Lemma 2 (Chernoff Bounds)

Let $X_1, \dots, X_n$ be n independent 0-1 random variables, not necessarily identically distributed. Then for $X = \sum_{i=1}^n X_i$ and $\mu = E[X]$, $L \le \mu \le U$, and $\delta > 0$

$$\Pr[X \ge (1+\delta)U] < \left(\frac{e^{\delta}}{(1+\delta)^{1+\delta}}\right)^U ,$$

and

$$\Pr[X \le (1-\delta)L] < \left(\frac{e^{-\delta}}{(1-\delta)^{1-\delta}}\right)^L ,$$

Lemma 3

For $0 \leq \delta \leq 1$ *we have that*

$$\left(\frac{e^{\delta}}{(1+\delta)^{1+\delta}}\right)^{U} \leq e^{-U\delta^2/3}$$

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Proof of Chernoff Bounds

Markovs Inequality:

Let X be random variable taking non-negative values.

Then

$$\Pr[X \geq a] \leq \mathrm{E}[X]/a$$

Trivial!

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Hence:

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$$\Pr[X \geq (1+\delta)U] \leq \frac{\mathrm{E}[X]}{(1+\delta)U} \approx \frac{1}{1+\delta}$$

That's awfully weak :(

Proof of Chernoff Bounds

Set $p_i = \Pr[X_i = 1]$. Assume $p_i > 0$ for all i.

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This may be a lot better (!?)

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Now, we apply Markov:

$$\begin{aligned}\Pr[X \geq (1+\delta)U] &= \Pr[e^{tX} \geq e^{t(1+\delta)U}] \\ &\leq \frac{\mathrm{E}[e^{tX}]}{e^{t(1+\delta)U}}\end{aligned}$$

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For $0 \leq \delta \leq 1$ *we have that*

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True for $\delta = 0$. Divide by U and take derivatives:

$$-\ln(1+\delta) \le -2\delta/3$$

Reason:

As long as derivative of left side is smaller than derivative of right side the inequality holds.

$$f(\delta) := -\ln(1+\delta) + 2\delta/3 \le 0$$

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$f(0) = 0$ and $f(1) = -\ln(2) + 2/3 < 0$

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True for $\delta = 0$. Divide by U and take derivatives:

$$-\ln(1+\delta) \leq -1/3 \iff \ln(1+\delta) \geq 1/3 \quad \text{(true)}$$

Reason:
As long as derivative of left side is smaller than derivative of right side the inequality holds.

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This holds for $0 \leq \delta < 1$.

Integer Multicommodity Flows

- Given s_i-t_i pairs in a graph.
- Connect each pair by a path such that not too many path use any given edge.

$$\begin{array}{llrcl}
\min & & W & & \\
\text{s.t.} & \forall i & \sum_{p \in \mathcal{P}_i} x_p & = & 1 \\
& & \sum_{p: e \in p} x_p & \leq & W \\
& & x_p & \in & \{0,1\}
\end{array}$$

Integer Multicommodity Flows

Randomized Rounding:
For each i choose one path from the set $\mathcal{P}_i$ at random according to the probability distribution given by the Linear Programming solution.

Theorem 5
If $W^* \geq c \ln n$ for some constant c, then with probability at least $n^{-c/3}$ the total number of paths using any edge is at most $W^* + \sqrt{cW^* \ln n}$.

Theorem 6
With probability at least $n^{-c/3}$ the total number of paths using any edge is at most $W^* + c \ln n$.

Integer Multicommodity Flows

Let X_e^i be a random variable that indicates whether the path for s_i-t_i uses edge e.

Then the number of paths using edge e is $Y_e = \sum_i X_e^i$.

$$E[Y_e] = \sum_i \sum_{p \in \mathcal{P}_i : e \in p} x_p^* = \sum_{p : e \in P} x_p^* \le W^*$$

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Integer Multicommodity Flows

Choose $\delta = \sqrt{(c \ln n)/W^*}$.

Then

$$\Pr[Y_e \geq (1+\delta)W^*] < e^{-W^*\delta^2/3} = \frac{1}{n^{c/3}}$$

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17.3 MAXSAT

Problem definition:

- n Boolean variables
- m clauses $C_1, \dots, C_m$. For example

$$C_7 = x_3 \vee \bar{x}_5 \vee \bar{x}_9$$

- Non-negative weight w_j for each clause C_j.
- Find an assignment of true/false to the variables sucht that the total weight of clauses that are satisfied is maximum.

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Terminology:

- A variable x_i and its negation $\bar{x}_i$ are called literals.
- Hence, each clause consists of a set of literals (i.e., no duplications: $x_i \vee x_i \vee \bar{x}_j$ is **not** a clause).
- We assume a clause does not contain x_i and $\bar{x}_i$ for any i.
- x_i is called a positive literal while the negation $\bar{x}_i$ is called a negative literal.
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- Clauses of length one are called unit clauses.

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- Clauses of length one are called unit clauses.

MAXSAT: Flipping Coins

Set each x_i independently to true with probability $\frac{1}{2}$ (and, hence, to false with probability $\frac{1}{2}$, as well).

Define random variable X_j with

$$X_j = \begin{cases} 1 & \text{if } C_j \text{ satisfied} \\ 0 & \text{otw.} \end{cases}$$

Then the total weight W of satisfied clauses is given by

$$W = \sum_j w_j X_j$$

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$$E[W]$$

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$$
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E[W] &= \sum_j w_j E[X_j] \\
&= \sum_j w_j \Pr[C_j \text{ is satisified}]
\end{aligned}
$$

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\end{aligned}
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&\ge \frac{1}{2} \sum_j w_j \\
&\ge \frac{1}{2} \mathrm{OPT}
\end{aligned}
$$

MAXSAT: LP formulation

- Let for a clause C_j, P_j be the set of positive literals and N_j the set of negative literals.

$$C_j = \bigvee_{j \in P_j} x_i \vee \bigvee_{j \in N_j} \bar{x}_i$$

$$\begin{array}{llrcl} \max & & \sum_j w_j z_j & & \\ \text{s.t.} & \forall j & \sum_{i \in P_j} y_i + \sum_{i \in N_j} (1 - y_i) & \geq & z_j \\ & \forall i & y_i & \in & \{0,1\} \\ & \forall j & z_j & \leq & 1 \end{array}$$

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MAXSAT: Randomized Rounding

Set each x_i independently to true with probability y_i (and, hence, to false with probability $(1 - y_i)$).

Lemma 7 (Geometric Mean $\leq$ Arithmetic Mean)

For any nonnegative $a_1, \ldots, a_k$

$$\left(\prod_{i=1}^{k} a_i\right)^{1/k} \leq \frac{1}{k}\sum_{i=1}^{k} a_i$$

Definition 8

A function f on an interval I is concave if for any two points s and r from I and any $\lambda \in [0,1]$ we have

$$f(\lambda s + (1-\lambda)r) \geq \lambda f(s) + (1-\lambda)f(r)$$

Lemma 9

Let f be a concave function on the interval $[0,1]$, with $f(0) = a$ and $f(1) = a + b$. Then

$$\begin{aligned} f(\lambda) &= f((1-\lambda)0 + \lambda 1) \\ &\geq (1-\lambda)f(0) + \lambda f(1) \\ &= a + \lambda b \end{aligned}$$

for $\lambda \in [0,1]$.

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$\Pr[C_j \text{ not satisfied}]$

$$\Pr[C_j \text{ not satisfied}] = \prod_{i\in P_j}(1-y_i)\prod_{i\in N_j} y_i$$

$$
\begin{aligned}
\Pr[C_j \text{ not satisfied}] &= \prod_{i\in P_j}(1-y_i)\prod_{i\in N_j} y_i \\
&\le \left[\frac{1}{\ell_j}\left(\sum_{i\in P_j}(1-y_i)+\sum_{i\in N_j} y_i\right)\right]^{\ell_j}
\end{aligned}
$$

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&= \left[1-\frac{1}{\ell_j}\left(\sum_{i\in P_j} y_i+\sum_{i\in N_j}(1-y_i)\right)\right]^{\ell_j}
\end{aligned}
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$$
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&= \left[1-\frac{1}{\ell_j}\left(\sum_{i\in P_j} y_i+\sum_{i\in N_j}(1-y_i)\right)\right]^{\ell_j} \\
&\le \left(1-\frac{z_j}{\ell_j}\right)^{\ell_j} \quad .
\end{aligned}
$$

The function $f(z) = 1 - (1 - \frac{z}{\ell})^{\ell}$ is concave. Hence,

$$\Pr[C_j \text{ satisfied}]$$

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$$
\begin{aligned}
\Pr[C_j \text{ satisfied}] &\geq 1 - \left(1 - \frac{z_j}{\ell_j}\right)^{\ell_j} \\
&\geq \left[1 - \left(1 - \frac{1}{\ell_j}\right)^{\ell_j}\right] \cdot z_j \ .
\end{aligned}
$$

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$$\geq \left[1 - \left(1 - \frac{1}{\ell_j}\right)^{\ell_j}\right] \cdot z_j \ .$$

$f''(z) = -\frac{\ell-1}{\ell}\left[1 - \frac{z}{\ell}\right]^{\ell-2} \leq 0$ for $z \in [0,1]$. Therefore, f is concave.

$$E[W]$$

$$E[W] = \sum_j w_j \Pr[C_j \text{ is satisfied}]$$

$$
\begin{aligned}
E[W] &= \sum_j w_j \Pr[C_j \text{ is satisfied}] \\
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&\geq \sum_j w_j z_j \left[1 - \left(1 - \frac{1}{\ell_j}\right)^{\ell_j}\right] \\
&\geq \left(1 - \frac{1}{e}\right) \mathrm{OPT} \ .
\end{aligned}
$$

MAXSAT: The better of two

Theorem 10

Choosing the better of the two solutions given by randomized rounding and coin flipping yields a $\frac{3}{4}$-approximation.

Let W_1 be the value of randomized rounding and W_2 the value obtained by coin flipping.

$$E[\max\{W_1, W_2\}]$$

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$$\geq E[\tfrac{1}{2}W_1 + \tfrac{1}{2}W_2]$$

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$$
\begin{aligned}
&E[\max\{W_1, W_2\}] \\
&\qquad \geq E[\tfrac{1}{2}W_1 + \tfrac{1}{2}W_2] \\
&\qquad \geq \frac{1}{2}\sum_j w_j z_j \left[1 - \left(1 - \frac{1}{\ell_j}\right)^{\ell_j}\right] + \frac{1}{2}\sum_j w_j \left(1 - \left(\frac{1}{2}\right)^{\ell_j}\right)
\end{aligned}
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Let W_1 be the value of randomized rounding and W_2 the value obtained by coin flipping.

$$
\begin{aligned}
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&\geq E[\tfrac{1}{2}W_1 + \tfrac{1}{2}W_2] \\
&\geq \frac{1}{2}\sum_j w_j z_j \left[1 - \left(1 - \frac{1}{\ell_j}\right)^{\ell_j}\right] + \frac{1}{2}\sum_j w_j \left(1 - \left(\frac{1}{2}\right)^{\ell_j}\right) \\
&\geq \sum_j w_j z_j \Bigg[\underbrace{\frac{1}{2}\left(1 - \left(1 - \frac{1}{\ell_j}\right)^{\ell_j}\right) + \frac{1}{2}\left(1 - \left(\frac{1}{2}\right)^{\ell_j}\right)}_{\geq \frac{3}{4} \text{for all integers}}\Bigg]
\end{aligned}
$$

Let W_1 be the value of randomized rounding and W_2 the value obtained by coin flipping.

$$
\begin{aligned}
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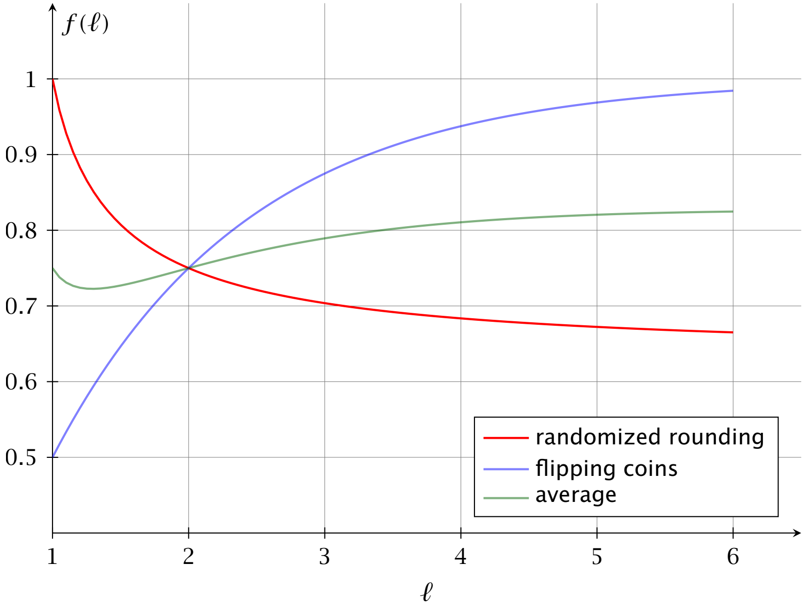
&\geq \sum_j w_j z_j \Bigg[\underbrace{\frac{1}{2}\left(1 - \left(1 - \frac{1}{\ell_j}\right)^{\ell_j}\right) + \frac{1}{2}\left(1 - \left(\frac{1}{2}\right)^{\ell_j}\right)}_{\geq \frac{3}{4} \text{for all integers}}\Bigg] \\
&\geq \frac{3}{4}\mathrm{OPT}
\end{aligned}
$$


$f(\ell)$
1
0.9
0.8
0.7
0.6
0.5
1
2
3
4
5
6
ℓ
randomized rounding
flipping coins
average

MAXSAT: Nonlinear Randomized Rounding

So far we used linear randomized rounding, i.e., the probability that a variable is set to 1/true was exactly the value of the corresponding variable in the linear program.

We could define a function $f : [0,1] \to [0,1]$ and set x_i to true with probability $f(y_i)$.

MAXSAT: Nonlinear Randomized Rounding

So far we used linear randomized rounding, i.e., the probability that a variable is set to 1/true was exactly the value of the corresponding variable in the linear program.

We could define a function $f : [0,1] \rightarrow [0,1]$ and set x_i to true with probability $f(y_i)$.

MAXSAT: Nonlinear Randomized Rounding

Let $f:[0,1]\to[0,1]$ be a function with

$$1-4^{-x}\le f(x)\le 4^{x-1}$$

Theorem 11

Rounding the LP-solution with a function f of the above form gives a $\frac{3}{4}$-approximation.

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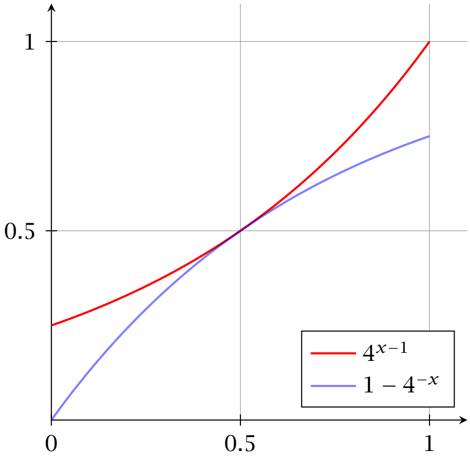
1
0.5
0
0.5
1
4^{x-1}
$1-4^{-x}$

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Can we do better?

Not if we compare ourselves to the value of an optimum LP-solution.

Definition 12 (Integrality Gap)

The integrality gap for an ILP is the worst-case ratio over all instances of the problem of the value of an optimal IP-solution to the value of an optimal solution to its linear programming relaxation.

Note that the integrality is less than one for maximization problems and larger than one for minimization problems (of course, equality is possible).

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Lemma 13

Our ILP-formulation for the MAXSAT problem has integrality gap at most $\frac{3}{4}$*.*

$$
\begin{array}{llrcl}
\max & & \sum_j w_j z_j & & \\
\text{s.t.} & \forall j & \sum_{i \in P_j} y_i + \sum_{i \in N_j} (1 - y_i) & \geq & z_j \\
& \forall i & y_i & \in & \{0,1\} \\
& \forall j & z_j & \leq & 1
\end{array}
$$

Consider: $(x_1 \vee x_2) \wedge (\bar{x}_1 \vee x_2) \wedge (x_1 \vee \bar{x}_2) \wedge (\bar{x}_1 \vee \bar{x}_2)$

- any solution can satisfy at most 3 clauses
- we can set $y_1 = y_2 = 1/2$ in the LP; this allows to set $z_1 = z_2 = z_3 = z_4 = 1$
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MaxCut

MaxCut
Given a weighted graph $G = (V, E, w)$, $w(v) \geq 0$, partition the vertices into two parts. Maximize the weight of edges between the parts.

Trivial 2-approximation

Semidefinite Programming

$$\begin{array}{rlrcl}
\max / \min & & \sum_{i,j} c_{ij} x_{ij} & & \\
\text{s.t.} & \forall k & \sum_{i,j,k} a_{ijk} x_{ij} & = & b_k \\
& \forall i,j & x_{ij} & = & x_{ji} \\
& & X = (x_{ij}) \text{ is psd.} & &
\end{array}$$

- linear objective, linear contraints
- we can constrain a square matrix of variables to be symmetric positive definite

Vector Programming

$$\begin{array}{rrrcl}
\max/\min & & \sum_{i,j} c_{ij}(v_i^t v_j) & & \\
\text{s.t.} & \forall k & \sum_{i,j,k} a_{ijk}(v_i^t v_j) & = & b_k \\
 & \forall i,j & x_{ij} & = & x_{ji} \\
 & & v_i \in \mathbb{R}^n & &
\end{array}$$

- variables are vectors in n-dimensional space
- objective functions and contraints are linear in inner products of the vectors

This is equivalent!

Fact [without proof]

We (essentially) can solve Semidefinite Programs in polynomial time...

Quadratic Programs

Quadratic Program for MaxCut:

$$\begin{array}{lllll} \max & & \frac{1}{2}\sum_{i,j} w_{ij}(1 - y_i y_j) & & \\ & \forall i & y_i & \in & \{-1,1\} \end{array}$$

This is exactly MaxCut!

Semidefinite Relaxation

$$\begin{array}{llrcl} \max & \frac{1}{2}\sum_{i,j} w_{ij}(1 - v_i^t v_j) & & & \\ & \forall i & v_i^t v_i & = & 1 \\ & \forall i & v_i & \in & \mathbb{R}^n \end{array}$$

- this is clearly a relaxation
- the solution will be vectors on the unit sphere

Rounding the SDP-Solution

- Choose a random vector r such that $r/\|r\|$ is uniformly distributed on the unit sphere.
- If $r^t v_i > 0$ set $y_i = 1$ else set $y_i = -1$

Rounding the SDP-Solution

Choose the i-th coordinate r_i as a Gaussian with mean 0 and variance 1, i.e., $r_i \sim \mathcal{N}(0,1)$.

Density function:

$$\varphi(x) = \frac{1}{\sqrt{2\pi}} e^{x^2/2}$$

Then

$$\begin{aligned}
&\Pr[r = (x_1, \dots, x_n)] \\
&\qquad = \frac{1}{(\sqrt{2\pi})^n} e^{x_1^2/2} \cdot e^{x_2^2/2} \cdot \ldots \cdot e^{x_n^2/2} \, \mathrm{d}x_1 \cdot \ldots \cdot \mathrm{d}x_n \\
&\qquad = \frac{1}{(\sqrt{2\pi})^n} e^{\frac{1}{2}(x_1^2 + \ldots + x_n^2)} \, \mathrm{d}x_1 \cdot \ldots \cdot \mathrm{d}x_n
\end{aligned}$$

Hence the probability for a point only depends on its distance to the origin.

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Rounding the SDP-Solution

Fact

The projection of r onto two unit vectors e_1 and e_2 are independent and are normally distributed with mean 0 and variance 1 iff e_1 and e_2 are orthogonal.

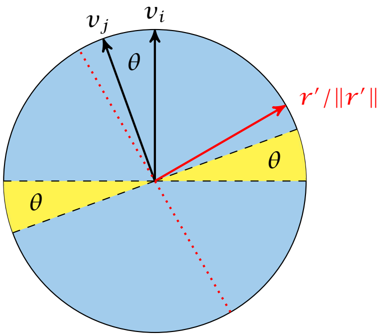
Note that this is clear if e_1 and e_2 are standard basis vectors.

Rounding the SDP-Solution

Corollary

If we project r onto a hyperplane its normalized projection ($r'/\|r'\|$) is uniformly distributed on the unit circle within the hyperplane.

Rounding the SDP-Solution



- if the normalized projection falls into the shaded region, v_i and v_j are rounded to different values
- this happens with probability θ/π

Rounding the SDP-Solution

- contribution of edge (i, j) to the SDP-relaxation:

$$\frac{1}{2} w_{ij}\left(1 - v_i^t v_j\right)$$

- (expected) contribution of edge (i, j) to the rounded instance $w_{ij} \arccos(v_i^t v_j)/\pi$
- ratio is at most

$$\min_{x \in [-1,1]} \frac{2 \arccos(x)}{\pi(1-x)} \geq 0.878$$

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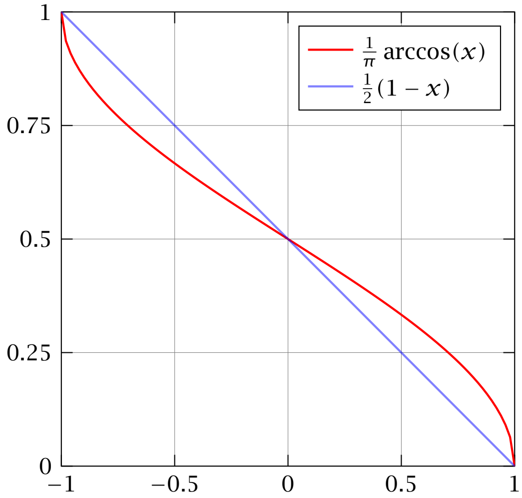

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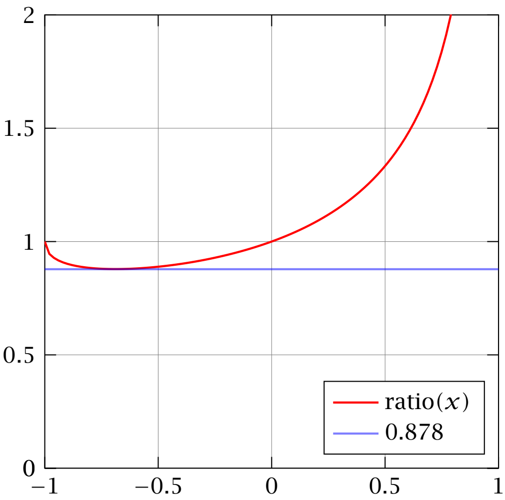

Rounding the SDP-Solution

Theorem 14

Given the unique games conjecture, there is no α*-approximation for the maximum cut problem with constant*

$$\alpha > \min_{x \in [-1,1]} \frac{2 \arccos(x)}{\pi(1-x)}$$

unless $\mathrm{P} = \mathrm{NP}$*.*