strong Duality

$$\exists y \in \mathbb{R}^m; v \in \mathbb{R}$$
$$\text{s.t.} \quad A^T y - cv \geq 0$$
$$b^T y - \alpha v < 0$$
$$y, v \geq 0$$

If the solution y, v has $v = 0$ we have that

$$\exists y \in \mathbb{R}^m$$
$$\text{s.t.} \quad A^T y \geq 0$$
$$b^T y < 0$$
$$y \geq 0$$

is feasible. By Farkas lemma this gives that LP P is infeasible.
Contradiction to the assumption of the lemma.

Proof of Strong Duality

Hence, there exists a solution y, v with $v > 0$.

We can rescale this solution (scaling both y and v) s.t. $v = 1$.

Then y is feasible for the dual but $b^T y < \alpha$. This means that $w < \alpha$.

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