Flows

Definition 2

An (s,t)-flow in a (complete) directed graph $G = (V, V \times V, c)$ is a function $f : V \times V \mapsto \mathbb{R}_0^+$ that satisfies

1. For each edge (x,y)

$$0 \leq f_{xy} \leq c_{xy} \quad .$$

(capacity constraints)

2. For each $v \in V \setminus \{s,t\}$

$$\sum_x f_{vx} = \sum_x f_{xv} \quad .$$

(flow conservation constraints)

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Definition 3
The value of an (s,t)-flow f is defined as

$$\mathrm{val}(f) = \sum_x f_{sx} - \sum_x f_{xs} \quad .$$

Maximum Flow Problem:
Find an (s,t)-flow with maximum value.

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LP-Formulation of Maxflow

$$
\begin{array}{llrcll}
\max & & \sum_z f_{sz} - \sum_z f_{zs} & & & \\
\text{s.t.} & \forall (z,w) \in V \times V & f_{zw} & \le & c_{zw} & \ell_{zw} \\
& \forall w \ne s,t & \sum_z f_{zw} - \sum_z f_{wz} & = & 0 & p_w \\
& & f_{zw} & \ge & 0 &
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\begin{array}{lllcl}
\min & & \sum_{(xy)} c_{xy}\ell_{xy} & & \\
\text{s.t.} & f_{xy}\ (x,y \ne s,t): & 1\ell_{xy} - 1p_x + 1p_y & \ge & 0 \\
 & f_{sy}\ (y \ne s,t): & 1\ell_{sy} \qquad + 1p_y & \ge & 1 \\
 & f_{xs}\ (x \ne s,t): & 1\ell_{xs} - 1p_x & \ge & -1 \\
 & f_{ty}\ (y \ne s,t): & 1\ell_{ty} \qquad + 1p_y & \ge & 0 \\
 & f_{xt}\ (x \ne s,t): & 1\ell_{xt} - 1p_x & \ge & 0 \\
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with $p_t = 0$ and $p_s = 1$.

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We can interpret the ℓ_{xy} value as assigning a length to every edge.

The value p_x for a variable, then can be seen as the distance of x to t (where the distance from s to t is required to be 1 since $p_s = 1$).

The constraint $p_x \leq \ell_{xy} + p_y$ then simply follows from triangle inequality ($d(x,t) \leq d(x,y) + d(y,t) \Rightarrow d(x,t) \leq \ell_{xy} + d(y,t)$).

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One can show that there is an optimum LP-solution for the dual problem that gives an integral assignment of variables.

This means $p_x = 1$ or $p_x = 0$ for our case. This gives rise to a cut in the graph with vertices having value 1 on one side and the other vertices on the other side. The objective function then evaluates the capacity of this cut.

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