Repetition: Primal Dual for Set Cover

Primal Relaxation:

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\min & & \sum_{i=1}^{k} w_i x_i & & \\
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If we would also fulfill dual slackness conditions

$$y_e > 0 \Rightarrow \sum_{j : e \in S_j} x_j = 1$$

then the solution would be optimal!!!

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right hand side of j-th dual constraint

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Feedback Vertex Set for Undirected Graphs

- Given a graph $G = (V, E)$ and non-negative weights $w_v \geq 0$ for vertex $v \in V$.

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- Each vertex can be viewed as a set that contains some cycles.

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- However, this encoding gives a Set Cover instance of non-polynomial size.
- The $O(\log n)$-approximation for Set Cover does not help us to get a good solution.

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Let $\mathbb{C}$ denote the set of all cycles (where a cycle is identified by its set of vertices)

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Primal Dual for Shortest Path

Given a graph $G = (V, E)$ with two nodes $s, t \in V$ and edge-weights $c : E \to \mathbb{R}^+$ find a shortest path between s and t w.r.t. edge-weights c.

$$\begin{array}{llrcl} \min & & \sum_e c(e) x_e & & \\ \text{s.t.} & \forall S \in \mathcal{S} & \sum_{e:\delta(S)} x_e & \geq & 1 \\ & \forall e \in E & x_e & \in & \{0,1\} \end{array}$$

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We can interpret the value y_S as the width of a moat surounding the set S.

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$$\sum_{e\in P} c(e) = \sum_{e\in P}\ \sum_{S:e\in\delta(S)} y_S = \sum_{S:s\in S,t\notin S} |P\cap\delta(S)|\cdot y_S \ .$$

If we can show that $y_S > 0$ implies $|P\cap\delta(S)| = 1$ gives

$$\sum_{e\in P} c(e) = \sum_S y_S \le \mathrm{OPT}$$

by weak duality.

Hence, we find a shortest path.

Steiner Forest Problem:

Given a graph $G = (V, E)$, together with source-target pairs s_i, t_i, $i = 1, \dots, k$, and a cost function $c : E \to \mathbb{R}^+$ on the edges. Find a subset $F \subseteq E$ of the edges such that for every $i \in \{1, \dots, k\}$ there is a path between s_i and t_i only using edges in F.

$$\begin{array}{lllll} \min & & \sum_e c(e) x_e & & \\ \text{s.t.} & \forall S \subseteq V : S \in S_i \text{ for some } i & \sum_{e \in \delta(S)} x_e & \geq & 1 \\ & \forall e \in E & x_e & \in & \{0,1\} \end{array}$$

Here S_i contains all sets S such that $s_i \in S$ and $t_i \notin S$.

If S contains two edges from P then there must exist a subpath P' of P that starts and ends with a vertex from S (and all interior vertices are not in S).

When we increased y_S, S was a connected component of the set of edges F' that we had chosen till this point.

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\max & & \sum_{S\,:\,\exists i \text{ s.t. } S\in S_i} y_S & & \\
\text{s.t.} & \forall e\in E & \sum_{S:e\in\delta(S)} y_S & \le & c(e)\\
 & & y_S & \ge & 0
\end{array}
$$

The difference to the dual of the shortest path problem is that we have many more variables (sets for which we can generate a moat of non-zero width).

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6: $F \leftarrow F \cup \{e'\}$
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EADS II
Harald Räcke
18.4 Steiner Forest

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If we show that $y_S > 0$ implies that $|\delta(S)\cap F| \le \alpha$ we are in good shape.

However, this is not true:

- Take a complete graph on $k+1$ vertices $v_0, v_1, \dots, v_k$.

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- We only set $y_{\{v_0\}} = 1$. All other dual variables stay 0.

Algorithm 1 FirstTry

```
y ← 0
F ← ∅
while not all s_i-t_i pairs connected in F do
    Let C be some connected component of (V,F) such that |C ∩ {s_i,t_i}| = 1 for some i.
    Increase y_C until there is an edge e' ∈ δ(C) s.t. ∑_{S∈S_i:e'∈δ(S)} y_S = c_{e'}
    F ← F ∪ {e'}
return ⋃_i P_i
```

$$\sum_{e\in F} c(e) = \sum_{e\in F}\sum_{S:e\in\delta(S)} y_S = \sum_S |\delta(S)\cap F|\cdot y_S \enspace .$$

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- $y_{\{v_0\}} > 0$ but $|\delta(\{v_0\})\cap F| = k$.

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6: $F \leftarrow F \cup \{e'\}$
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Algorithm 1 SecondTry

$y \leftarrow 0$; $F \leftarrow \emptyset$; $\ell \leftarrow 0$
while not all s_i-t_i pairs connected in F **do**
  $\ell \leftarrow \ell + 1$
  Let $\mathfrak{C}$ be set of all connected components C of (V, F) such that $|C \cap \{s_i, t_i\}| = 1$ for some i.
  Increase y_C for all $C \in \mathfrak{C}$ uniformly until for some edge $e_\ell \in \delta(C')$, $C' \in \mathfrak{C}$ s.t. $\sum_{S: e_\ell \in \delta(S)} y_S = c_{e_\ell}$
  $F \leftarrow F \cup \{e_\ell\}$
$F' \leftarrow F$
for $k \leftarrow \ell$ downto 1 **do** // reverse deletion
  if $F' - e_k$ is feasible solution **then**
    remove e_k from F'
return F'

$$\sum_{e \in F} c(e) = \sum_{e \in F} \sum_{S: e \in \delta(S)} y_S = \sum_S |\delta(S) \cap F| \cdot y_S \ .$$

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The reverse deletion step is not strictly necessary this way. It would also be sufficient to simply delete all unnecessary edges in any order.

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Example

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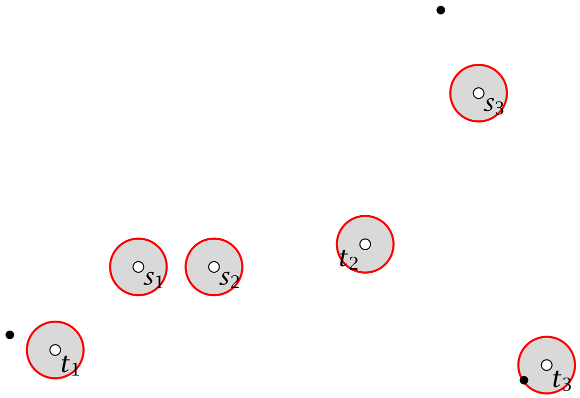
Example



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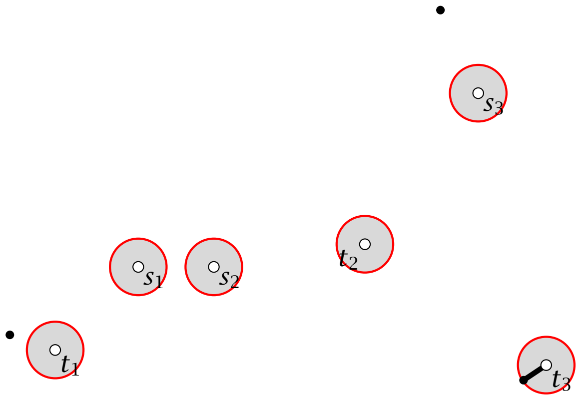

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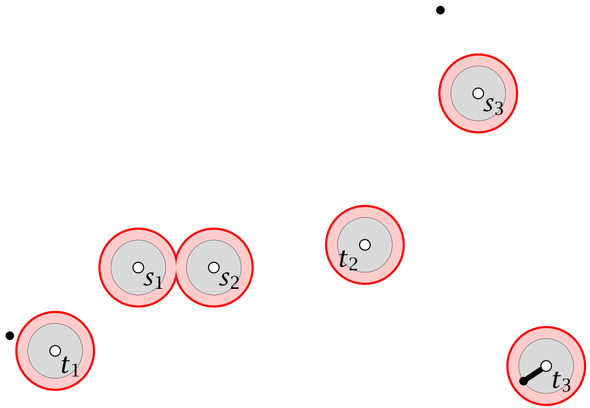

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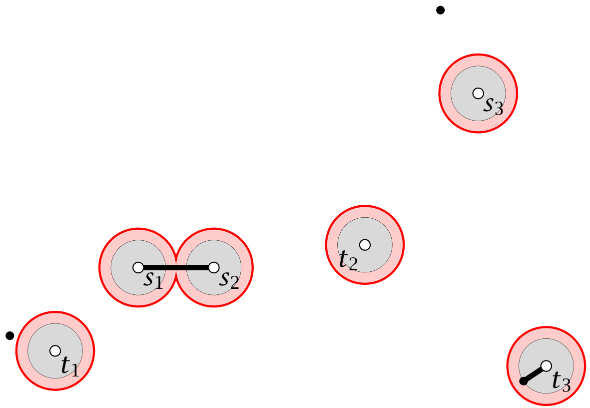

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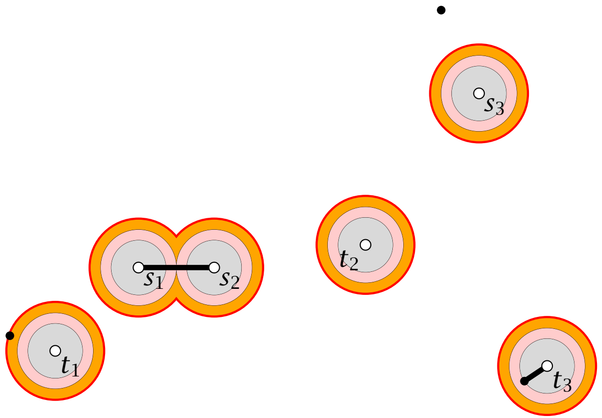

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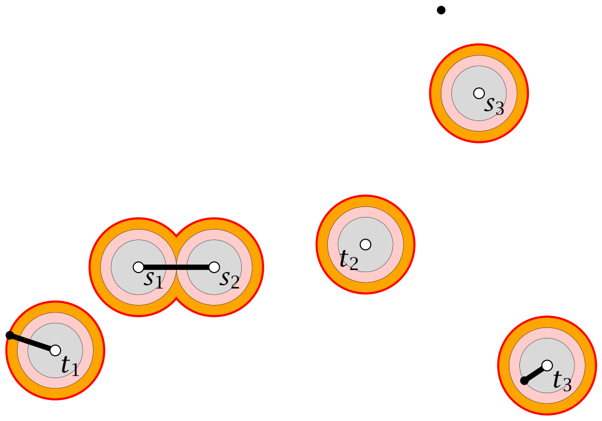

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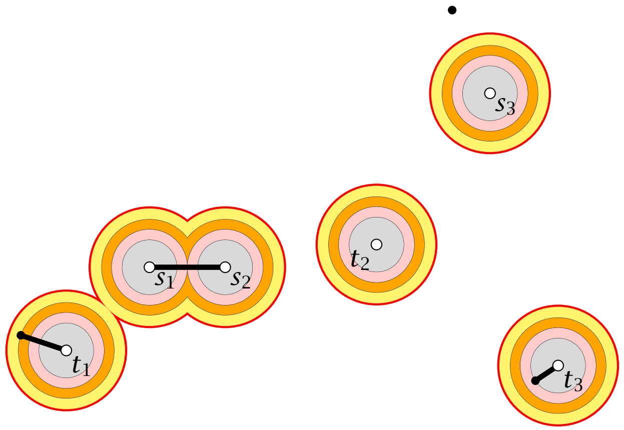

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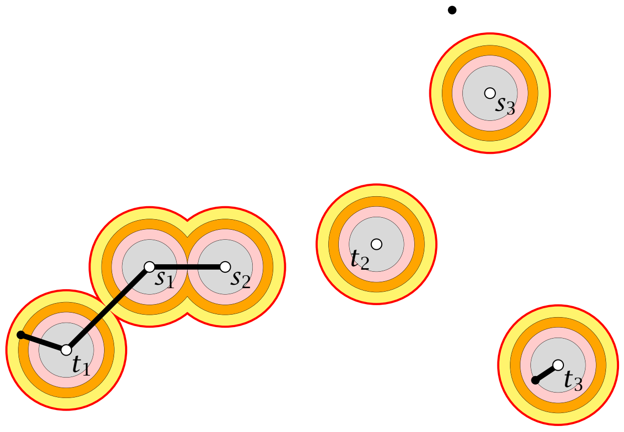

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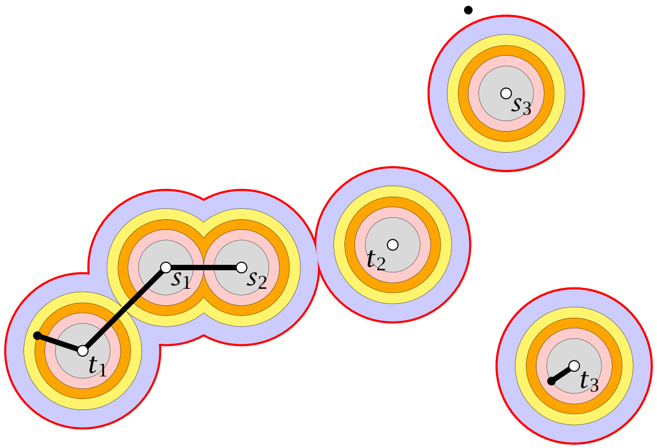

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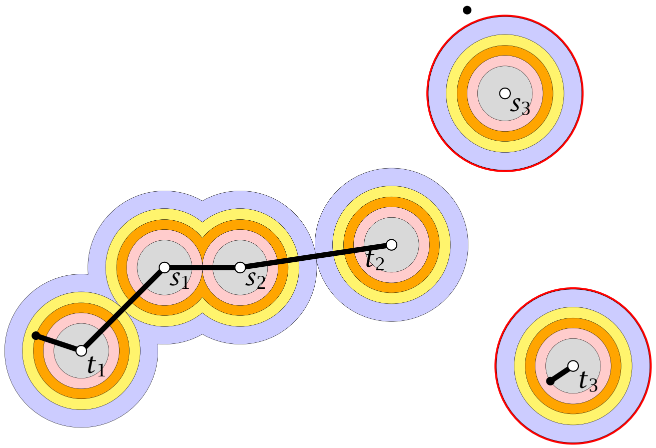

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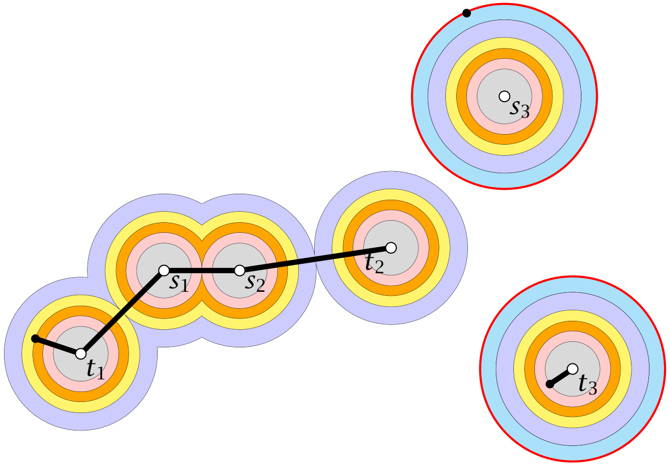

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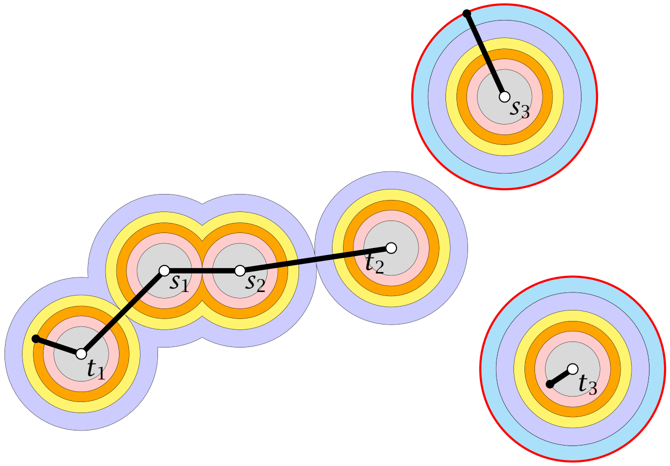

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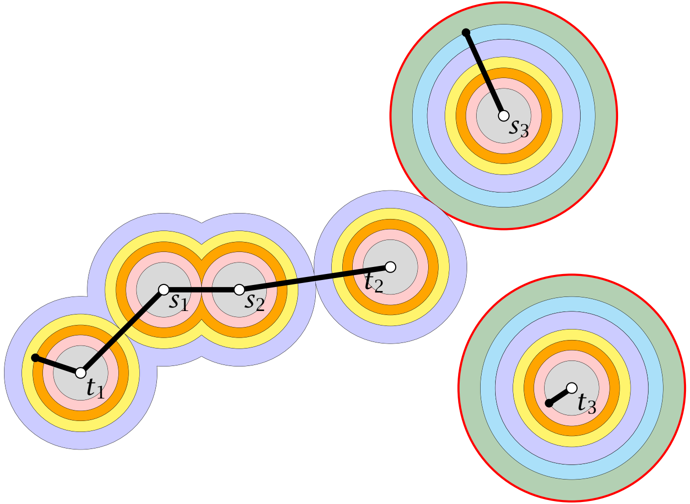

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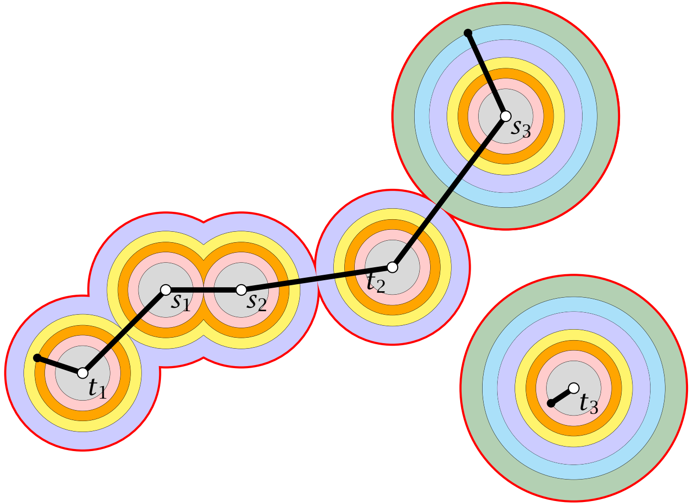

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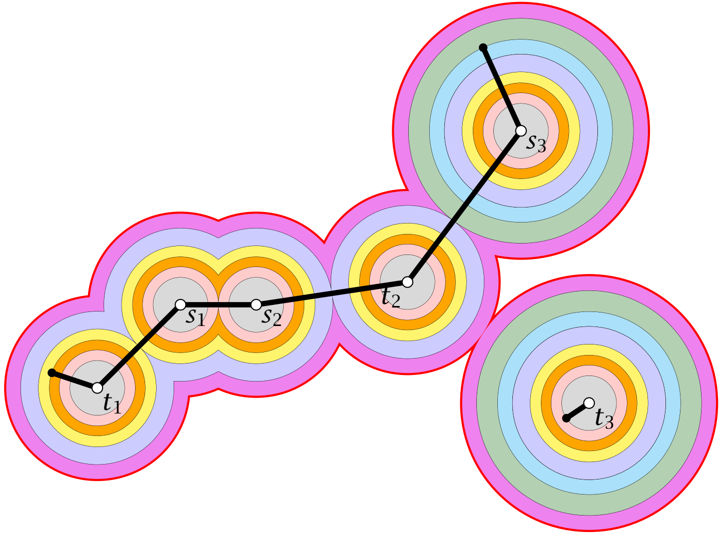

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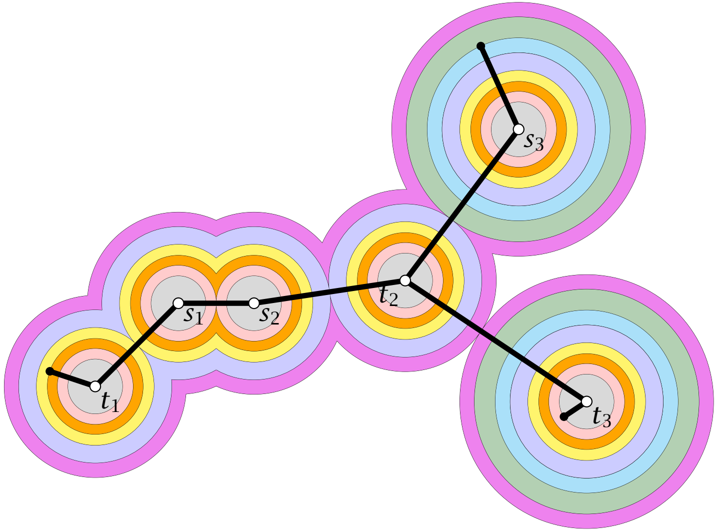

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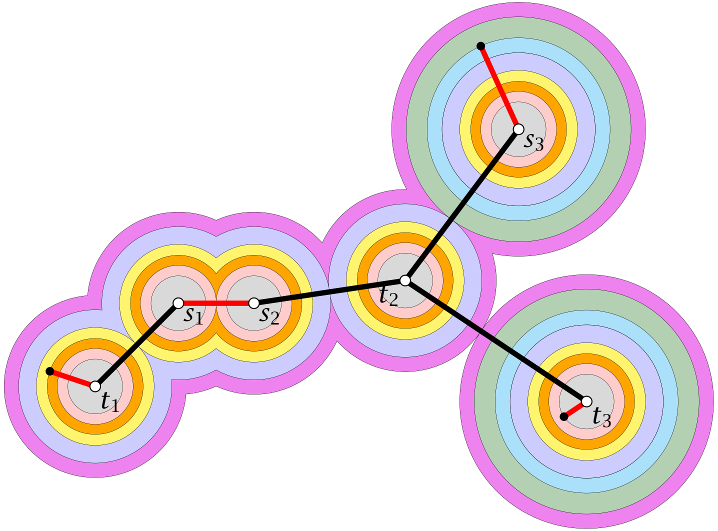

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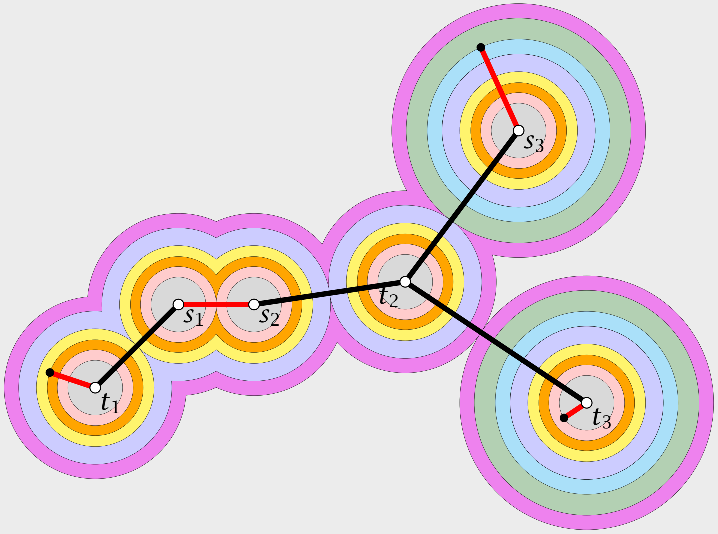
Lemma 4
For any $\mathfrak{C}$ in any iteration of the algorithm

$$\sum_{C\in\mathfrak{C}} |\delta(C) \cap F'| \le 2|\mathfrak{C}|$$

This means that the number of times a moat from $\mathfrak{C}$ is crossed in the final solution is at most twice the number of moats.

Proof: later...

Example



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We want to show that

$$\sum_S |F'\cap\delta(S)|\cdot y_S \le 2\sum_S y_S$$

Lemma 4
For any $\mathbb{C}$ in any iteration of the algorithm

$$\sum_{C\in\mathbb{C}} |\delta(C)\cap F'| \le 2|\mathbb{C}|$$

This means that the number of times a moat from $\mathbb{C}$ is crossed in the final solution is at most twice the number of moats.

Proof: later...

$$\sum_{e\in F'} c_e = \sum_{e\in F'} \sum_{S:e\in\delta(S)} y_S = \sum_S |F' \cap \delta(S)| \cdot y_S \ .$$

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- In the i-th iteration the increase of the left-hand side is

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  and the increase of the right hand side is $2\epsilon|\mathfrak{C}|$.
- Hence, by the previous lemma the inequality holds after the iteration if it holds in the beginning of the iteration.

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- At any point during the algorithm the set of edges forms a forest (why?).
- Fix iteration i. Let F_i be the set of edges in F at the beginning of the iteration.
- Let $H = F' - F_i$.
- All edges in H are necessary for the solution.

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- Contract all edges in F_i into single vertices V'.
- We can consider the forest H on the set of vertices V'.
- Let $\deg(v)$ be the degree of a vertex $v \in V'$ within this forest.
- Color a vertex $v \in V'$ red if it corresponds to a component from $\mathfrak{C}$ (an active component). Otw. color it blue. (Let B the set of blue vertices (with non-zero degree) and R the set of red vertices)
- We have

$$\sum_{v \in R} \deg(v) \geq \sum_{C \in \mathfrak{C}} |\delta(C) \cap F'| \stackrel{?}{\leq} 2|\mathfrak{C}| = 2|R|$$

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- Every blue vertex with non-zero degree must have degree at least two.

- Contract all edges in F_i into single vertices V'.
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  - But this means that the cluster corresponding to b must separate a source-target pair.

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  - But this means that the cluster corresponding to b must separate a source-target pair.
  - But then it must be a red node.

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