

Repetition: Primal Dual for Set Cover

Primal Relaxation:

$$\begin{array}{ll} \min & \sum_{i=1}^k w_i x_i \\ \text{s.t.} & \forall u \in U \quad \sum_{i:u \in S_i} x_i \geq 1 \\ & \forall i \in \{1, \dots, k\} \quad x_i \geq 0 \end{array}$$

Dual Formulation:

$$\begin{array}{ll} \max & \sum_{u \in U} y_u \\ \text{s.t.} & \forall i \in \{1, \dots, k\} \quad \sum_{u:u \in S_i} y_u \leq w_i \\ & y_u \geq 0 \end{array}$$

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$$\begin{aligned} \sum_j w_j x_j &= \sum_j \sum_{e \in S_j} y_e = \sum_e |\{j : e \in S_j\}| \cdot y_e \\ &\leq f \cdot \sum_e y_e \leq f \cdot \text{OPT} \end{aligned}$$

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If we would also fulfill **dual slackness conditions**

$$y_e > 0 \Rightarrow \sum_{j: e \in S_j} x_j = 1$$

then the solution would be **optimal!!!!**

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This is sufficient to show that the solution is an f -approximation.

Suppose we have a primal/dual pair

$$\begin{array}{ll} \min & \sum_j c_j x_j \\ \text{s.t.} & \forall i \quad \sum_j a_{ij} x_j \geq b_i \\ & \forall j \quad x_j \geq 0 \end{array}$$

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and solutions that fulfill approximate slackness conditions:

$$x_j > 0 \Rightarrow \sum_i a_{ij} y_i \geq \frac{1}{\alpha} c_j$$
$$y_i > 0 \Rightarrow \sum_j a_{ij} x_j \leq \beta b_i$$

Then

$$\sum_j c_j x_j$$

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↑

primal cost

A diagram consisting of two rectangular boxes. The top box contains the mathematical expression $\sum_j c_j x_j$ in a purple font. A blue arrow points upwards from the bottom box to the bottom center of the top box. The bottom box contains the text "primal cost" in a blue font.

Then

right hand side of j -th
dual constraint

$$\sum_j c_j x_j$$

primal cost

Then

$$\boxed{\sum_j c_j x_j} \leq \alpha \sum_j \left(\sum_i a_{ij} y_i \right) x_j$$

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$$\boxed{\text{primal cost}} = \alpha \sum_i \left(\sum_j a_{ij} x_j \right) y_i$$

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Feedback Vertex Set for Undirected Graphs

- ▶ Given a graph $G = (V, E)$ and non-negative weights $w_v \geq 0$ for vertex $v \in V$.

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- ▶ Given a graph $G = (V, E)$ and non-negative weights $w_v \geq 0$ for vertex $v \in V$.
- ▶ Choose a minimum cost subset of vertices s.t. every cycle contains at least one vertex.

We can encode this as an instance of Set Cover

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- ▶ Each vertex can be viewed as a set that contains some cycles.
- ▶ However, this encoding gives a Set Cover instance of non-polynomial size.
- ▶ The $O(\log n)$ -approximation for Set Cover does not help us to get a good solution.

Let \mathcal{C} denote the set of all cycles (where a cycle is identified by its set of vertices)

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Primal Relaxation:

$$\begin{array}{ll} \min & \sum_v w_v x_v \\ \text{s.t.} & \forall C \in \mathcal{C} \quad \sum_{v \in C} x_v \geq 1 \\ & \forall v \quad x_v \geq 0 \end{array}$$

Dual Formulation:

$$\begin{array}{ll} \max & \sum_{C \in \mathcal{C}} y_C \\ \text{s.t.} & \forall v \in V \quad \sum_{C: v \in C} y_C \leq w_v \\ & \forall C \quad y_C \geq 0 \end{array}$$

If we perform the previous dual technique for Set Cover we get the following:

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- ▶ Start with $x = 0$ and $y = 0$
- ▶ While there is a cycle C that is not covered (does not contain a chosen vertex).
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 - ▶ set $x_v = 1$.

Then

$$\sum_v w_v x_v$$

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$$\begin{aligned}\sum_v w_v x_v &= \sum_v \sum_{C:v \in C} y_C x_v \\ &= \sum_{v \in S} \sum_{C:v \in C} y_C\end{aligned}$$

where S is the set of vertices we choose.

Then

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where S is the set of vertices we choose.

If every cycle is short we get a good approximation ratio, but this is unrealistic.

Algorithm 1 FeedbackVertexSet

- 1: $y \leftarrow 0$
- 2: $x \leftarrow 0$
- 3: **while** exists cycle C in G **do**
- 4: increase y_C until there is $v \in C$ s.t. $\sum_{C:v \in C} y_C = w_v$
- 5: $x_v = 1$
- 6: remove v from G
- 7: repeatedly remove vertices of degree 1 from G

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Always choose a short cycle that is not covered. If we always find a cycle of length at most α we get an α -approximation.

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Observation:

For any path P of vertices of degree 2 in G the algorithm chooses at most one vertex from P .

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If we always choose a cycle for which the number of vertices of degree at least 3 is at most α we get a 2α -approximation.

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Theorem 2

In any graph with no vertices of degree 1, there always exists a cycle that has at most $\mathcal{O}(\log n)$ vertices of degree 3 or more. We can find such a cycle in linear time.

This means we have

$$y_C > 0 \Rightarrow |S \cap C| \leq \mathcal{O}(\log n) .$$

Primal Dual for Shortest Path

Given a graph $G = (V, E)$ with two nodes $s, t \in V$ and edge-weights $c : E \rightarrow \mathbb{R}^+$ find a shortest path between s and t w.r.t. edge-weights c .

$$\begin{array}{ll} \min & \sum_e c(e)x_e \\ \text{s.t.} & \forall S \in \mathcal{S} \quad \sum_{e \in \delta(S)} x_e \geq 1 \\ & \forall e \in E \quad x_e \in \{0, 1\} \end{array}$$

Here $\delta(S)$ denotes the set of edges with exactly one end-point in S , and $\mathcal{S} = \{S \subseteq V : s \in S, t \notin S\}$.

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The Dual:

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Primal Dual for Shortest Path

We can interpret the value γ_S as the width of a moat surrounding the set S .

Each set can have its own moat but all moats must be disjoint.

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Algorithm 1 PrimalDualShortestPath

- 1: $\gamma \leftarrow 0$
- 2: $F \leftarrow \emptyset$
- 3: **while** there is no s - t path in (V, F) **do**
- 4: Let C be the connected component of (V, F) containing s
- 5: Increase γ_C until there is an edge $e' \in \delta(C)$ such that $\sum_{S:e' \in \delta(S)} \gamma_S = c(e')$.
- 6: $F \leftarrow F \cup \{e'\}$
- 7: **Let P be an s - t path in (V, F)**
- 8: **return P**

Lemma 3

At each point in time the set F forms a tree.

Proof:

Initially, F is empty. We take the current unvisited component C from U , that contains s , and add s to F . Then we visit C and add all edges $e \in E$ that connect s to some point t of the new component C . The set F is a tree.

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At each point in time the set F forms a tree.

Proof:

- ▶ In each iteration we take the current connected component from (V, F) that contains s (call this component C) and add some edge from $\delta(C)$ to F .
- ▶ Since, at most one end-point of the new edge is in C the edge cannot close a cycle.

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$$\sum_{e \in P} c(e)$$

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If we can show that $y_S > 0$ implies $|P \cap \delta(S)| = 1$ gives

$$\sum_{e \in P} c(e) = \sum_S y_S \leq \text{OPT}$$

by weak duality.

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Hence, we find a shortest path.

If S contains two edges from P then there must exist a subpath P' of P that starts and ends with a vertex from S (and all interior vertices are not in S).

When we increased γ_S , S was a connected component of the set of edges F' that we had chosen till this point.

$F' \cup P'$ contains a cycle. Hence, also the final set of edges contains a cycle.

This is a contradiction.

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Steiner Forest Problem:

Given a graph $G = (V, E)$, together with source-target pairs s_i, t_i , $i = 1, \dots, k$, and a cost function $c : E \rightarrow \mathbb{R}^+$ on the edges. Find a subset $F \subseteq E$ of the edges such that for every $i \in \{1, \dots, k\}$ there is a path between s_i and t_i only using edges in F .

$$\begin{array}{ll} \min & \sum_e c(e)x_e \\ \text{s.t.} & \forall S \subseteq V : S \in S_i \text{ for some } i \quad \sum_{e \in \delta(S)} x_e \geq 1 \\ & \forall e \in E \quad x_e \in \{0, 1\} \end{array}$$

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$$\begin{array}{ll} \min & \sum_e c(e)x_e \\ \text{s.t.} & \forall S \subseteq V : S \in \mathcal{S}_i \text{ for some } i \quad \sum_{e \in \delta(S)} x_e \geq 1 \\ & \forall e \in E \quad x_e \in \{0, 1\} \end{array}$$

Here \mathcal{S}_i contains all sets S such that $s_i \in S$ and $t_i \notin S$.

$$\begin{array}{ll}
 \max & \sum_{S: \exists i \text{ s.t. } S \in S_i} \gamma_S \\
 \text{s.t.} & \forall e \in E \quad \sum_{S: e \in \delta(S)} \gamma_S \leq c(e) \\
 & \gamma_S \geq 0
 \end{array}$$

The difference to the dual of the shortest path problem is that we have many more variables (sets for which we can generate a moat of non-zero width).

Algorithm 1 FirstTry

- 1: $\gamma \leftarrow 0$
- 2: $F \leftarrow \emptyset$
- 3: **while** not all s_i-t_i pairs connected in F **do**
- 4: Let C be some connected component of (V, F) such that $|C \cap \{s_i, t_i\}| = 1$ for some i .
- 5: Increase γ_C until there is an edge $e' \in \delta(C)$ s.t.
 $\sum_{S \in S_i: e' \in \delta(S)} \gamma_S = c_{e'}$
- 6: $F \leftarrow F \cup \{e'\}$
- 7: **return** $\bigcup_i P_i$

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If we show that $y_S > 0$ implies that $|\delta(S) \cap F| \leq \alpha$ we are in good shape.

However, this is not true:

- ▶ Take a complete graph on $k + 1$ vertices v_0, v_1, \dots, v_k .

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However, this is not true:

- ▶ Take a complete graph on $k + 1$ vertices v_0, v_1, \dots, v_k .
- ▶ The i -th pair is $v_0 - v_i$.

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If we show that $y_S > 0$ implies that $|\delta(S) \cap F| \leq \alpha$ we are in good shape.

However, this is not true:

- ▶ Take a complete graph on $k + 1$ vertices v_0, v_1, \dots, v_k .
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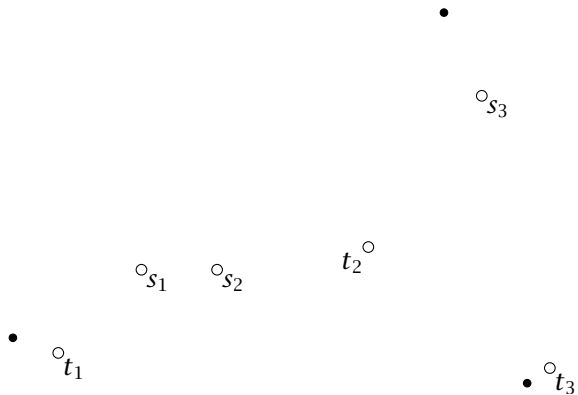
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- ▶ The final set F contains all edges $\{v_0, v_i\}$, $i = 1, \dots, k$.
- ▶ $y_{\{v_0\}} > 0$ but $|\delta(\{v_0\}) \cap F| = k$.

Algorithm 1 SecondTry

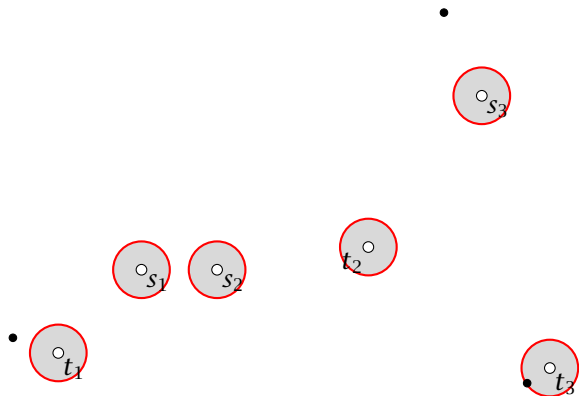
```
1:  $\gamma \leftarrow 0; F \leftarrow \emptyset; \ell \leftarrow 0$ 
2: while not all  $s_i-t_i$  pairs connected in  $F$  do
3:    $\ell \leftarrow \ell + 1$ 
4:   Let  $\mathfrak{C}$  be set of all connected components  $C$  of  $(V, F)$ 
      such that  $|C \cap \{s_i, t_i\}| = 1$  for some  $i$ .
5:   Increase  $\gamma_C$  for all  $C \in \mathfrak{C}$  uniformly until for some edge
       $e_\ell \in \delta(C')$ ,  $C' \in \mathfrak{C}$  s.t.  $\sum_{S: e_\ell \in \delta(S)} \gamma_S = c_{e_\ell}$ 
6:    $F \leftarrow F \cup \{e_\ell\}$ 
7:  $F' \leftarrow F$ 
8: for  $k \leftarrow \ell$  downto 1 do // reverse deletion
9:   if  $F' - e_k$  is feasible solution then
10:     remove  $e_k$  from  $F'$ 
11: return  $F'$ 
```


The reverse deletion step is not strictly necessary this way. It would also be sufficient to simply delete all unnecessary edges in any order.

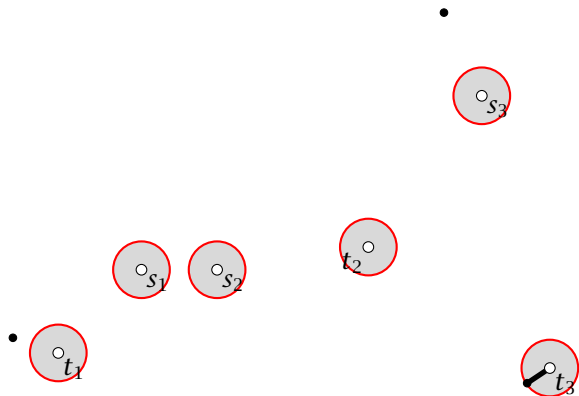
Example



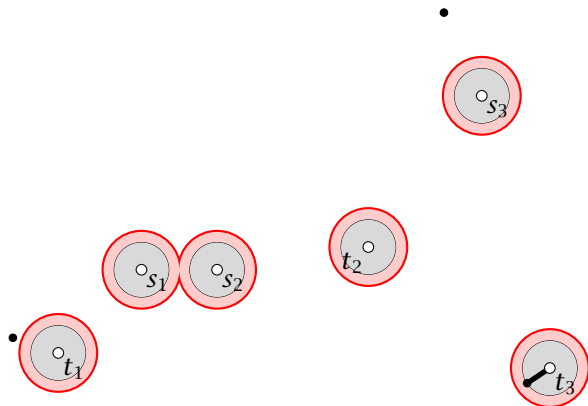
Example



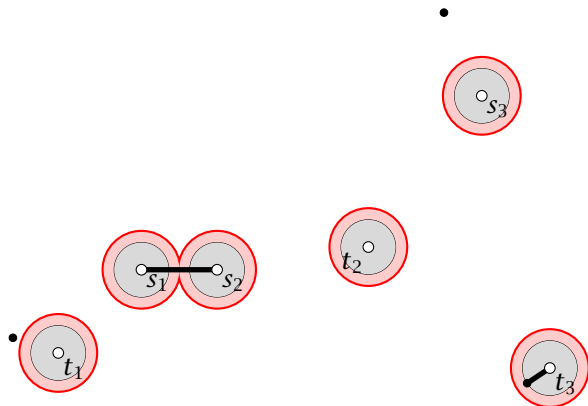
Example



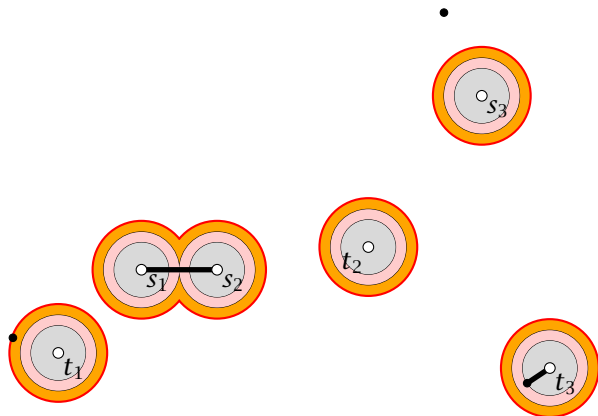
Example



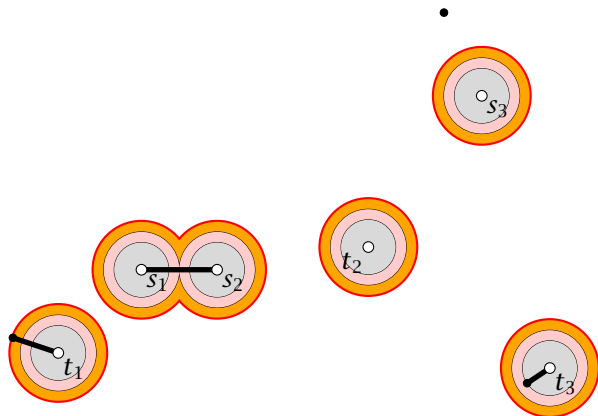
Example



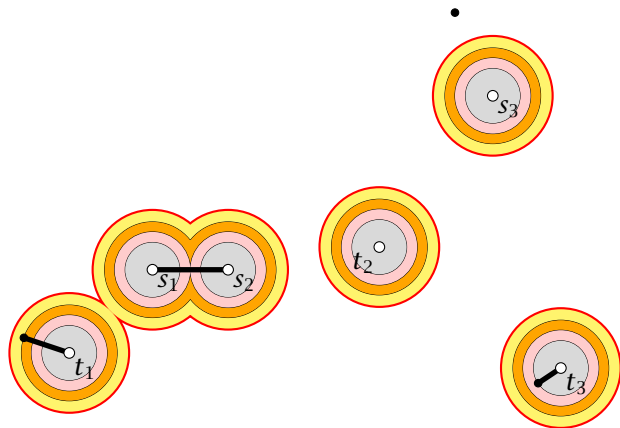
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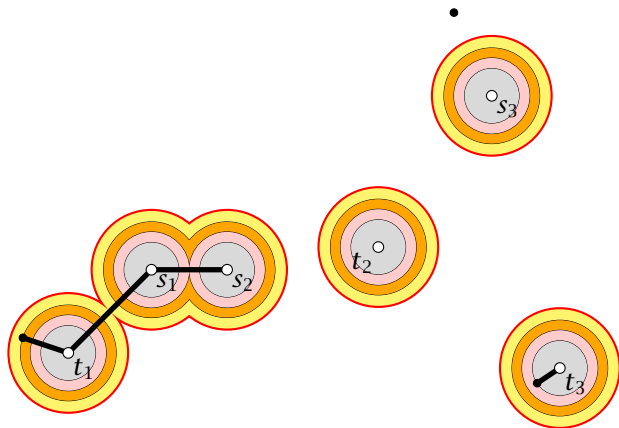
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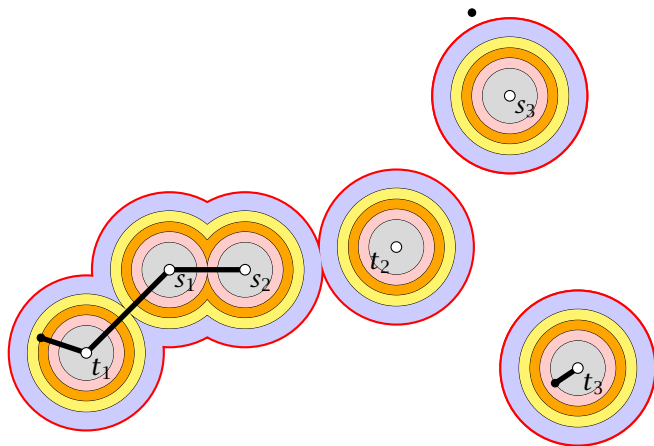
Example



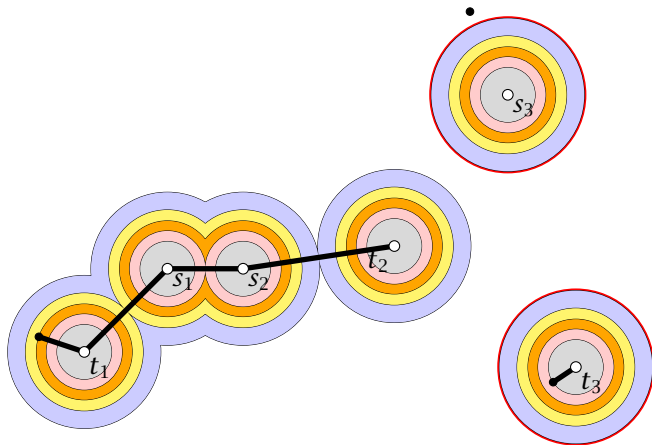
Example



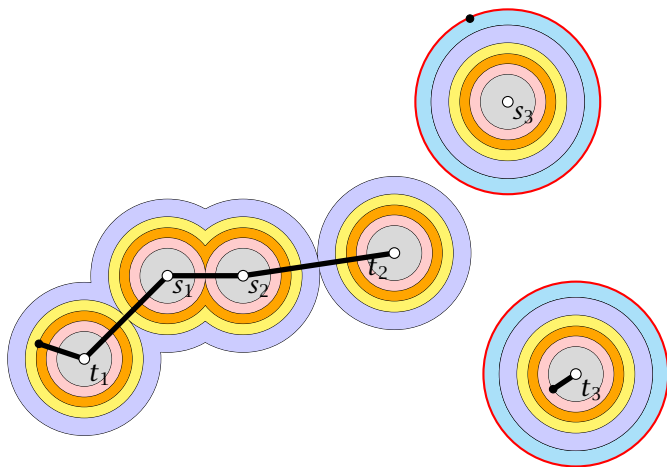
Example



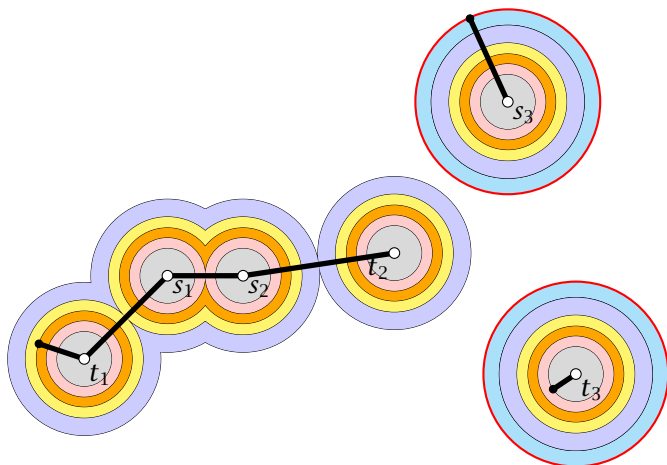
Example



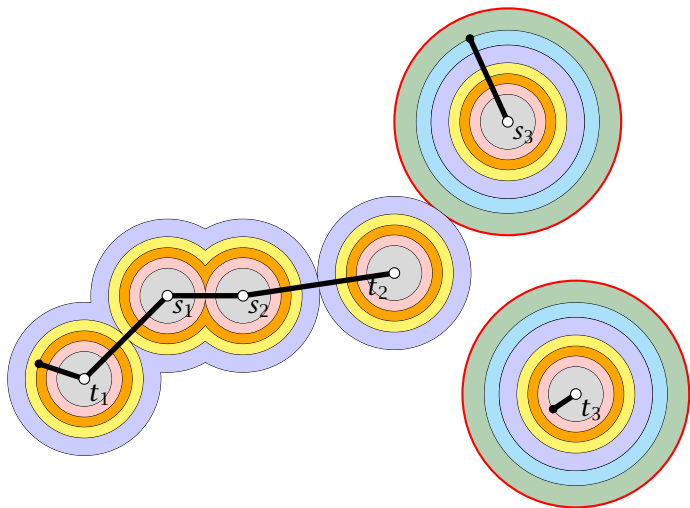
Example



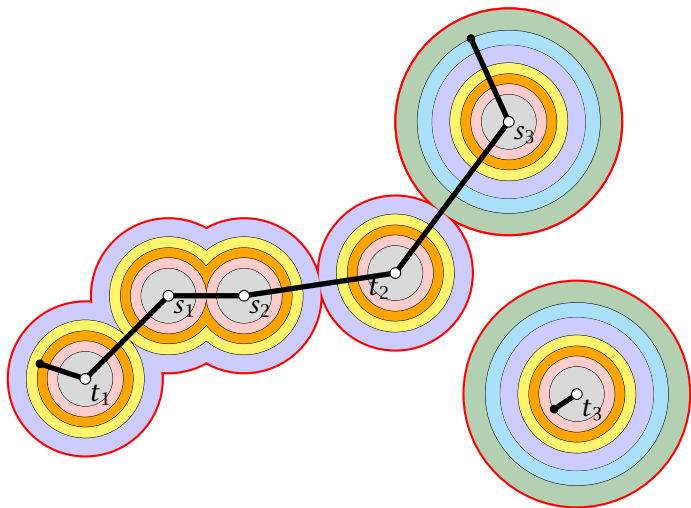
Example



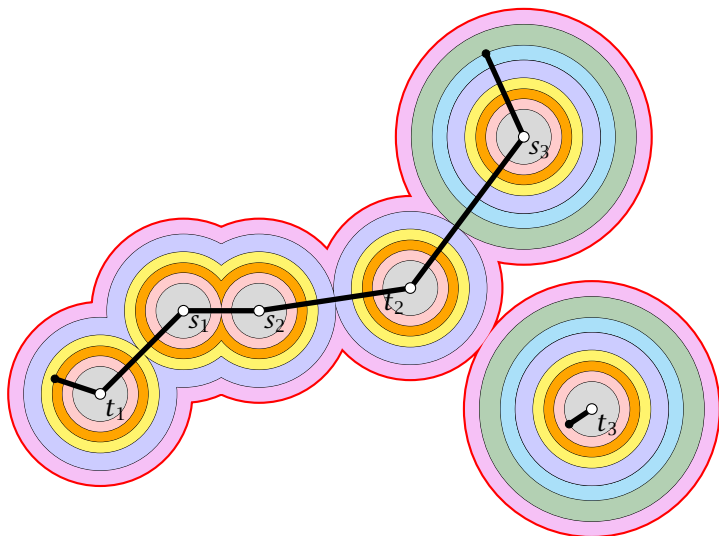
Example



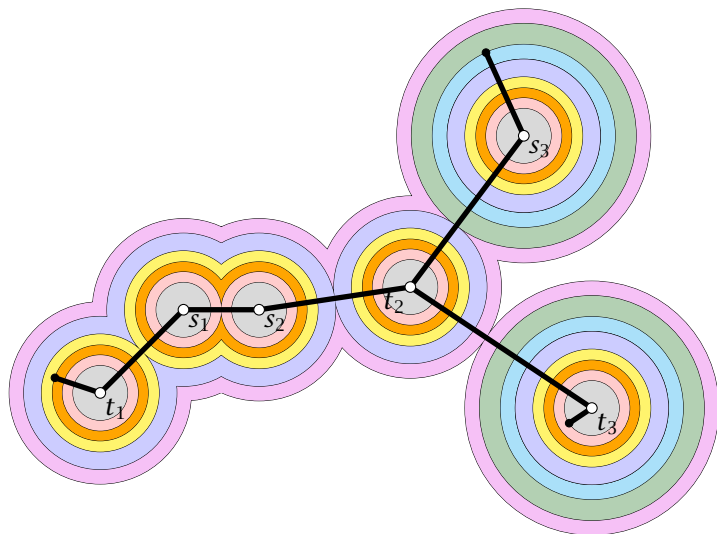
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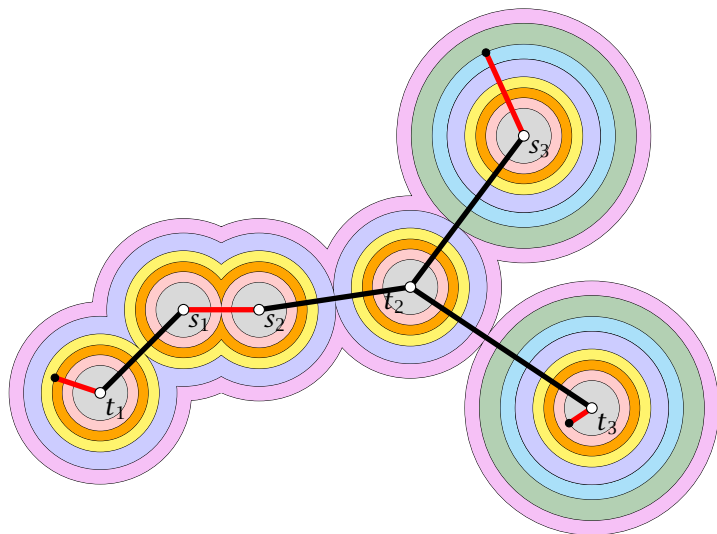
Example



Example



Example



Lemma 4

For any \mathcal{C} in any iteration of the algorithm

$$\sum_{C \in \mathcal{C}} |\delta(C) \cap F'| \leq 2|\mathcal{C}|$$

This means that the number of times a moat from \mathcal{C} is crossed in the final solution is at most twice the number of moats.

Proof: later...

$$\sum_{e \in F'} c_e = \sum_{e \in F'} \sum_{S: e \in \delta(S)} \gamma_S = \sum_S |F' \cap \delta(S)| \cdot \gamma_S .$$

We want to show that

$$\sum_S |F' \cap \delta(S)| \cdot \gamma_S \leq 2 \sum_S \gamma_S$$

in the i -th iteration the increase of the left-hand side is

$$\sum_{S \in \mathcal{S}_i} |F' \cap \delta(S)| \cdot \gamma_S + \sum_{S \in \mathcal{S}_i} \gamma_S$$

and the increase of the right hand side is $\sum_{S \in \mathcal{S}_i} \gamma_S$.

Hence, by the previous lemma the inequality holds after the iteration if it holds in the beginning of the iteration.

$$\sum_{e \in F'} c_e = \sum_{e \in F'} \sum_{S: e \in \delta(S)} \gamma_S = \sum_S |F' \cap \delta(S)| \cdot \gamma_S.$$

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by the induction hypothesis the increase of the left-hand side is

$$\sum_{S \in \mathcal{S}} |F' \cap \delta(S)| \cdot \gamma_S - \sum_{S \in \mathcal{S}} |F \cap \delta(S)| \cdot \gamma_S$$

and the increase of the right hand side is $2 \sum_{S \in \mathcal{S}} \gamma_S$.

Since, by the previous lemma, the inequality holds after the iteration if it holds in the beginning of the iteration,

$$\sum_{e \in F'} c_e = \sum_{e \in F'} \sum_{S: e \in \delta(S)} \gamma_S = \sum_S |F' \cap \delta(S)| \cdot \gamma_S .$$

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Since, by the previous lemma, the inequality holds after the removal of e , it holds in the beginning of the induction.

$$\sum_{e \in F'} c_e = \sum_{e \in F'} \sum_{S: e \in \delta(S)} y_S = \sum_S |F' \cap \delta(S)| \cdot y_S .$$

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- ▶ In the i -th iteration the increase of the left-hand side is

$$\epsilon \sum_{C \in \mathcal{C}} |F' \cap \delta(C)|$$

and the increase of the right hand side is $2\epsilon|\mathcal{C}|$.

- ▶ Hence, by the previous lemma the inequality holds after the iteration if it holds in the beginning of the iteration.

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Lemma 5

For any set of connected components \mathcal{C} in any iteration of the algorithm

$$\sum_{C \in \mathcal{C}} |\delta(C) \cap F'| \leq 2|\mathcal{C}|$$

Proof:

Let \mathcal{C} be any set of connected components of $G \setminus F'$.

Observe that

each edge $e \in E$ is contained in at most two connected components of \mathcal{C} .

Therefore, we have

that the sum of the degrees of all vertices in \mathcal{C} is at most $2|E|$.

Since $\delta(C) \cap F'$ is a subset of the boundary of C , we have

that $|\delta(C) \cap F'| \leq \deg(C)$ for every $C \in \mathcal{C}$.

Summing over all $C \in \mathcal{C}$, we obtain

Lemma 5

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Proof:

- ▶ At any point during the algorithm the set of edges forms a forest (why?).
- ▶ Fix iteration i . Let F_i be the set of edges in F at the beginning of the iteration.
- ▶ Let $H = F' - F_i$.
- ▶ All edges in H are necessary for the solution.

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- ▶ Contract all edges in F_i into single vertices V' .
- ▶ We can consider the forest H on the set of vertices V' .
- ▶ Let $\deg(v)$ be the degree of a vertex $v \in V'$ within this forest.
- ▶ Color a vertex $v \in V'$ red if it corresponds to a component from \mathbb{C} (an active component). Otw. color it blue. (Let B the set of blue vertices (with non-zero degree) and R the set of red vertices)
- ▶ We have

$$\sum_{v \in R} \deg(v) \geq \sum_{C \in \mathbb{C}} |\delta(C) \cap F'| \stackrel{?}{\leq} 2|\mathbb{C}| = 2|R|$$

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$$\begin{aligned}\sum_{v \in R} \deg(v) &= \sum_{v \in R \cup B} \deg(v) - \sum_{v \in B} \deg(v) \\ &\leq 2(|R| + |B|) - 2|B|\end{aligned}$$

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 - ▶ Suppose not. The single edge connecting $b \in B$ comes from H , and, hence, is necessary.
 - ▶ But this means that the cluster corresponding to b must separate a source-target pair.

- ▶ Suppose that no node in B has degree one.
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$$\begin{aligned}\sum_{v \in R} \deg(v) &= \sum_{v \in R \cup B} \deg(v) - \sum_{v \in B} \deg(v) \\ &\leq 2(|R| + |B|) - 2|B| = 2|R|\end{aligned}$$

- ▶ Every blue vertex with non-zero degree must have degree at least two.
 - ▶ Suppose not. The single edge connecting $b \in B$ comes from H , and, hence, is necessary.
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 - ▶ But then it must be a red node.