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Given $A \in \mathbb{Z}^{m\times n}$, $b \in \mathbb{Z}^m$. Does there exist $x \in \mathbb{R}$ with $Ax = b$, $x \geq 0$?

The Bit Model

Input size

- The number of bits to represent a number $a \in \mathbb{Z}$ is

$$\lceil \log_2(|a|) \rceil + 1$$

- Let for an $m \times n$ matrix M, $L(M)$ denote the number of bits required to encode all the numbers in M.

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- In the following we assume that input matrices are encoded in a standard way, where each number is encoded in binary and then suitable separators are added in order to separate distinct number from each other.
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Size of a Basic Feasible Solution

Lemma 2
Let $M \in \mathbb{Z}^{m\times m}$ be an invertible matrix and let $b \in \mathbb{Z}^m$. Further define $L = \langle M\rangle + \langle b\rangle + n\log_2 n$. Then a solution to $Mx = b$ has rational components x_j of the form $\frac{D_j}{D}$, where $|D_j| \le 2^L$ and $|D| \le 2^L$.

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Cramers rules says that we can compute x_j as

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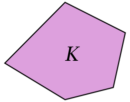

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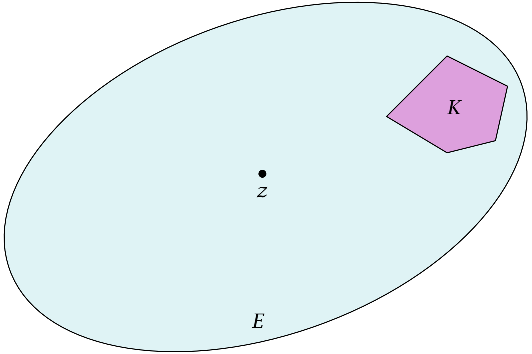

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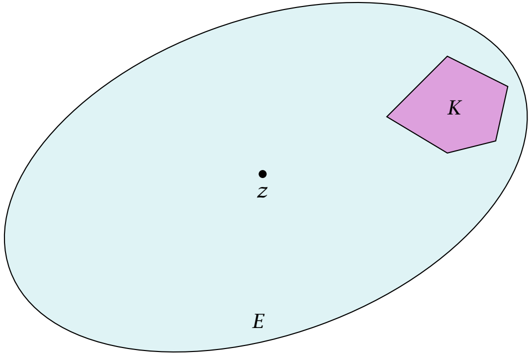

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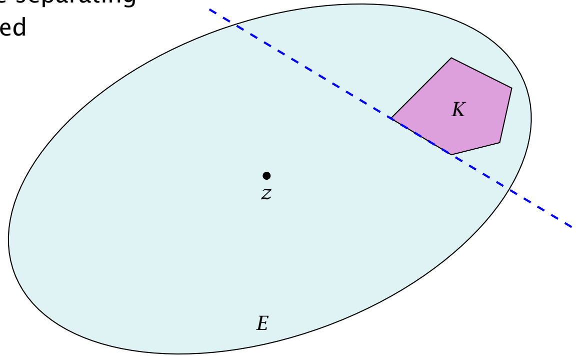

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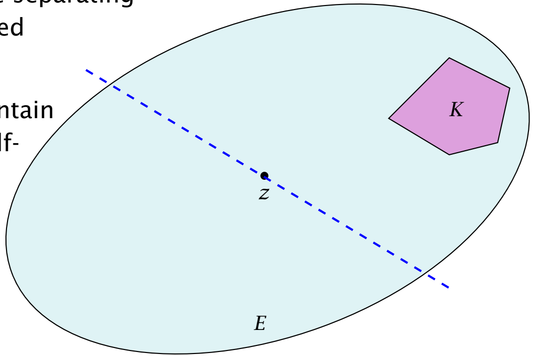

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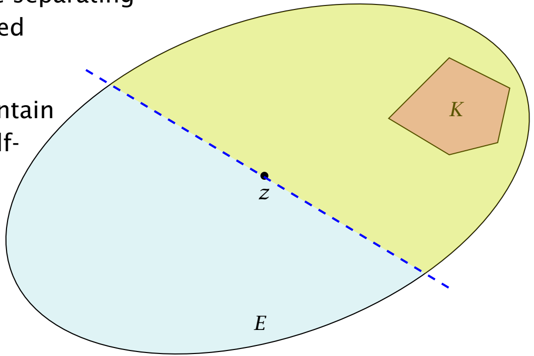

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Let $M_{\max} = n2^{2L'}$ be an upper bound on the objective value of a basic feasible solution.

We can add a constraint $c^T x \geq M_{\max} + 1$ and check for feasibility.

Ellipsoid Method

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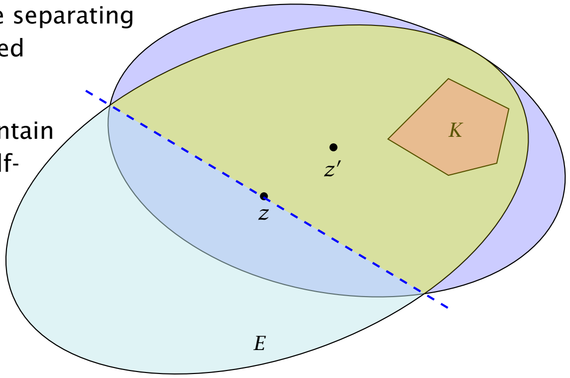

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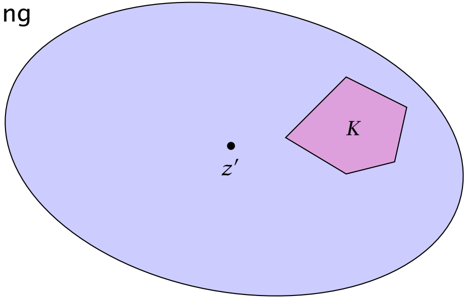

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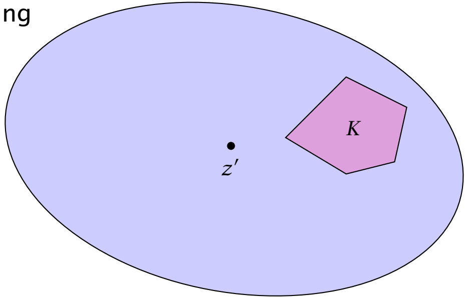

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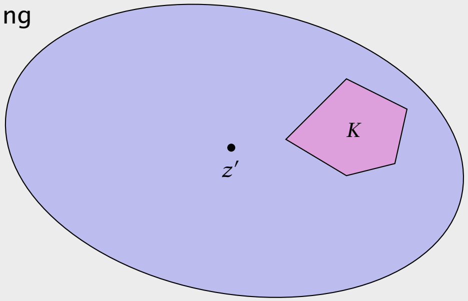

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A ball in $\mathbb{R}^n$ with center c and radius r is given by

$$\begin{aligned} B(c,r) &= \{x \mid (x-c)^T(x-c) \le r^2\} \\ &= \{x \mid \sum_i (x-c)_i^2/r^2 \le 1\} \end{aligned}$$

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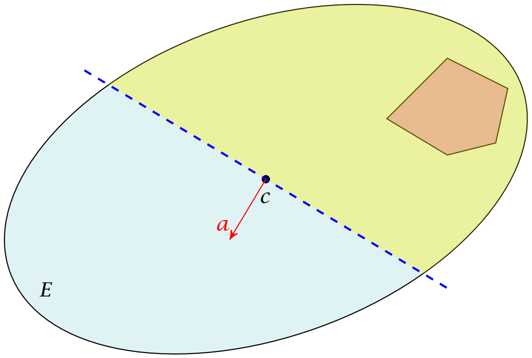

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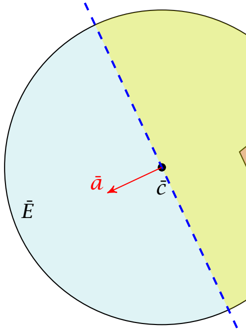

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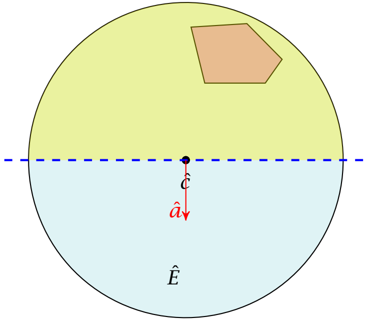

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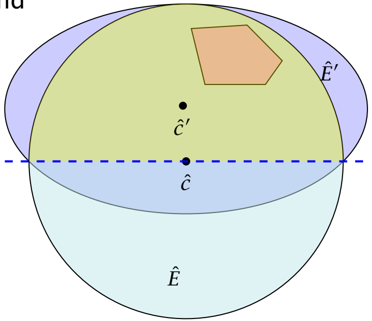

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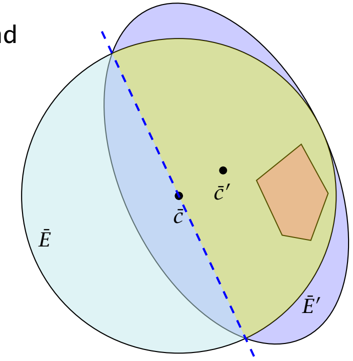

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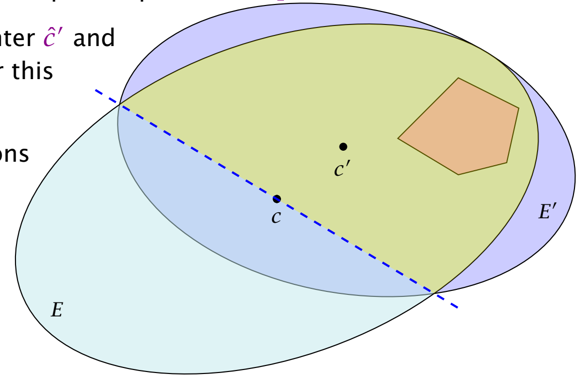

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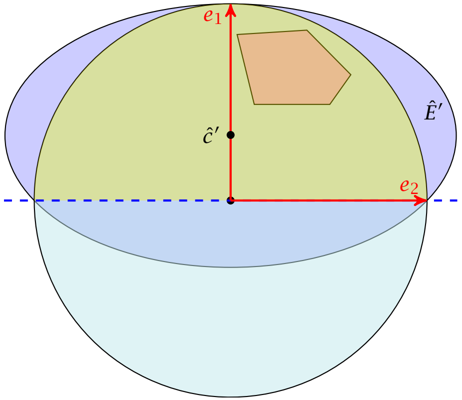

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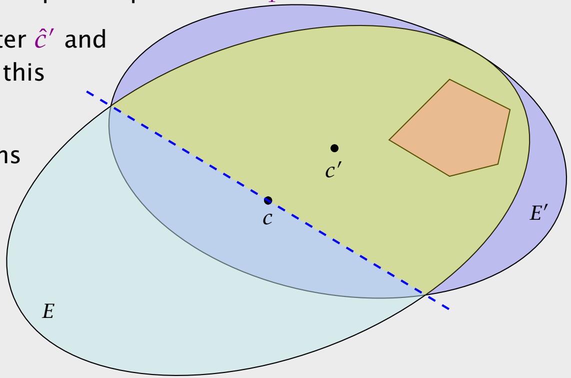

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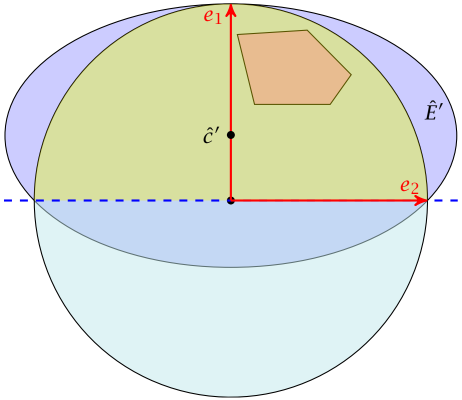

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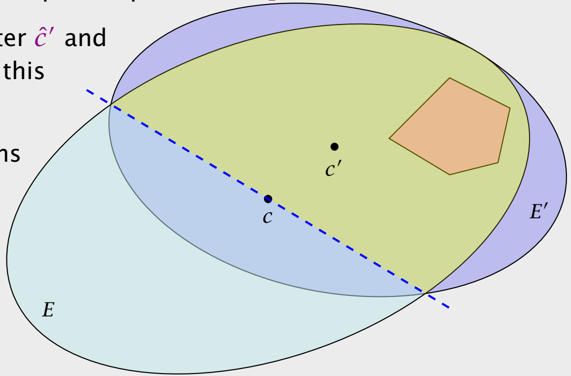

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- To obtain the matrix $\hat{Q}'^{-1}$ for our ellipsoid $\hat{E}'$ note that $\hat{E}'$ is axis-parallel.
- Let a denote the radius along the x_1-axis and let b denote the (common) radius for the other axes.
- The matrix

$$\hat{L}' = \begin{pmatrix} a & 0 & \dots & 0 \\ 0 & b & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & b \end{pmatrix}$$

maps the unit ball (via function $\hat{f}'(x) = \hat{L}'x$) to an axis-parallel ellipsoid with radius a in direction x_1 and b in all other directions.

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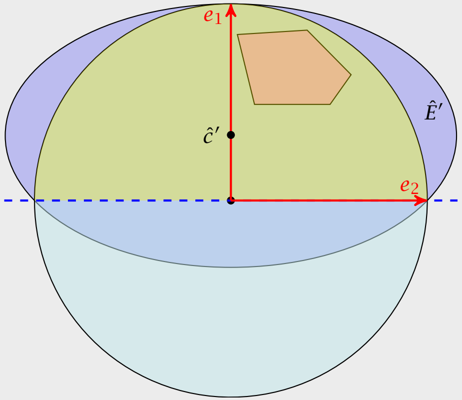

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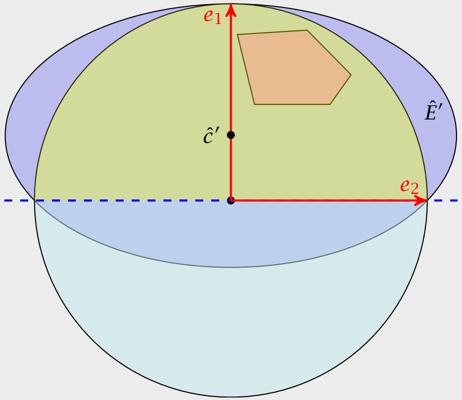

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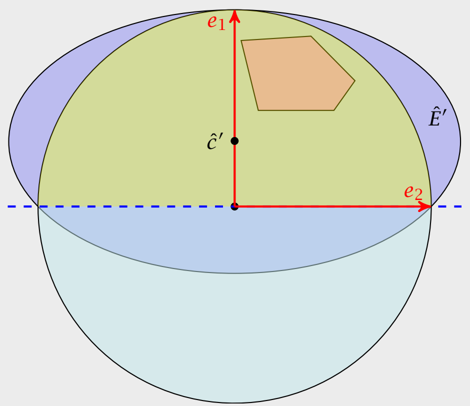

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- As $\hat{Q}' = \hat{L}'\hat{L}'^t$ the matrix $\hat{Q}'^{-1}$ is of the form

$$\hat{Q}'^{-1} = \begin{pmatrix} \frac{1}{a^2} & 0 & \dots & 0 \\ 0 & \frac{1}{b^2} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & \frac{1}{b^2} \end{pmatrix}$$

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- Let a denote the radius along the x_1-axis and let b denote the (common) radius for the other axes.
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$$\hat{L}' = \begin{pmatrix} a & 0 & \dots & 0 \\ 0 & b & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & b \end{pmatrix}$$

maps the unit ball (via function $\hat{f}'(x) = \hat{L}'x$) to an axis-parallel ellipsoid with radius a in direction x_1 and b in all other directions.

The Easy Case

- $(e_1 - \hat{c}')^T \hat{Q}'^{-1}(e_1 - \hat{c}') = 1$ gives

$$\begin{pmatrix} 1-t \\ 0 \\ \vdots \\ 0 \end{pmatrix}^T \cdot \begin{pmatrix} \frac{1}{a^2} & 0 & \dots & 0 \\ 0 & \frac{1}{b^2} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & \frac{1}{b^2} \end{pmatrix} \cdot \begin{pmatrix} 1-t \\ 0 \\ \vdots \\ 0 \end{pmatrix} = 1$$

- This gives $(1-t)^2 = a^2$.

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- As $\hat{Q}' = \hat{L}'\hat{L}'^t$ the matrix $\hat{Q}'^{-1}$ is of the form

$$\hat{Q}'^{-1} = \begin{pmatrix} \frac{1}{a^2} & 0 & \dots & 0 \\ 0 & \frac{1}{b^2} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & \frac{1}{b^2} \end{pmatrix}$$

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- For $i \neq 1$ the equation $(e_i - \hat{c}')^T \hat{Q}'^{-1}(e_i - \hat{c}') = 1$ looks like (here $i = 2$)

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$$\frac{1}{b^2} = 1 - \frac{t^2}{a^2}$$

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So far we have

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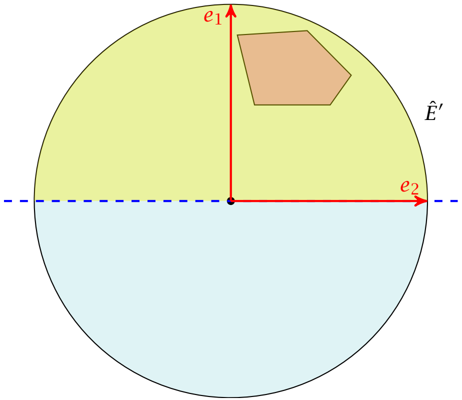
We still have many choices for t:



Choose t such that the volume of $\hat{E}'$ is minimal!!!

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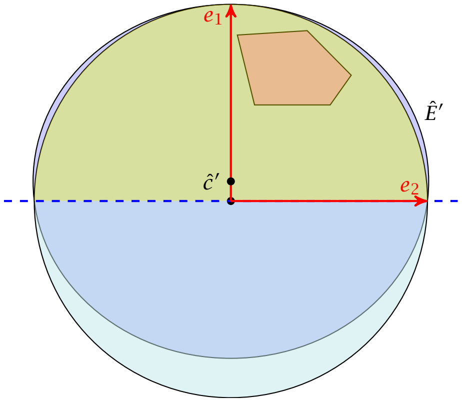

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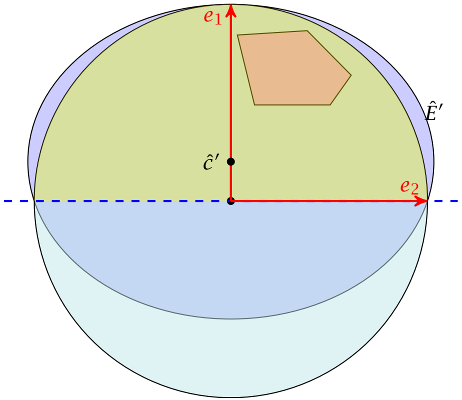

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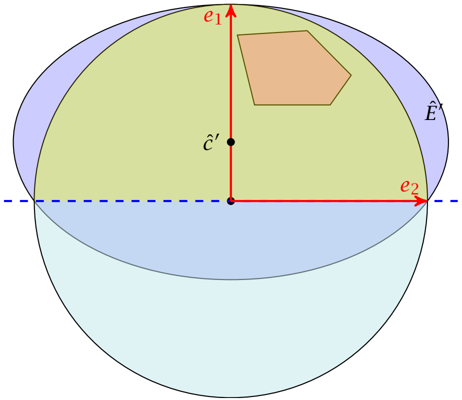

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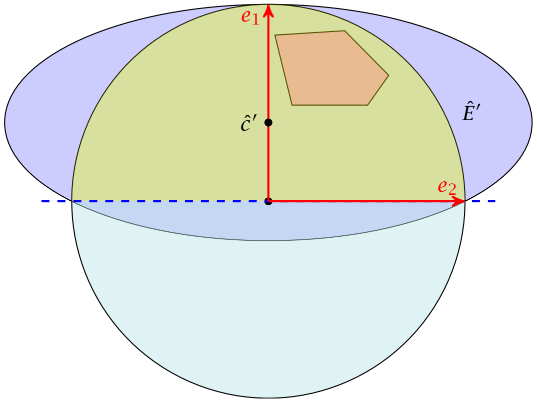

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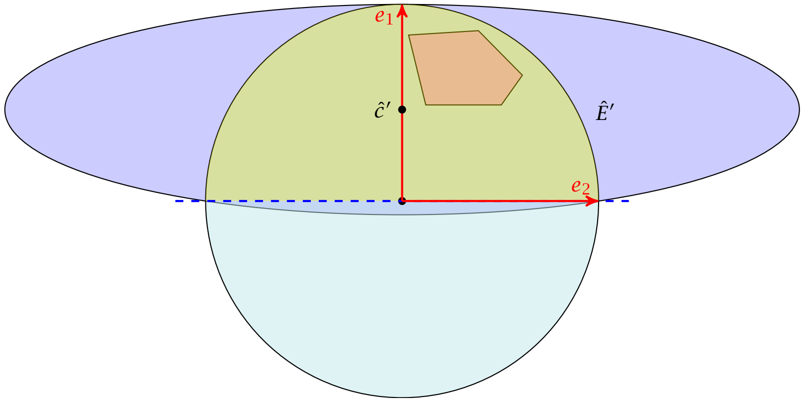

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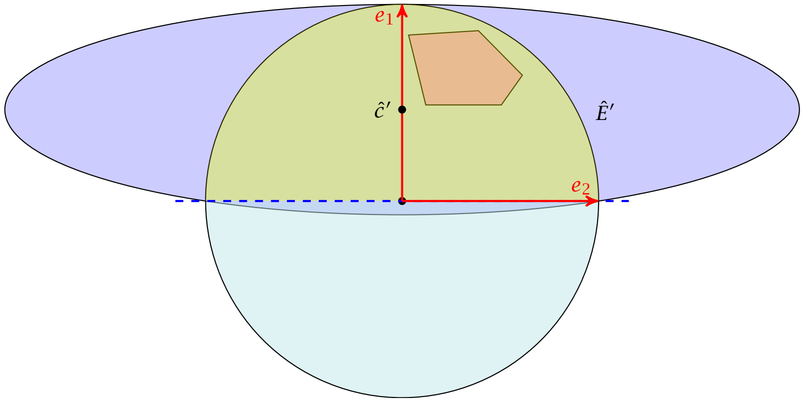

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We want to choose t such that the volume of $\hat{E}'$ is minimal.

Lemma 6

Let L be an affine transformation and $K \subseteq \mathbb{R}^n$. Then

$$\mathrm{vol}(L(K)) = |\det(L)| \cdot \mathrm{vol}(K) \ .$$

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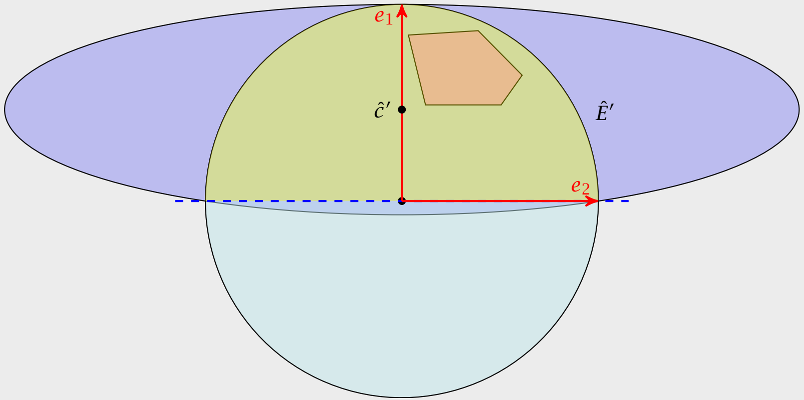

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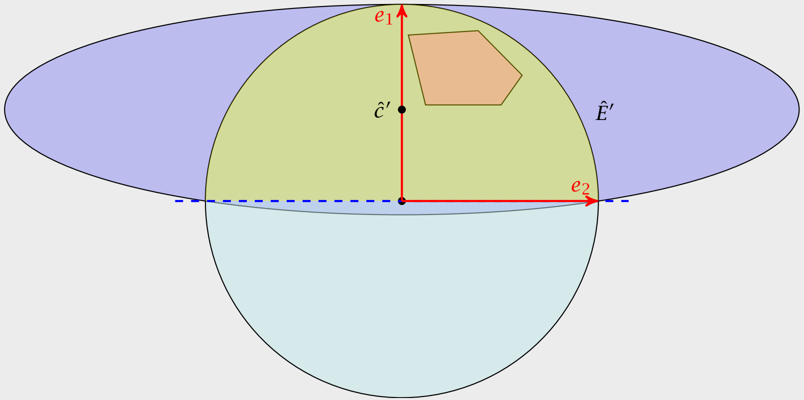

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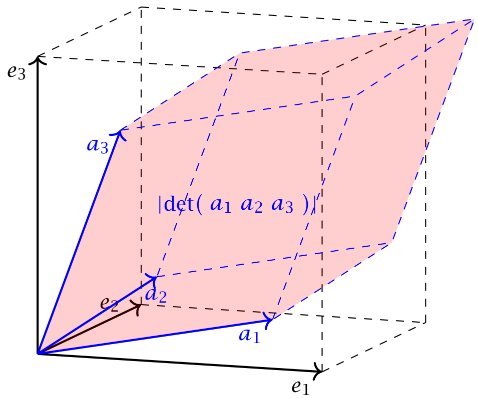
n-dimensional volume



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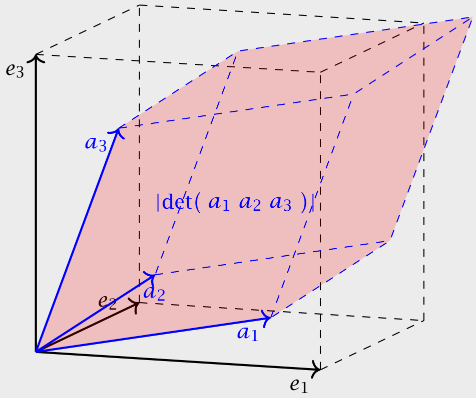

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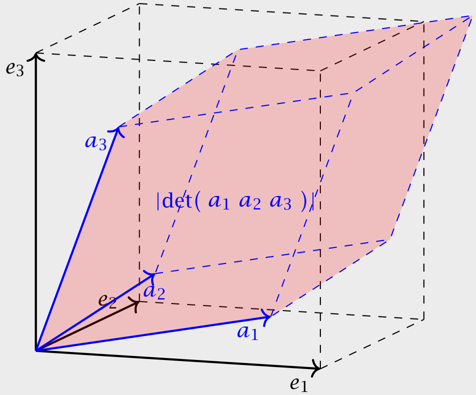

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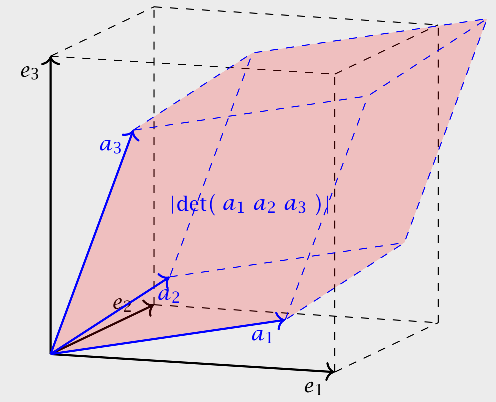

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N = denominator

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derivative of numerator

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$$-(n-1)(\sqrt{1-2t})^{n-2}$$

outer derivative

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&\quad -(n-1)(\sqrt{1-2t})^{n-2}\cdot\underbrace{\frac{1}{2\sqrt{1-2t}}\cdot(-2)}_{\text{inner derivative}}
\end{aligned}
$$

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The Easy Case

$$\frac{\mathrm{d}\,\mathrm{vol}(\hat{E}')}{\mathrm{d}\,t} = \frac{\mathrm{d}}{\mathrm{d}\,t}\left(\Phi\frac{(1-t)^n}{(\sqrt{1-2t})^{n-1}}\right)$$
$$= \frac{\Phi}{N^2}\cdot\Bigg((-1)\cdot n(1-t)^{n-1}\cdot(\sqrt{1-2t})^{n-1}$$
$$-(n-1)(\sqrt{1-2t})^{n-2}\cdot\frac{1}{2\sqrt{1-2t}}\cdot(-2)\cdot\underbrace{(1-t)^n}_{\text{numerator}}\Bigg)$$

The Easy Case

$$\begin{aligned}
\mathrm{vol}(\hat{E}') &= \mathrm{vol}(B(0,1))\cdot|\det(\hat{L}')|\\
&= \mathrm{vol}(B(0,1))\cdot ab^{n-1}\\
&= \mathrm{vol}(B(0,1))\cdot(1-t)\cdot\left(\frac{1-t}{\sqrt{1-2t}}\right)^{n-1}\\
&= \mathrm{vol}(B(0,1))\cdot\frac{(1-t)^n}{(\sqrt{1-2t})^{n-1}}
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The Easy Case

$$\begin{aligned}\mathrm{vol}(\hat{E}') &= \mathrm{vol}(B(0,1))\cdot|\det(\hat{L}')|\\ &= \mathrm{vol}(B(0,1))\cdot ab^{n-1}\\ &= \mathrm{vol}(B(0,1))\cdot(1-t)\cdot\left(\frac{1-t}{\sqrt{1-2t}}\right)^{n-1}\\ &= \mathrm{vol}(B(0,1))\cdot\frac{(1-t)^n}{(\sqrt{1-2t})^{n-1}}\end{aligned}$$

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&\qquad \cancel{-}(n-1)\cancel{(\sqrt{1-2t})^{n-2}}\cdot\frac{1}{\cancel{2}\cancel{\sqrt{1-2t}}}\cdot\cancel{(-2)}\cdot\overset{1-t}{\cancel{(1-t)^n}}\Bigg)\\
&= \frac{\Phi}{N^2}\cdot(\sqrt{1-2t})^{n-3}\cdot(1-t)^{n-1}\\
&\qquad\cdot\Big((n-1)(1-t)-n(1-2t)\Big)
\end{aligned}
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&= \frac{\Phi}{N^2}\cdot\Bigg((-1)\cdot n\cancel{(1-t)^{n-1}}\cdot\overset{1-2t}{\cancel{(\sqrt{1-2t})^{n-1}}}\\
&\qquad \cancel{2}(n-1)\cancel{(\sqrt{1-2t})^{n-2}}\cdot\frac{1}{\cancel{2}\cancel{\sqrt{1-2t}}}\cdot(\cancel{-2})\cdot\overset{1-t}{\cancel{(1-t)^n}}\Bigg)\\
&= \frac{\Phi}{N^2}\cdot(\sqrt{1-2t})^{n-3}\cdot(1-t)^{n-1}\\
&\qquad\cdot\Big((n-1)(1-t)-n(1-2t)\Big)\\
&= \frac{\Phi}{N^2}\cdot(\sqrt{1-2t})^{n-3}\cdot(1-t)^{n-1}\cdot\Big((n+1)t-1\Big)
\end{aligned}
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\end{aligned}
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We use the shortcut $\Phi := \mathrm{vol}(B(0,1))$.

The Easy Case

- We obtain the minimum for $t = \frac{1}{n+1}$.
- For this value we obtain

$$a$$

The Easy Case

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\begin{aligned}
\frac{\mathrm{d}\,\mathrm{vol}(\hat{E}')}{\mathrm{d}t} &= \frac{\mathrm{d}}{\mathrm{d}t}\left(\Phi\frac{(1-t)^n}{(\sqrt{1-2t})^{n-1}}\right) \\
&= \frac{\Phi}{N^2}\cdot\Bigg((-1)\cdot n\cancel{(1-t)^{n-1}}\cdot \overset{1-2t}{\cancel{(\sqrt{1-2t})^{n-1}}} \\
&\qquad \cancel{-}(n-1)\cancel{(\sqrt{1-2t})^{n-2}}\cdot\frac{1}{\cancel{2}\cancel{\sqrt{1-2t}}}\cdot\cancel{(-2)}\cdot\overset{1-t}{\cancel{(1-t)^n}}\Bigg) \\
&= \frac{\Phi}{N^2}\cdot(\sqrt{1-2t})^{n-3}\cdot(1-t)^{n-1} \\
&\qquad\cdot\Big((n-1)(1-t)-n(1-2t)\Big) \\
&= \frac{\Phi}{N^2}\cdot(\sqrt{1-2t})^{n-3}\cdot(1-t)^{n-1}\cdot\Big((n+1)t-1\Big)
\end{aligned}
$$

The Easy Case

- We obtain the minimum for $t = \frac{1}{n+1}$.
- For this value we obtain

$$a = 1 - t$$

The Easy Case

$$\begin{aligned}
\frac{\mathrm{d}\,\mathrm{vol}(\hat{E}')}{\mathrm{d}t} &= \frac{\mathrm{d}}{\mathrm{d}t}\left(\Phi\frac{(1-t)^n}{(\sqrt{1-2t})^{n-1}}\right)\\
&= \frac{\Phi}{N^2}\cdot\Bigg((-1)\cdot n\cancel{(1-t)^{n-1}}\cdot\overset{1-2t}{\cancel{(\sqrt{1-2t})^{n-1}}}\\
&\qquad\cancel{-}(n-1)\cancel{(\sqrt{1-2t})^{n-2}}\cdot\frac{1}{\cancel{2}\cancel{\sqrt{1-2t}}}\cdot(\cancel{-2})\cdot\overset{1-t}{\cancel{(1-t)^n}}\Bigg)\\
&= \frac{\Phi}{N^2}\cdot(\sqrt{1-2t})^{n-3}\cdot(1-t)^{n-1}\\
&\qquad\cdot\Big((n-1)(1-t)-n(1-2t)\Big)\\
&= \frac{\Phi}{N^2}\cdot(\sqrt{1-2t})^{n-3}\cdot(1-t)^{n-1}\cdot\Big((n+1)t-1\Big)
\end{aligned}$$

The Easy Case

- We obtain the minimum for $t = \frac{1}{n+1}$.
- For this value we obtain

$$a = 1 - t = \frac{n}{n+1}$$

The Easy Case

$$\begin{aligned}
\frac{\mathrm{d}\,\mathrm{vol}(\hat{E}')}{\mathrm{d}t} &= \frac{\mathrm{d}}{\mathrm{d}t}\left(\Phi\frac{(1-t)^n}{(\sqrt{1-2t})^{n-1}}\right)\\
&= \frac{\Phi}{N^2}\cdot\Bigg((-1)\cdot n\cancel{(1-t)^{n-1}}\cdot \overset{1-2t}{\cancel{(\sqrt{1-2t})^{n-1}}}\\
&\quad \cancel{-}(n-1)\cancel{(\sqrt{1-2t})^{n-2}}\cdot\frac{1}{\cancel{2}\cancel{\sqrt{1-2t}}}\cdot(\cancel{-2})\cdot \overset{1-t}{\cancel{(1-t)^n}}\Bigg)\\
&= \frac{\Phi}{N^2}\cdot(\sqrt{1-2t})^{n-3}\cdot(1-t)^{n-1}\\
&\qquad\cdot\Big((n-1)(1-t)-n(1-2t)\Big)\\
&= \frac{\Phi}{N^2}\cdot(\sqrt{1-2t})^{n-3}\cdot(1-t)^{n-1}\cdot\Big((n+1)t-1\Big)
\end{aligned}$$

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- We obtain the minimum for $t = \frac{1}{n+1}$.
- For this value we obtain

$$a = 1 - t = \frac{n}{n+1} \text{ and } b =$$

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&= \frac{\Phi}{N^2}\cdot(\sqrt{1-2t})^{n-3}\cdot(1-t)^{n-1} \\
&\qquad \cdot\Big((n-1)(1-t)-n(1-2t)\Big) \\
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\end{aligned}$$

The Easy Case

- We obtain the minimum for $t = \frac{1}{n+1}$.
- For this value we obtain

$$a = 1 - t = \frac{n}{n+1} \text{ and } b = \frac{1-t}{\sqrt{1-2t}}$$

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&= \frac{\Phi}{N^2}\cdot(\sqrt{1-2t})^{n-3}\cdot(1-t)^{n-1} \\
&\qquad \cdot\Big((n-1)(1-t) - n(1-2t)\Big) \\
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To see the equation for b, observe that

$$b^2$$

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&= \frac{\Phi}{N^2}\cdot(\sqrt{1-2t})^{n-3}\cdot(1-t)^{n-1}\\
&\qquad\cdot\Big((n-1)(1-t)-n(1-2t)\Big)\\
&= \frac{\Phi}{N^2}\cdot(\sqrt{1-2t})^{n-3}\cdot(1-t)^{n-1}\cdot\Big((n+1)t-1\Big)
\end{aligned}$$

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- We obtain the minimum for $t = \frac{1}{n+1}$.
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$$b^2 = \frac{(1-t)^2}{1-2t}$$

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&\quad \cancel{-}(n-1)\cancel{(\sqrt{1-2t})^{n-2}}\cdot\frac{1}{\cancel{2}\cancel{\sqrt{1-2t}}}\cdot\cancel{(-2)}\cdot\cancel{(1-t)^n}^{\,1-t}\Bigg) \\
&= \frac{\Phi}{N^2}\cdot(\sqrt{1-2t})^{n-3}\cdot(1-t)^{n-1} \\
&\qquad \cdot\Big((n-1)(1-t)-n(1-2t)\Big) \\
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$$b^2 = \frac{(1-t)^2}{1-2t} = \frac{(1-\frac{1}{n+1})^2}{1-\frac{2}{n+1}}$$

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&\qquad \cancel{-}(n-1)\cancel{(\sqrt{1-2t})^{n-2}}\cdot\frac{1}{\cancel{2}\cancel{\sqrt{1-2t}}}\cdot\cancel{(-2)}\cdot\cancel{(1-t)^n}^{\,1-t}\Bigg) \\
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\end{aligned}$$

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To see the equation for b, observe that

$$b^2 = \frac{(1-t)^2}{1-2t} = \frac{(1-\frac{1}{n+1})^2}{1-\frac{2}{n+1}} = \frac{(\frac{n}{n+1})^2}{\frac{n-1}{n+1}}$$

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&= \frac{\Phi}{N^2}\cdot\Bigg((-1)\cdot n\overbrace{\cancel{(1-t)^{n-1}}}\cdot\overbrace{\cancel{(\sqrt{1-2t})^{n-1}}}^{1-2t}\\
&\qquad \cancel{-}(n-1)\cancel{(\sqrt{1-2t})^{n-2}}\cdot\frac{1}{\cancel{2}\cancel{\sqrt{1-2t}}}\cdot\cancel{(-2)}\cdot\overbrace{\cancel{(1-t)^{n}}}^{1-t}\Bigg)\\
&= \frac{\Phi}{N^2}\cdot(\sqrt{1-2t})^{n-3}\cdot(1-t)^{n-1}\\
&\qquad\cdot\Big((n-1)(1-t)-n(1-2t)\Big)\\
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\end{aligned}$$

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- We obtain the minimum for $t = \frac{1}{n+1}$.
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To see the equation for b, observe that

$$b^2 = \frac{(1-t)^2}{1-2t} = \frac{(1-\frac{1}{n+1})^2}{1-\frac{2}{n+1}} = \frac{(\frac{n}{n+1})^2}{\frac{n-1}{n+1}} = \frac{n^2}{n^2-1}$$

The Easy Case

$$\begin{aligned}
\frac{\mathrm{d}\,\mathrm{vol}(\hat{E}')}{\mathrm{d}t} &= \frac{\mathrm{d}}{\mathrm{d}t}\left(\Phi\frac{(1-t)^n}{(\sqrt{1-2t})^{n-1}}\right)\\
&= \frac{\Phi}{N^2}\cdot\Bigg((-1)\cdot n\cancel{(1-t)^{n-1}}\cdot \overset{1-2t}{\cancel{(\sqrt{1-2t})^{n-1}}}\\
&\quad \cancel{+}(n-1)\cancel{(\sqrt{1-2t})^{n-2}}\cdot\frac{1}{\cancel{2}\cancel{\sqrt{1-2t}}}\cdot\cancel{(-2)}\cdot\overset{1-t}{\cancel{(1-t)^n}}\Bigg)\\
&= \frac{\Phi}{N^2}\cdot(\sqrt{1-2t})^{n-3}\cdot(1-t)^{n-1}\\
&\qquad\cdot\Big((n-1)(1-t)-n(1-2t)\Big)\\
&= \frac{\Phi}{N^2}\cdot(\sqrt{1-2t})^{n-3}\cdot(1-t)^{n-1}\cdot\Big((n+1)t-1\Big)
\end{aligned}$$

The Easy Case

Let $\gamma_n = \frac{\mathrm{vol}(\hat{E}')}{\mathrm{vol}(B(0,1))} = ab^{n-1}$ be the ratio by which the volume changes:

$$\gamma_n^2$$

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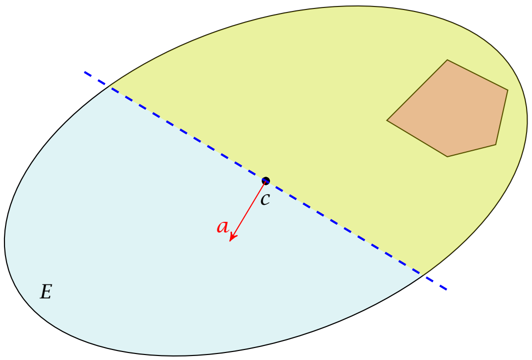
How to Compute the New Ellipsoid



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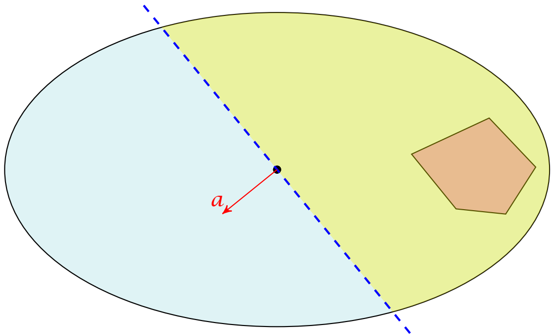

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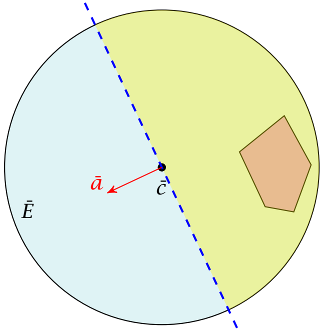

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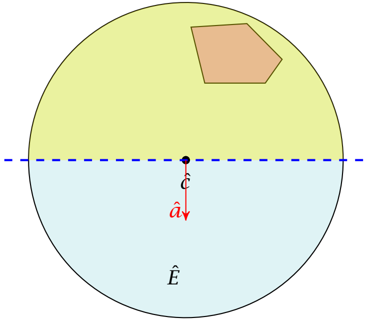

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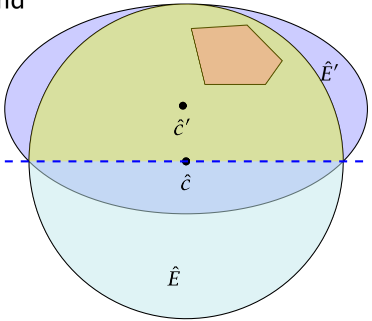

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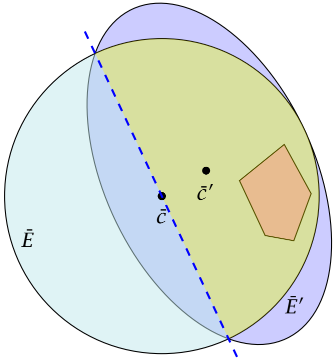

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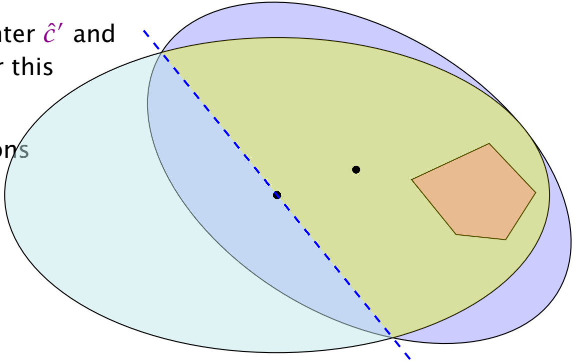

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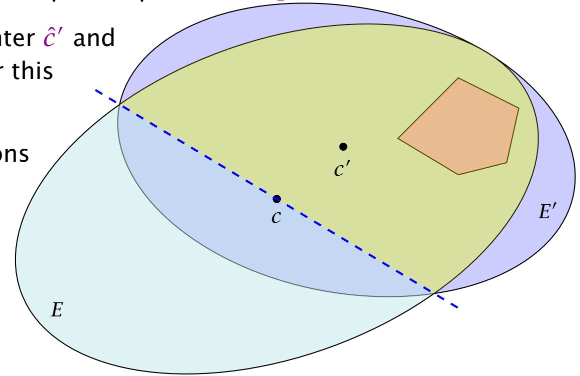

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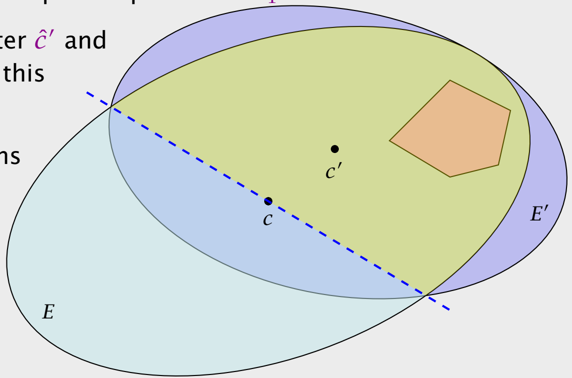

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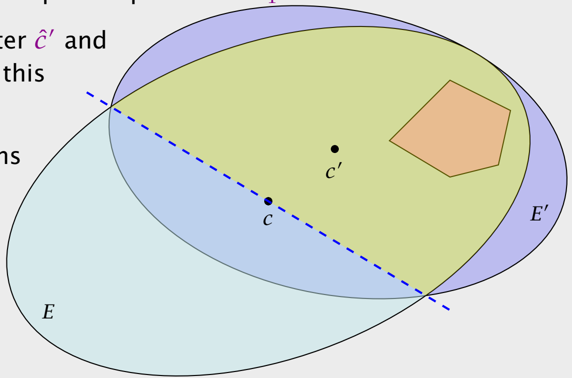

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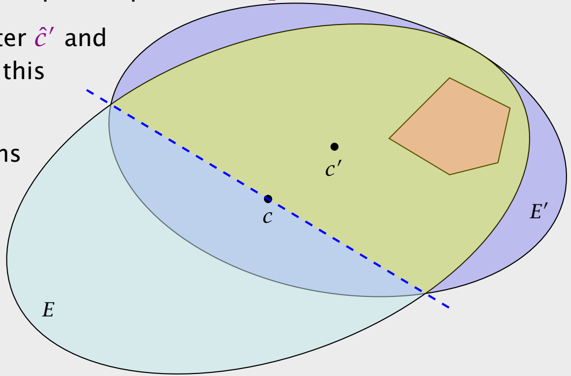

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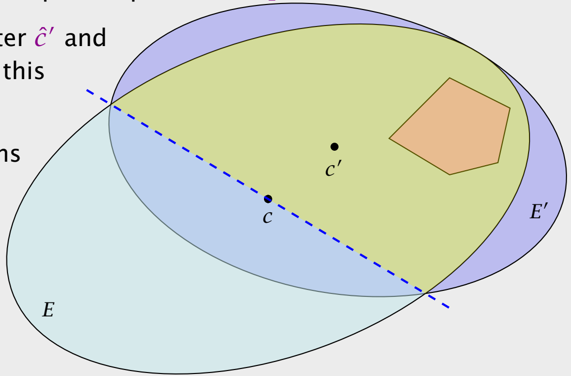

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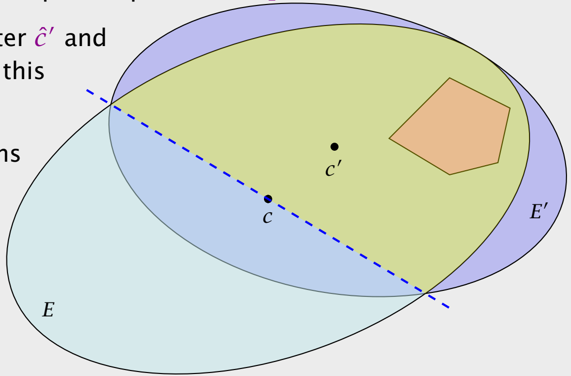

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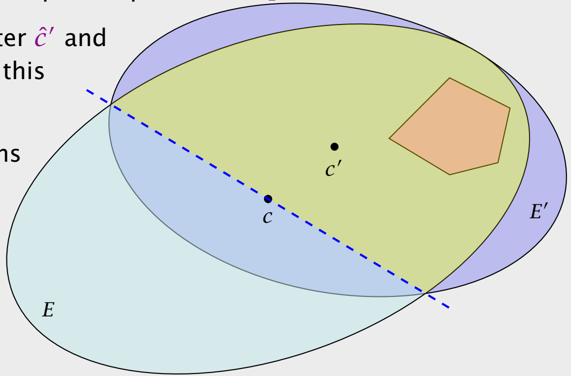
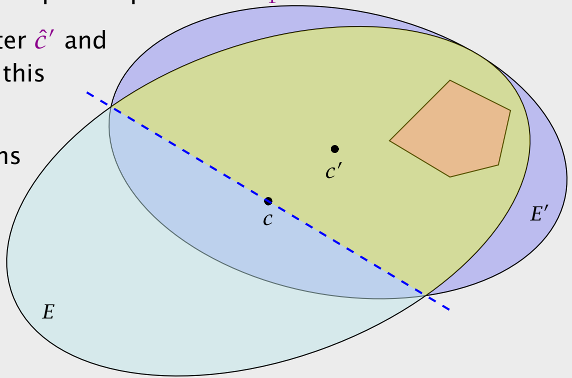


Our progress is the same:

$$e^{-\frac{1}{2(n+1)}} \geq \frac{\mathrm{vol}(\hat{E}')}{\mathrm{vol}(B(0,1))} = \frac{\mathrm{vol}(\hat{E}')}{\mathrm{vol}(\hat{E})} = \frac{\mathrm{vol}(R(\hat{E}'))}{\mathrm{vol}(R(\hat{E}))}$$
$$= \frac{\mathrm{vol}(\bar{E}')}{\mathrm{vol}(\bar{E})} = \frac{\mathrm{vol}(f(\bar{E}'))}{\mathrm{vol}(f(\bar{E}))} = \frac{\mathrm{vol}(E')}{\mathrm{vol}(E)}$$

How to Compute the New Ellipsoid

- Use f^{-1} (recall that $f = Lx + t$ is the affine transformation of the unit ball) to rotate/distort the ellipsoid (back) into the unit ball.
- Use a rotation R^{-1} to rotate the unit ball such that the normal vector of the halfspace is parallel to e_1.
- Compute the new center $\hat{c}'$ and the new matrix $\hat{Q}'$ for this simplified setting.
- Use the transformations R and f to get the new center c' and the new matrix Q' for the original ellipsoid E.



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Here it is important that mapping a set with affine function $f(x) = Lx + t$ changes the volume by factor $\det(L)$.

How to Compute the New Ellipsoid

- Use f^{-1} (recall that $f = Lx + t$ is the affine transformation of the unit ball) to rotate/distort the ellipsoid (back) into the unit ball.
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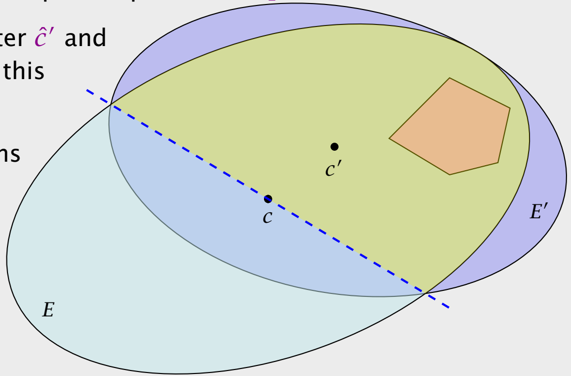

The Ellipsoid Algorithm

How to Compute The New Parameters?

Our progress is the same:

$$e^{-\frac{1}{2(n+1)}} \geq \frac{\mathrm{vol}(\hat{E}')}{\mathrm{vol}(B(0,1))} = \frac{\mathrm{vol}(\hat{E}')}{\mathrm{vol}(\hat{E})} = \frac{\mathrm{vol}(R(\hat{E}'))}{\mathrm{vol}(R(\hat{E}))}$$
$$= \frac{\mathrm{vol}(\bar{E}')}{\mathrm{vol}(\bar{E})} = \frac{\mathrm{vol}(f(\bar{E}'))}{\mathrm{vol}(f(\bar{E}))} = \frac{\mathrm{vol}(E')}{\mathrm{vol}(E)}$$

Here it is important that mapping a set with affine function $f(x) = Lx + t$ changes the volume by factor $\det(L)$.

The Ellipsoid Algorithm

How to Compute The New Parameters?

The transformation function of the (old) ellipsoid: $f(x) = Lx + c$;

Our progress is the same:

$$\begin{aligned} e^{-\frac{1}{2(n+1)}} &\geq \frac{\mathrm{vol}(\hat{E}')}{\mathrm{vol}(B(0,1))} = \frac{\mathrm{vol}(\hat{E}')}{\mathrm{vol}(\hat{E})} = \frac{\mathrm{vol}(R(\hat{E}'))}{\mathrm{vol}(R(\hat{E}))} \\ &= \frac{\mathrm{vol}(\bar{E}')}{\mathrm{vol}(\bar{E})} = \frac{\mathrm{vol}(f(\bar{E}'))}{\mathrm{vol}(f(\bar{E}))} = \frac{\mathrm{vol}(E')}{\mathrm{vol}(E)} \end{aligned}$$

Here it is important that mapping a set with affine function $f(x) = Lx + t$ changes the volume by factor $\det(L)$.

The Ellipsoid Algorithm

How to Compute The New Parameters?

The transformation function of the (old) ellipsoid: $f(x) = Lx + c$;

The halfspace to be intersected: $H = \{x \mid a^T(x - c) \leq 0\}$;

Our progress is the same:

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The Ellipsoid Algorithm

How to Compute The New Parameters?

The transformation function of the (old) ellipsoid: $f(x) = Lx + c$;

The halfspace to be intersected: $H = \{x \mid a^T(x - c) \le 0\}$;

$$f^{-1}(H) = \{f^{-1}(x) \mid a^T(x - c) \le 0\}$$

Our progress is the same:

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The Ellipsoid Algorithm

How to Compute The New Parameters?

The transformation function of the (old) ellipsoid: $f(x) = Lx + c$;

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The Ellipsoid Algorithm

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The Ellipsoid Algorithm

How to Compute The New Parameters?

The transformation function of the (old) ellipsoid: $f(x) = Lx + c$;

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Our progress is the same:

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Here it is important that mapping a set with affine function $f(x) = Lx + t$ changes the volume by factor $\det(L)$.

The Ellipsoid Algorithm

How to Compute The New Parameters?

The transformation function of the (old) ellipsoid: $f(x) = Lx + c$;

The halfspace to be intersected: $H = \{x \mid a^T(x - c) \le 0\}$;

$$\begin{aligned} f^{-1}(H) &= \{f^{-1}(x) \mid a^T(x-c) \le 0\} \\ &= \{f^{-1}(f(y)) \mid a^T(f(y)-c) \le 0\} \\ &= \{y \mid a^T(f(y)-c) \le 0\} \\ &= \{y \mid a^T(Ly+c-c) \le 0\} \\ &= \{y \mid (a^TL)y \le 0\} \end{aligned}$$

Our progress is the same:

$$\begin{aligned} e^{-\frac{1}{2(n+1)}} \ge \frac{\mathrm{vol}(\hat{E}')}{\mathrm{vol}(B(0,1))} &= \frac{\mathrm{vol}(\hat{E}')}{\mathrm{vol}(\hat{E})} = \frac{\mathrm{vol}(R(\hat{E}'))}{\mathrm{vol}(R(\hat{E}))} \\ &= \frac{\mathrm{vol}(\bar{E}')}{\mathrm{vol}(\bar{E})} = \frac{\mathrm{vol}(f(\bar{E}'))}{\mathrm{vol}(f(\bar{E}))} = \frac{\mathrm{vol}(E')}{\mathrm{vol}(E)} \end{aligned}$$

Here it is important that mapping a set with affine function $f(x) = Lx + t$ changes the volume by factor $\det(L)$.

The Ellipsoid Algorithm

How to Compute The New Parameters?

The transformation function of the (old) ellipsoid: $f(x) = Lx + c$;

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$$
\begin{aligned}
f^{-1}(H) &= \{f^{-1}(x) \mid a^T(x - c) \le 0\} \\
&= \{f^{-1}(f(y)) \mid a^T(f(y) - c) \le 0\} \\
&= \{y \mid a^T(f(y) - c) \le 0\} \\
&= \{y \mid a^T(Ly + c - c) \le 0\} \\
&= \{y \mid (a^T L)y \le 0\}
\end{aligned}
$$

This means $\bar{a} = L^T a$.

Our progress is the same:

$$
\begin{aligned}
e^{-\frac{1}{2(n+1)}} \ge \frac{\mathrm{vol}(\hat{E}')}{\mathrm{vol}(B(0,1))} &= \frac{\mathrm{vol}(\hat{E}')}{\mathrm{vol}(\hat{E})} = \frac{\mathrm{vol}(R(\hat{E}'))}{\mathrm{vol}(R(\hat{E}))} \\
&= \frac{\mathrm{vol}(\bar{E}')}{\mathrm{vol}(\bar{E})} = \frac{\mathrm{vol}(f(\bar{E}'))}{\mathrm{vol}(f(\bar{E}))} = \frac{\mathrm{vol}(E')}{\mathrm{vol}(E)}
\end{aligned}
$$

Here it is important that mapping a set with affine function $f(x) = Lx + t$ changes the volume by factor $\det(L)$.

The Ellipsoid Algorithm

After rotating back (applying R^{-1}) the normal vector of the halfspace points in negative x_1-direction. Hence,

$$R^{-1}\Big(\frac{L^Ta}{\|L^Ta\|}\Big) = -e_1 \quad \Rightarrow \quad -\frac{L^Ta}{\|L^Ta\|} = R\cdot e_1$$

The Ellipsoid Algorithm

How to Compute The New Parameters?

The transformation function of the (old) ellipsoid: $f(x) = Lx + c$;

The halfspace to be intersected: $H = \{x \mid a^T(x-c) \le 0\}$;

$$\begin{aligned} f^{-1}(H) &= \{f^{-1}(x) \mid a^T(x-c) \le 0\} \\ &= \{f^{-1}(f(y)) \mid a^T(f(y)-c) \le 0\} \\ &= \{y \mid a^T(f(y)-c) \le 0\} \\ &= \{y \mid a^T(Ly+c-c) \le 0\} \\ &= \{y \mid (a^TL)y \le 0\} \end{aligned}$$

This means $\bar{a} = L^Ta$.

The Ellipsoid Algorithm

After rotating back (applying R^{-1}) the normal vector of the halfspace points in negative x_1-direction. Hence,

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Hence,

$$\bar{c}'$$

The Ellipsoid Algorithm

How to Compute The New Parameters?

The transformation function of the (old) ellipsoid: $f(x) = Lx + c$;

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This means $\bar{a} = L^Ta$.

The Ellipsoid Algorithm

After rotating back (applying R^{-1}) the normal vector of the halfspace points in negative x_1-direction. Hence,

$$R^{-1}\left(\frac{L^T a}{\|L^T a\|}\right) = -e_1 \quad \Rightarrow \quad -\frac{L^T a}{\|L^T a\|} = R \cdot e_1$$

Hence,

$$\bar{c}' = R \cdot \hat{c}'$$

The Ellipsoid Algorithm

How to Compute The New Parameters?

The transformation function of the (old) ellipsoid: $f(x) = Lx + c$;

The halfspace to be intersected: $H = \{x \mid a^T(x - c) \le 0\}$;

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This means $\bar{a} = L^T a$.

The Ellipsoid Algorithm

After rotating back (applying R^{-1}) the normal vector of the halfspace points in negative x_1-direction. Hence,

$$R^{-1}\left(\frac{L^T a}{\|L^T a\|}\right) = -e_1 \quad \Rightarrow \quad -\frac{L^T a}{\|L^T a\|} = R \cdot e_1$$

Hence,

$$\bar{c}' = R \cdot \hat{c}' = R \cdot \frac{1}{n+1} e_1$$

The Ellipsoid Algorithm

How to Compute The New Parameters?

The transformation function of the (old) ellipsoid: $f(x) = Lx + c$;

The halfspace to be intersected: $H = \{x \mid a^T(x - c) \le 0\}$;

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&= \{y \mid a^T(f(y) - c) \le 0\} \\
&= \{y \mid a^T(Ly + c - c) \le 0\} \\
&= \{y \mid (a^T L)y \le 0\}
\end{aligned}$$

This means $\bar{a} = L^T a$.

The Ellipsoid Algorithm

After rotating back (applying R^{-1}) the normal vector of the halfspace points in negative x_1-direction. Hence,

$$R^{-1}\left(\frac{L^Ta}{\|L^Ta\|}\right) = -e_1 \quad \Rightarrow \quad -\frac{L^Ta}{\|L^Ta\|} = R \cdot e_1$$

Hence,

$$\bar{c}' = R \cdot \hat{c}' = R \cdot \frac{1}{n+1} e_1 = -\frac{1}{n+1} \frac{L^Ta}{\|L^Ta\|}$$

The Ellipsoid Algorithm

How to Compute The New Parameters?

The transformation function of the (old) ellipsoid: $f(x) = Lx + c$;

The halfspace to be intersected: $H = \{x \mid a^T(x - c) \le 0\}$;

$$\begin{aligned} f^{-1}(H) &= \{f^{-1}(x) \mid a^T(x-c) \le 0\} \\ &= \{f^{-1}(f(y)) \mid a^T(f(y)-c) \le 0\} \\ &= \{y \mid a^T(f(y)-c) \le 0\} \\ &= \{y \mid a^T(Ly+c-c) \le 0\} \\ &= \{y \mid (a^TL)y \le 0\} \end{aligned}$$

This means $\bar{a} = L^Ta$.

The Ellipsoid Algorithm

After rotating back (applying R^{-1}) the normal vector of the halfspace points in negative x_1-direction. Hence,

$$R^{-1}\Big(\frac{L^T a}{\|L^T a\|}\Big) = -e_1 \quad \Rightarrow \quad -\frac{L^T a}{\|L^T a\|} = R \cdot e_1$$

Hence,

$$\bar{c}' = R \cdot \hat{c}' = R \cdot \frac{1}{n+1} e_1 = -\frac{1}{n+1} \frac{L^T a}{\|L^T a\|}$$

$$c'$$

The Ellipsoid Algorithm

How to Compute The New Parameters?

The transformation function of the (old) ellipsoid: $f(x) = Lx + c$;

The halfspace to be intersected: $H = \{x \mid a^T(x - c) \leq 0\}$;

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f^{-1}(H) &= \{f^{-1}(x) \mid a^T(x-c) \leq 0\} \\
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&= \{y \mid a^T(f(y)-c) \leq 0\} \\
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&= \{y \mid (a^T L)y \leq 0\}
\end{aligned}$$

This means $\bar{a} = L^T a$.

The Ellipsoid Algorithm

After rotating back (applying R^{-1}) the normal vector of the halfspace points in negative x_1-direction. Hence,

$$R^{-1}\left(\frac{L^T a}{\|L^T a\|}\right) = -e_1 \quad \Rightarrow \quad -\frac{L^T a}{\|L^T a\|} = R \cdot e_1$$

Hence,

$$\bar{c}' = R \cdot \hat{c}' = R \cdot \frac{1}{n+1} e_1 = -\frac{1}{n+1} \frac{L^T a}{\|L^T a\|}$$

$$c' = f(\bar{c}')$$

The Ellipsoid Algorithm

How to Compute The New Parameters?

The transformation function of the (old) ellipsoid: $f(x) = Lx + c$;

The halfspace to be intersected: $H = \{x \mid a^T(x-c) \le 0\}$;

$$\begin{aligned} f^{-1}(H) &= \{f^{-1}(x) \mid a^T(x-c) \le 0\} \\ &= \{f^{-1}(f(y)) \mid a^T(f(y)-c) \le 0\} \\ &= \{y \mid a^T(f(y)-c) \le 0\} \\ &= \{y \mid a^T(Ly+c-c) \le 0\} \\ &= \{y \mid (a^T L)y \le 0\} \end{aligned}$$

This means $\bar{a} = L^T a$.

The Ellipsoid Algorithm

After rotating back (applying R^{-1}) the normal vector of the halfspace points in negative x_1-direction. Hence,

$$R^{-1}\Big(\frac{L^T a}{\|L^T a\|}\Big) = -e_1 \quad \Rightarrow \quad -\frac{L^T a}{\|L^T a\|} = R \cdot e_1$$

Hence,

$$\bar{c}' = R \cdot \hat{c}' = R \cdot \frac{1}{n+1} e_1 = -\frac{1}{n+1} \frac{L^T a}{\|L^T a\|}$$

$$c' = f(\bar{c}') = L \cdot \bar{c}' + c$$

The Ellipsoid Algorithm

How to Compute The New Parameters?

The transformation function of the (old) ellipsoid: $f(x) = Lx + c$;

The halfspace to be intersected: $H = \{x \mid a^T(x - c) \le 0\}$;

$$\begin{aligned} f^{-1}(H) &= \{f^{-1}(x) \mid a^T(x-c) \le 0\} \\ &= \{f^{-1}(f(y)) \mid a^T(f(y)-c) \le 0\} \\ &= \{y \mid a^T(f(y)-c) \le 0\} \\ &= \{y \mid a^T(Ly+c-c) \le 0\} \\ &= \{y \mid (a^T L)y \le 0\} \end{aligned}$$

This means $\bar{a} = L^T a$.

The Ellipsoid Algorithm

After rotating back (applying R^{-1}) the normal vector of the halfspace points in negative x_1-direction. Hence,

$$R^{-1}\Big(\frac{L^Ta}{\|L^Ta\|}\Big) = -e_1 \quad \Rightarrow \quad -\frac{L^Ta}{\|L^Ta\|} = R\cdot e_1$$

Hence,

$$\bar{c}' = R\cdot\hat{c}' = R\cdot\frac{1}{n+1}e_1 = -\frac{1}{n+1}\frac{L^Ta}{\|L^Ta\|}$$

$$\begin{aligned} c' &= f(\bar{c}') = L\cdot\bar{c}' + c \\ &= -\frac{1}{n+1}L\frac{L^Ta}{\|L^Ta\|} + c \end{aligned}$$

The Ellipsoid Algorithm

How to Compute The New Parameters?

The transformation function of the (old) ellipsoid: $f(x) = Lx + c$;

The halfspace to be intersected: $H = \{x \mid a^T(x-c) \le 0\}$;

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This means $\bar{a} = L^Ta$.

The Ellipsoid Algorithm

After rotating back (applying R^{-1}) the normal vector of the halfspace points in negative x_1-direction. Hence,

$$R^{-1}\Big(\frac{L^Ta}{\|L^Ta\|}\Big) = -e_1 \quad \Rightarrow \quad -\frac{L^Ta}{\|L^Ta\|} = R\cdot e_1$$

Hence,

$$\bar{c}' = R\cdot\hat{c}' = R\cdot\frac{1}{n+1}e_1 = -\frac{1}{n+1}\frac{L^Ta}{\|L^Ta\|}$$

$$\begin{aligned} c' &= f(\bar{c}') = L\cdot\bar{c}'+c \\ &= -\frac{1}{n+1}L\frac{L^Ta}{\|L^Ta\|}+c \\ &= c-\frac{1}{n+1}\frac{Qa}{\sqrt{a^TQa}} \end{aligned}$$

The Ellipsoid Algorithm

How to Compute The New Parameters?

The transformation function of the (old) ellipsoid: $f(x) = Lx+c$;

The halfspace to be intersected: $H = \{x \mid a^T(x-c)\le 0\}$;

$$\begin{aligned} f^{-1}(H) &= \{f^{-1}(x) \mid a^T(x-c)\le 0\} \\ &= \{f^{-1}(f(y)) \mid a^T(f(y)-c)\le 0\} \\ &= \{y \mid a^T(f(y)-c)\le 0\} \\ &= \{y \mid a^T(Ly+c-c)\le 0\} \\ &= \{y \mid (a^TL)y\le 0\} \end{aligned}$$

This means $\bar{a} = L^Ta$.

For computing the matrix Q' of the new ellipsoid we assume in the following that $\hat{E}', \bar{E}'$ and E' refer to the ellispoids centered in the origin.

The Ellipsoid Algorithm

After rotating back (applying R^{-1}) the normal vector of the halfspace points in negative x_1-direction. Hence,

$$R^{-1}\Big(\frac{L^T a}{\|L^T a\|}\Big) = -e_1 \quad \Rightarrow \quad -\frac{L^T a}{\|L^T a\|} = R \cdot e_1$$

Hence,

$$\bar{c}' = R \cdot \hat{c}' = R \cdot \frac{1}{n+1} e_1 = -\frac{1}{n+1} \frac{L^T a}{\|L^T a\|}$$

$$\begin{aligned} c' &= f(\bar{c}') = L \cdot \bar{c}' + c \\ &= -\frac{1}{n+1} L \frac{L^T a}{\|L^T a\|} + c \\ &= c - \frac{1}{n+1} \frac{Qa}{\sqrt{a^T Q a}} \end{aligned}$$

Recall that

$$\hat{Q}' = \begin{pmatrix} a^2 & 0 & \dots & 0 \\ 0 & b^2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & b^2 \end{pmatrix}$$

This gives

$$\hat{Q}' = \frac{n^2}{n^2-1}\Big(I - \frac{2}{n+1}e_1 e_1^T\Big)$$

because for $a^2 = n^2/(n+1)^2$ and $b^2 = n^2/n^2-1$

For computing the matrix Q' of the new ellipsoid we assume in the following that $\hat{E}', \bar{E}'$ and E' refer to the ellispoids centered in the origin.

Recall that

$$\hat{Q}' = \begin{pmatrix} a^2 & 0 & \dots & 0 \\ 0 & b^2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & b^2 \end{pmatrix}$$

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$$= \frac{n^2(n+1) - 2n^2}{(n-1)(n+1)^2} = \frac{n^2(n-1)}{(n-1)(n+1)^2} = a^2$$

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For computing the matrix Q' of the new ellipsoid we assume in the following that $\hat{E}', \bar{E}'$ and E' refer to the ellispoids centered in the origin.

9 The Ellipsoid Algorithm

$\bar{E}'$

Recall that

$$\hat{Q}' = \begin{pmatrix} a^2 & 0 & \dots & 0 \\ 0 & b^2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & b^2 \end{pmatrix}$$

This gives

$$\hat{Q}' = \frac{n^2}{n^2-1}\Big(I - \frac{2}{n+1}e_1e_1^T\Big)$$

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9 The Ellipsoid Algorithm

$$\bar{E}' = R(\hat{E}')$$

Recall that

$$\hat{Q}' = \begin{pmatrix} a^2 & 0 & \dots & 0 \\ 0 & b^2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & b^2 \end{pmatrix}$$

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9 The Ellipsoid Algorithm

$$\bar{E}' = R(\hat{E}') = \{R(x) \mid x^T \hat{Q}'^{-1} x \le 1\}$$

Recall that

$$\hat{Q}' = \begin{pmatrix} a^2 & 0 & \dots & 0 \\ 0 & b^2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & b^2 \end{pmatrix}$$

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9 The Ellipsoid Algorithm

$$\begin{aligned}\bar{E}' &= R(\hat{E}') \\ &= \{R(x) \mid x^T\hat{Q}'^{-1}x \le 1\} \\ &= \{y \mid (R^{-1}y)^T\hat{Q}'^{-1}R^{-1}y \le 1\}\end{aligned}$$

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9 The Ellipsoid Algorithm

$$\begin{aligned}\bar{E}' &= R(\hat{E}')\\ &= \{R(x) \mid x^T \hat{Q}'^{-1} x \le 1\}\\ &= \{y \mid (R^{-1}y)^T \hat{Q}'^{-1} R^{-1} y \le 1\}\\ &= \{y \mid y^T (R^T)^{-1} \hat{Q}'^{-1} R^{-1} y \le 1\}\end{aligned}$$

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9 The Ellipsoid Algorithm

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&= \{y \mid y^T (R^T)^{-1} \hat{Q}'^{-1} R^{-1} y \le 1\} \\
&= \{y \mid y^T (\underbrace{R\hat{Q}'R^T}_{\bar{Q}'})^{-1} y \le 1\}
\end{aligned}
$$

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\hat{Q}' = \begin{pmatrix} a^2 & 0 & \dots & 0 \\ 0 & b^2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & b^2 \end{pmatrix}
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\hat{Q}' = \frac{n^2}{n^2-1}\Big(I - \frac{2}{n+1} e_1 e_1^T\Big)
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$$
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&= \frac{n^2(n+1) - 2n^2}{(n-1)(n+1)^2} = \frac{n^2(n-1)}{(n-1)(n+1)^2} = a^2
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9 The Ellipsoid Algorithm

Hence,

$\bar{Q}'$

9 The Ellipsoid Algorithm

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\bar{E}' &= R(\hat{E}') \\
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9 The Ellipsoid Algorithm

Hence,

$$\bar{Q}' = R\hat{Q}'R^T$$

9 The Ellipsoid Algorithm

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9 The Ellipsoid Algorithm

Hence,

$$\begin{aligned}\bar{Q}' &= R\hat{Q}'R^T \\ &= R \cdot \frac{n^2}{n^2-1}\Big(I - \frac{2}{n+1}e_1e_1^T\Big) \cdot R^T\end{aligned}$$

9 The Ellipsoid Algorithm

$$\begin{aligned}\bar{E}' &= R(\hat{E}') \\ &= \{R(x) \mid x^T\hat{Q}'^{-1}x \le 1\} \\ &= \{y \mid (R^{-1}y)^T\hat{Q}'^{-1}R^{-1}y \le 1\} \\ &= \{y \mid y^T(R^T)^{-1}\hat{Q}'^{-1}R^{-1}y \le 1\} \\ &= \{y \mid y^T(\underbrace{R\hat{Q}'R^T}_{\bar{Q}'})^{-1}y \le 1\}\end{aligned}$$

9 The Ellipsoid Algorithm

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&= \frac{n^2}{n^2-1}\Big(R \cdot R^T - \frac{2}{n+1}(Re_1)(Re_1)^T\Big)
\end{aligned}$$

9 The Ellipsoid Algorithm

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9 The Ellipsoid Algorithm

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&= \frac{n^2}{n^2-1}\Big(R\cdot R^T-\frac{2}{n+1}(Re_1)(Re_1)^T\Big) \\
&= \frac{n^2}{n^2-1}\Big(I-\frac{2}{n+1}\frac{L^Taa^TL}{\|L^Ta\|^2}\Big)
\end{aligned}$$

9 The Ellipsoid Algorithm

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9 The Ellipsoid Algorithm

E'

9 The Ellipsoid Algorithm

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&= \frac{n^2}{n^2-1}\Big(I - \frac{2}{n+1}\frac{L^T a a^T L}{\|L^T a\|^2}\Big)
\end{aligned}
$$

9 The Ellipsoid Algorithm

$$E' = L(\bar{E}')$$

9 The Ellipsoid Algorithm

Hence,

$$\begin{aligned}\bar{Q}' &= R\hat{Q}'R^T \\ &= R \cdot \frac{n^2}{n^2-1}\Big(I - \frac{2}{n+1}e_1e_1^T\Big) \cdot R^T \\ &= \frac{n^2}{n^2-1}\Big(R \cdot R^T - \frac{2}{n+1}(Re_1)(Re_1)^T\Big) \\ &= \frac{n^2}{n^2-1}\Big(I - \frac{2}{n+1}\frac{L^Taa^TL}{\|L^Ta\|^2}\Big)\end{aligned}$$

9 The Ellipsoid Algorithm

$$\begin{aligned} E' &= L(\bar{E}') \\ &= \{L(x) \mid x^T \bar{Q}'^{-1} x \le 1\} \end{aligned}$$

9 The Ellipsoid Algorithm

Hence,

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9 The Ellipsoid Algorithm

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9 The Ellipsoid Algorithm

Hence,

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&= \frac{n^2}{n^2-1}\Big(I - \frac{2}{n+1} \frac{L^T a a^T L}{\|L^T a\|^2}\Big)
\end{aligned}
$$

9 The Ellipsoid Algorithm

$$\begin{aligned} E' &= L(\bar{E}') \\ &= \{L(x) \mid x^T \bar{Q}'^{-1} x \le 1\} \\ &= \{y \mid (L^{-1}y)^T \bar{Q}'^{-1} L^{-1} y \le 1\} \\ &= \{y \mid y^T (L^T)^{-1} \bar{Q}'^{-1} L^{-1} y \le 1\} \end{aligned}$$

9 The Ellipsoid Algorithm

Hence,

$$\begin{aligned} \bar{Q}' &= R\hat{Q}'R^T \\ &= R \cdot \frac{n^2}{n^2-1}\Big(I - \frac{2}{n+1} e_1 e_1^T\Big) \cdot R^T \\ &= \frac{n^2}{n^2-1}\Big(R \cdot R^T - \frac{2}{n+1}(Re_1)(Re_1)^T\Big) \\ &= \frac{n^2}{n^2-1}\Big(I - \frac{2}{n+1}\frac{L^T a a^T L}{\|L^T a\|^2}\Big) \end{aligned}$$

9 The Ellipsoid Algorithm

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\end{aligned}$$

9 The Ellipsoid Algorithm

Hence,

$$\begin{aligned}
\bar{Q}' &= R\hat{Q}'R^T \\
&= R \cdot \frac{n^2}{n^2-1}\Big(I - \frac{2}{n+1} e_1 e_1^T\Big) \cdot R^T \\
&= \frac{n^2}{n^2-1}\Big(R \cdot R^T - \frac{2}{n+1}(Re_1)(Re_1)^T\Big) \\
&= \frac{n^2}{n^2-1}\Big(I - \frac{2}{n+1}\frac{L^T a a^T L}{\|L^T a\|^2}\Big)
\end{aligned}$$

9 The Ellipsoid Algorithm

Hence,

$$Q'$$

9 The Ellipsoid Algorithm

$$\begin{aligned}
E' &= L(\bar{E}') \\
&= \{L(x) \mid x^T \bar{Q}'^{-1} x \le 1\} \\
&= \{y \mid (L^{-1}y)^T \bar{Q}'^{-1} L^{-1} y \le 1\} \\
&= \{y \mid y^T (L^T)^{-1} \bar{Q}'^{-1} L^{-1} y \le 1\} \\
&= \{y \mid y^T (\underbrace{L\bar{Q}'L^T}_{Q'})^{-1} y \le 1\}
\end{aligned}$$

9 The Ellipsoid Algorithm

Hence,

$$Q' = L\bar{Q}'L^T$$

9 The Ellipsoid Algorithm

$$\begin{aligned}
E' &= L(\bar{E}') \\
&= \{L(x) \mid x^T \bar{Q}'^{-1} x \le 1\} \\
&= \{y \mid (L^{-1}y)^T \bar{Q}'^{-1} L^{-1} y \le 1\} \\
&= \{y \mid y^T (L^T)^{-1} \bar{Q}'^{-1} L^{-1} y \le 1\} \\
&= \{y \mid y^T (\underbrace{L\bar{Q}'L^T}_{Q'})^{-1} y \le 1\}
\end{aligned}$$

9 The Ellipsoid Algorithm

Hence,

$$\begin{aligned} Q' &= L\bar{Q}'L^T \\ &= L \cdot \frac{n^2}{n^2-1}\Big(I - \frac{2}{n+1}\frac{L^T a a^T L}{a^T Q a}\Big) \cdot L^T \end{aligned}$$

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9 The Ellipsoid Algorithm

Hence,

$$\begin{aligned} Q' &= L\bar{Q}'L^T \\ &= L \cdot \frac{n^2}{n^2-1}\Big(I - \frac{2}{n+1}\frac{L^T a a^T L}{a^T Q a}\Big) \cdot L^T \\ &= \frac{n^2}{n^2-1}\Big(Q - \frac{2}{n+1}\frac{Q a a^T Q}{a^T Q a}\Big) \end{aligned}$$

9 The Ellipsoid Algorithm

$$\begin{aligned} E' &= L(\bar{E}') \\ &= \{L(x) \mid x^T \bar{Q}'^{-1} x \le 1\} \\ &= \{y \mid (L^{-1}y)^T \bar{Q}'^{-1} L^{-1} y \le 1\} \\ &= \{y \mid y^T (L^T)^{-1} \bar{Q}'^{-1} L^{-1} y \le 1\} \\ &= \{y \mid y^T \underbrace{(L\bar{Q}'L^T)}_{Q'}{}^{-1} y \le 1\} \end{aligned}$$

Incomplete Algorithm

Algorithm 1 ellipsoid-algorithm

input: point $c \in \mathbb{R}^n$, convex set $K \subseteq \mathbb{R}^n$
output: point $x \in K$ or "K is empty"
$Q \leftarrow$???
repeat
    if $c \in K$ **then return** c
    else
        choose a violated hyperplane a
        $c \leftarrow c - \frac{1}{n+1}\frac{Qa}{\sqrt{a^TQa}}$
        $Q \leftarrow \frac{n^2}{n^2-1}\Big(Q - \frac{2}{n+1}\frac{Qaa^TQ}{a^TQa}\Big)$
    endif
until ???
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9 The Ellipsoid Algorithm

Hence,

$$
\begin{aligned}
Q' &= L\bar{Q}'L^T \\
&= L\cdot\frac{n^2}{n^2-1}\Big(I - \frac{2}{n+1}\frac{L^Taa^TL}{a^TQa}\Big)\cdot L^T \\
&= \frac{n^2}{n^2-1}\Big(Q - \frac{2}{n+1}\frac{Qaa^TQ}{a^TQa}\Big)
\end{aligned}
$$

Repeat: Size of basic solutions

Lemma 7

Let $P = \{x \in \mathbb{R}^n \mid Ax \leq b\}$ be a bounded polyhedron. Let $\langle a_{\max} \rangle$ be the maximum encoding length of an entry in A, b. Then every entry x_j in a basic solution fulfills $|x_j| = \frac{D_j}{D}$ with $D_j, D \leq 2^{2n\langle a_{\max}\rangle + 2n\log_2 n}$.

In the following we use $\delta := 2^{2n\langle a_{\max}\rangle + 2n\log_2 n}$.

Note that here we have $P = \{x \mid Ax \leq b\}$. The previous lemmas we had about the size of feasible solutions were slightly different as they were for different polytopes.

Incomplete Algorithm

Algorithm 1 ellipsoid-algorithm

input: point $c \in \mathbb{R}^n$, convex set $K \subseteq \mathbb{R}^n$
output: point $x \in K$ or "K is empty"
$Q \leftarrow$???
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Proof:
Let $\bar{A} = \begin{bmatrix} A & -A & I_m \end{bmatrix}$, b, be the matrix and right-hand vector after transforming the system to standard form.

The determinant of the matrices $\bar{A}_B$ and $\bar{M}_j$ (matrix obt. when replacing the j-th column of $\bar{A}_B$ by b) can become at most

$$\begin{aligned}\det(\bar{A}_B), \det(\bar{M}_j) &\le \|\vec{\ell}_{\max}\|^{2n} \\ &\le (\sqrt{2n} \cdot 2^{\langle a_{\max}\rangle})^{2n} \le 2^{2n\langle a_{\max}\rangle + 2n\log_2 n} ,\end{aligned}$$

where $\vec{\ell}_{\max}$ is the longest column-vector that can be obtained after deleting all but $2n$ rows and columns from $\bar{A}$.

This holds because columns from I_m selected when going from $\bar{A}$ to $\bar{A}_B$ do not increase the determinant. Only the at most $2n$ columns from matrices A and $-A$ that $\bar{A}$ consists of contribute.

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How do we find the first ellipsoid?

For feasibility checking we can assume that the polytop P is bounded; it is sufficient to consider basic solutions.

Every entry x_i in a basic solution fulfills $|x_i| \le \delta$.

Hence, P is contained in the cube $-\delta \le x_i \le \delta$.

A vector in this cube has at most distance $R := \sqrt{n}\delta$ from the origin.

Starting with the ball $E_0 := B(0, R)$ ensures that P is completely contained in the initial ellipsoid. This ellipsoid has volume at most $R^n \operatorname{vol}(B(0,1)) \le (n\delta)^n \operatorname{vol}(B(0,1))$.

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When can we terminate?

Let $P := \{x \mid Ax \le b\}$ with $A \in \mathbb{Z}$ and $b \in \mathbb{Z}$ be a bounded polytop. Let $\langle a_{\max} \rangle$ be the encoding length of the largest entry in A or b.

Consider the following polyhedron

$$P_\lambda := \left\{ x \mid Ax \le b + \frac{1}{\lambda}\begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} \right\} ,$$

where $\lambda = \delta^2 + 1$.

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P_λ is feasible if and only if P is feasible.

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$\Longrightarrow$:

Consider the polyhedrons

$$\bar{P} = \left\{x \mid \begin{bmatrix} A & -A & I_m \end{bmatrix} x = b; x \geq 0\right\}$$

and

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P is feasible if and only if $\bar{P}$ is feasible, and P_λ feasible if and only if $\bar{P}_\lambda$ feasible.

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Let $\bar{A} = \begin{bmatrix} A & -A & I_m \end{bmatrix}$.

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(The other x-values are zero)

The only reason that this basic feasible solution is not feasible for $\bar{P}$ is that one of the basic variables becomes negative.

Hence, there exists i with

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$$x_B = \bar{A}_B^{-1}b + \frac{1}{\lambda}\bar{A}_B^{-1}\begin{pmatrix}1\\ \vdots \\ 1\end{pmatrix}$$

(The other x-values are zero)

The only reason that this basic feasible solution is not feasible for $\bar{P}$ is that one of the basic variables becomes negative.

Hence, there exists i with

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Algorithm 1 ellipsoid-algorithm

input: point $c \in \mathbb{R}^n$, convex set $K \subseteq \mathbb{R}^n$, radii R and r
    with $K \subseteq B(c,R)$, and $B(x,r) \subseteq K$ for some x
output: point $x \in K$ or "K is empty"
$Q \leftarrow \mathrm{diag}(R^2,\dots,R^2)$ // i.e., $L = \mathrm{diag}(R,\dots,R)$
repeat
    if $c \in K$ **then return** c
    else
        choose a violated hyperplane a
        $c \leftarrow c - \frac{1}{n+1}\frac{Qa}{\sqrt{a^TQa}}$
        $Q \leftarrow \frac{n^2}{n^2-1}\left(Q - \frac{2}{n+1}\frac{Qaa^TQ}{a^TQa}\right)$
    endif
until $\det(Q) \leq r^{2n}$ // i.e., $\det(L) \leq r^n$
return "K is empty"

How many iterations do we need until the volume becomes too small?

$$e^{-\frac{i}{2(n+1)}} \cdot \mathrm{vol}(B(0,R)) < \mathrm{vol}(B(0,r))$$

Hence,

$$\begin{aligned}
i &> 2(n+1)\ln\left(\frac{\mathrm{vol}(B(0,R))}{\mathrm{vol}(B(0,r))}\right)\\
&= 2(n+1)\ln\left(n^n\delta^n \cdot \delta^{3n}\right)\\
&= 8n(n+1)\ln(\delta) + 2(n+1)n\ln(n)\\
&= \mathcal{O}(\mathrm{poly}(n,\langle a_{\max}\rangle))
\end{aligned}$$

Separation Oracle:

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- certifies that $x \in K$,
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We will usually assume that A is a polynomial-time algorithm.

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The Ellipsoid algorithm requires $\mathcal{O}(\mathrm{poly}(n) \cdot \log(R/r))$ iterations. Each iteration is polytime for a polynomial-time Separation oracle.

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$Q \leftarrow \mathrm{diag}(R^2, \dots, R^2)$ // i.e., $L = \mathrm{diag}(R, \dots, R)$
repeat
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