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Problem set 11
June 24, 2016
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Online and Approximation Algorithms

Due July 1, 2016 before 10:00

Exercise 1 (Lower bound for Bin Packing - 10 points)

Consider the Partition problem. Given n non-negative numbers $a_1, a_2, \ldots, a_n$ such that $\sum_{i=1}^{n} a_i = A$. The Partition problem is to decide whether it is possible to identify a subset $B \subset A$ such that $\sum_{i \in B} a_i = \frac{1}{2}A$. This problem is NP-hard.
Use a reduction from the Partition problem to show that there is no polynomial-time approximation algorithm that can approximate Bin Packing to a ratio better than 3/2.

Exercise 2 (Next-Fit Bin Packing - 10 points)

In the Bin Packing problem, we are given a set of items $I = \{1, 2, \ldots, n\}$ which have to be to packed into bins. Each item $i \in I$ has a volume $0 \leq v_i \leq 1$ and it has to be assigned entirely to a single bin. The capacity of a bin is equal to 1. We would like to pack the items in a way that the total volume of the items stored in a bin does not exceed its capacity. Our objective is to minimize the number of used bins.
Algorithm *Next-Fit* takes an arbitrary order of the items and it packs them one by one. It maintains one *open bin* and, for each item $i \in I$, if i fits in the open bin, then Next-Fit puts it in. Otherwise, the bin is closed and a new bin is opened. Show that Next-Fit is a 2-approximation.
Also show that we cannot expect a better approximation ratio for it.

Exercise 3 (Fractional Knapsack - 10 points)

In the Fractional Knapsack problem, we are given n items with weights $w_1, w_2, \ldots, w_n$ and values $v_1, v_2, \ldots, v_n$. The objective is to pack items of maximum total value in a knapsack of capacity B and we are allowed to take any fraction of each item. A fraction $x_i \in [0, 1]$ of item i has weight $x_i \cdot w_i$ and value $x_i \cdot v_i$. Propose an optimal algorithm for the Fractional Knapsack problem and show that it solves the problem optimally in polynomial time.

Exercise 4 (PTAS for Bin Covering - 10 points + 5 bonus points)

Consider the Bin Covering problem. Given n items with sizes $a_1, a_2, \ldots, a_n \in (0, 1]$, maximize the number of bins opened such that every bin has items summing to at least 1. We consider the special case in which item sizes are bounded from below by c, for a fixed constant $c > 0$.
The goal of this exercise is to develop an asymptotic PTAS for this problem.

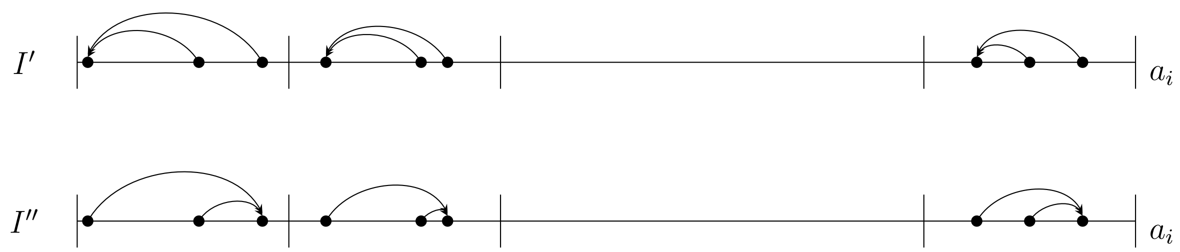
(a) As a first step, prove that when throwing b identical balls in m distinct bins, the total number of possible outcomes is $\binom{b+m-1}{b}$.

(b) Let $K > 0$ be a fixed constant. Consider now the Bin Covering problem given above and assume that there are K different item sizes. Let a bin type be defined as the number of items of each size it contains. Use the fact from (a) to prove that the number of different bin types is bounded by $\binom{\lceil \frac{1}{c} \rceil + K}{\lceil \frac{1}{c} \rceil}$.

(c) Denote $T = \binom{\lceil \frac{1}{c} \rceil + K}{\lceil \frac{1}{c} \rceil}$. Prove that, since we open at most n bins in the Bin Covering problem, the number of possible feasible packings for this problem is bounded by $\binom{n+T}{T}$. Also show that since this is polynomial in n, we can solve this problem optimally.

(d) Let $\epsilon > 0$ be a constant. Starting with a problem instance I for the Bin Covering problem with all items of size at least c, construct instances I' and I'' with a constant number K of item sizes as follows.

We sort the items by non-decreasing order and assume that $a_1 \leq a_2 \leq \ldots \leq a_n$. The items are now partitioned into K groups, each having at most $q = \lceil n/K \rceil$ items. Instance I' is then created by rounding down the size of each item to the size of the smallest item in the group. Instance I'' is created by rounding up the size of each item to that of the largest item in its group.



Note that both I' and I'' can be solved optimally and $\text{OPT}(I') \leq \text{OPT}(I) \leq \text{OPT}(I'')$.

Show that $\text{OPT}(I') \geq \text{OPT}(I'') - q$ and use this and a well-chosen value for K to show that $\text{OPT}(I') \geq (1-\epsilon)\text{OPT}(I)$, thereby establishing a PTAS for this special case of Bin Covering.