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Given a standard minimization LP

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# Computing a Lower Bound

Let $s$ denote the smallest common multiple of all denominators of entries in $A, b$.

Multiply entries in $A, b$ by $s$ to obtain integral entries. This does not change the feasible region.

Add slack variables to $A$; denote the resulting matrix with $\bar{A}$.

If $B$ is an optimal basis then $x_B$ with $\bar{A}_B x_B = \bar{b}$, gives an optimal assignment to the basis variables (non-basic variables are $0$).



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**Theorem 2 (Cramers Rule)**
*Let $M$ be a matrix with $\det(M) \neq 0$. Then the solution to the system $Mx = b$ is given by*

$$x_i = \frac{\det(M_j)}{\det(M)} ,$$

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Let $Z$ be the maximum absolute entry occuring in $\bar{A}, \bar{b}$ or $c$. Let $C$ denote the matrix obtained from $\bar{A}_B$ by replacing the $j$-th column with vector $\bar{b}$ (for some $j$).

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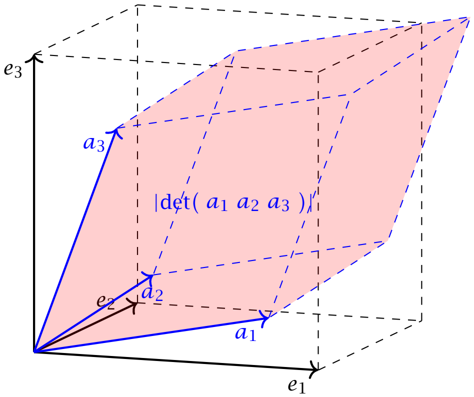


Hadamards inequality says that the volume of the red parallelepiped (Spat) is smaller than the volume in the black cube (if $\|e_1\| = \|a_1\|$, $\|e_2\| = \|a_2\|$, $\|e_3\| = \|a_3\|$).



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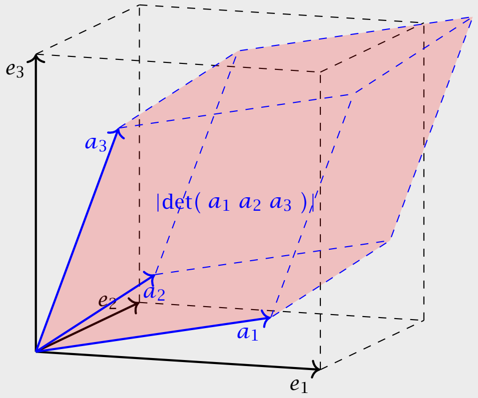


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Compute an optimum basis for the new LP.

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1: **if** $d = 1$ **then** solve 1-dimensional problem and return;

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since $\sum_{i\ge 1}\frac{i^2}{i!}$ is a constant.



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