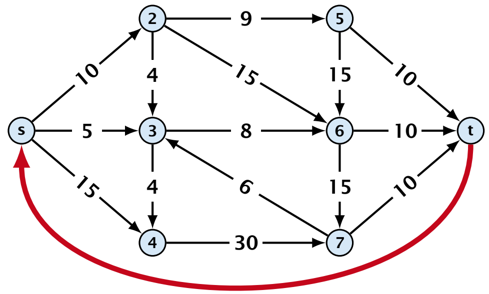
Mincost Flow

Problem Definition:

$$\begin{array}{ll} \text{min} & \sum_e c(e) f(e) \\ \text{s.t.} & \forall e \in E: \;\; 0 \le f(e) \le u(e) \\ & \forall v \in V: \;\; f(v) = b(v) \end{array}$$

- $G = (V, E)$ is a directed graph.
- $u : E \to \mathbb{R}_0^+ \cup \{\infty\}$ is the capacity function.
- $c : E \to \mathbb{R}$ is the cost function (note that $c(e)$ may be negative).
- $b : V \to \mathbb{R}$, $\sum_{v \in V} b(v) = 0$ is a demand function.

Solve Maxflow Using Mincost Flow



- Given a flow network for a standard maxflow problem.
- Set $b(v) = 0$ for every node. Keep the capacity function u for all edges. Set the cost $c(e)$ for every edge to 0.
- Add an edge from t to s with infinite capacity and cost -1.
- Then, $\text{val}(f^*) = -\text{cost}(f_{\text{min}})$, where f^* is a maxflow, and f_{min} is a mincost-flow.

Solve Maxflow Using Mincost Flow

Solve decision version of maxflow:

- Given a flow network for a standard maxflow problem, and a value k.
- Set $b(v) = 0$ for every node apart from s or t. Set $b(s) = -k$ and $b(t) = k$.
- Set edge-costs to zero, and keep the capacities.
- There exists a maxflow of value k if and only if the mincost-flow problem is feasible.

Generalization

Our model:

$$\begin{array}{ll} \text{min} & \sum_e c(e) f(e) \\ \text{s.t.} & \forall e \in E: \;\; 0 \le f(e) \le u(e) \\ & \forall v \in V: \;\; f(v) = b(v) \end{array}$$

where $b : V \to \mathbb{R}$, $\sum_v b(v) = 0$; $u : E \to \mathbb{R}_0^+ \cup \{\infty\}$; $c : E \to \mathbb{R}$;

A more general model?

$$\begin{array}{ll} \text{min} & \sum_e c(e) f(e) \\ \text{s.t.} & \forall e \in E: \;\; \ell(e) \le f(e) \le u(e) \\ & \forall v \in V: \;\; a(v) \le f(v) \le b(v) \end{array}$$

where $a : V \to \mathbb{R}$, $b : V \to \mathbb{R}$; $\ell : E \to \mathbb{R} \cup \{-\infty\}$, $u : E \to \mathbb{R} \cup \{\infty\}$ $c : E \to \mathbb{R}$;

Generalization

Differences

- Flow along an edge e may have non-zero lower bound $\ell(e)$.
- Flow along e may have negative upper bound $u(e)$.
- The demand at a node v may have lower bound $a(v)$ and upper bound $b(v)$ instead of just lower bound = upper bound = $b(v)$.

Reduction I

$$\begin{array}{lll} \text{min} & \sum_e c(e)f(e) & \\ \text{s.t.} & \forall e \in E: & \ell(e) \leq f(e) \leq u(e) \\ & \forall v \in V: & a(v) \leq f(v) \leq b(v) \end{array}$$

We can assume that $a(v) = b(v)$:

Add new node r.

Add edge (r, v) for all $v \in V$.

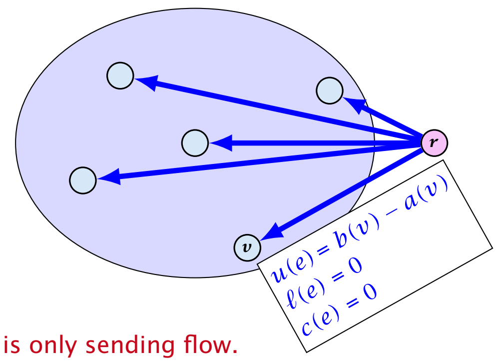
Set $\ell(e) = c(e) = 0$ for these edges.

Set $u(e) = b(v) - a(v)$ for edge (r, v).

Set $a(v) = b(v)$ for all $v \in V$.

Set $b(r) = -\sum_{v \in V} b(v)$.

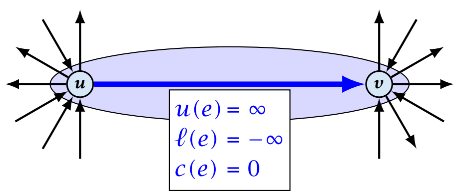
$-\sum_v b(v)$ is negative; hence r is only sending flow.



Reduction II

$$\begin{array}{lll} \text{min} & \sum_e c(e)f(e) & \\ \text{s.t.} & \forall e \in E: & \ell(e) \leq f(e) \leq u(e) \\ & \forall v \in V: & f(v) = b(v) \end{array}$$

We can assume that either $\ell(e) \neq -\infty$ or $u(e) \neq \infty$:

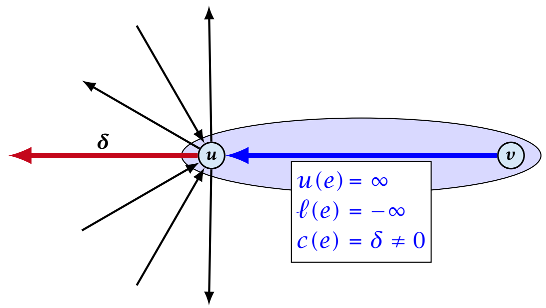


If $c(e) = 0$ we can contract the edge/identify nodes u and v.

If $c(e) \neq 0$ we can transform the graph so that $c(e) = 0$.

Reduction II

We can transform any network so that a particular edge has cost $c(e) = 0$:

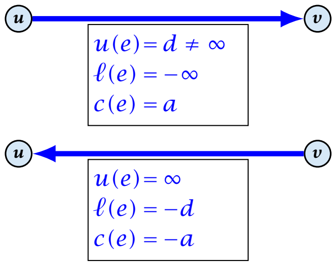


Additionally we set $b(u) = 0$.

Reduction III

$$\begin{array}{lll} \min & \sum_e c(e)f(e) & \\ \text{s.t.} & \forall e \in E: & \ell(e) \le f(e) \le u(e) \\ & \forall v \in V: & f(v) = b(v) \end{array}$$

We can assume that $\ell(e) \neq -\infty$:

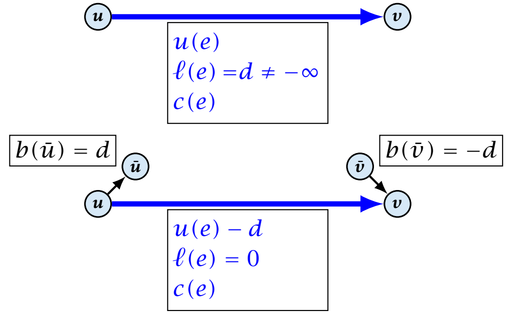


Replace the edge by an edge in opposite direction.

Reduction IV

$$\begin{array}{lll} \min & \sum_e c(e)f(e) & \\ \text{s.t.} & \forall e \in E: & \ell(e) \le f(e) \le u(e) \\ & \forall v \in V: & f(v) = b(v) \end{array}$$

We can assume that $\ell(e) = 0$:

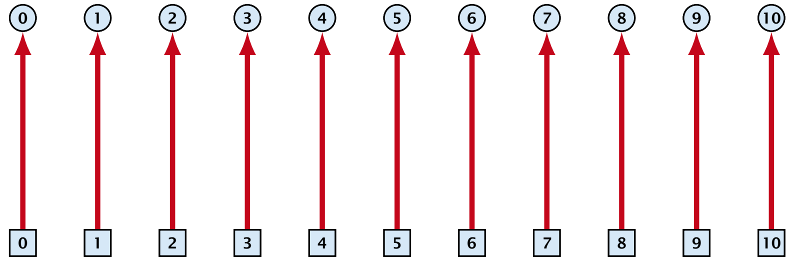


The added edges have infinite capacity and cost $c(e)/2$.

Applications

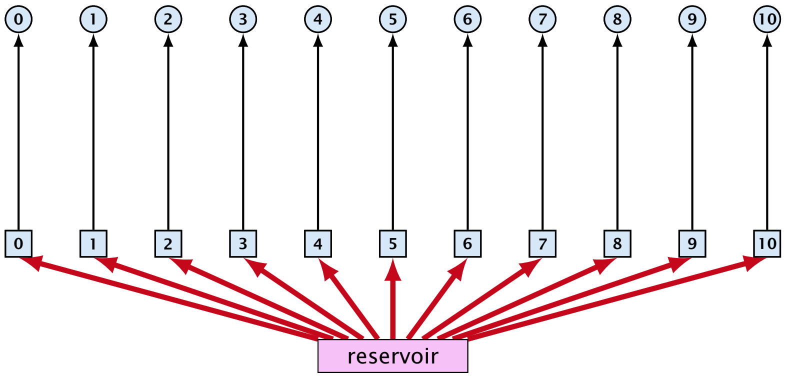
Caterer Problem

- She needs to supply r_i napkins on N successive days.
- She can buy new napkins at p cents each.
- She can launder them at a fast laundry that takes m days and cost f cents a napkin.
- She can use a slow laundry that takes $k > m$ days and costs s cents each.
- At the end of each day she should determine how many to send to each laundry and how many to buy in order to fulfill demand.
- Minimize cost.



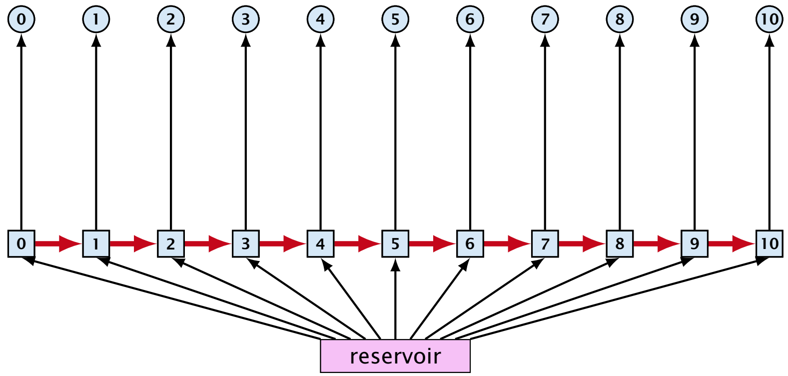
day edges:

upper bound: $u(e_i) = \infty$;
lower bound: $\ell(e_i) = r_i$;
cost: $c(e) = 0$



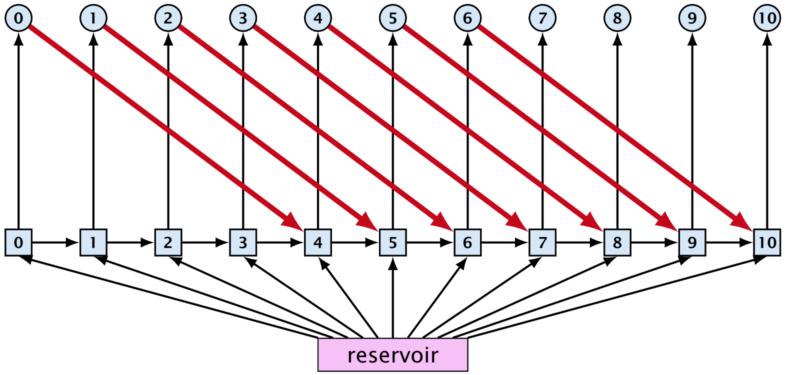
buy edges:

upper bound: $u(e_i) = \infty$;
lower bound: $\ell(e_i) = 0$;
cost: $c(e) = p$



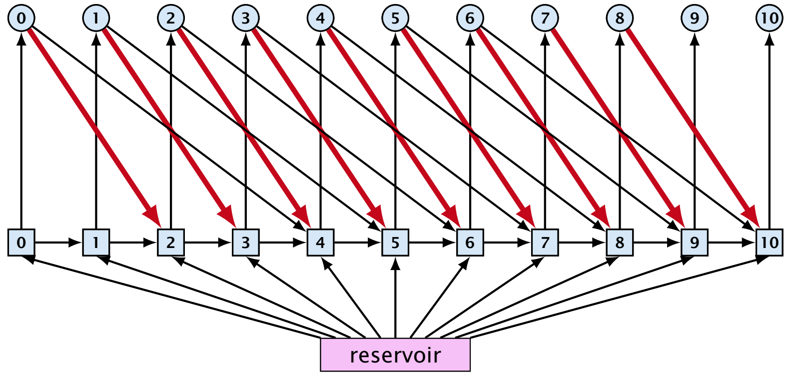
forward edges:

upper bound: $u(e_i) = \infty$;
lower bound: $\ell(e_i) = 0$;
cost: $c(e) = 0$



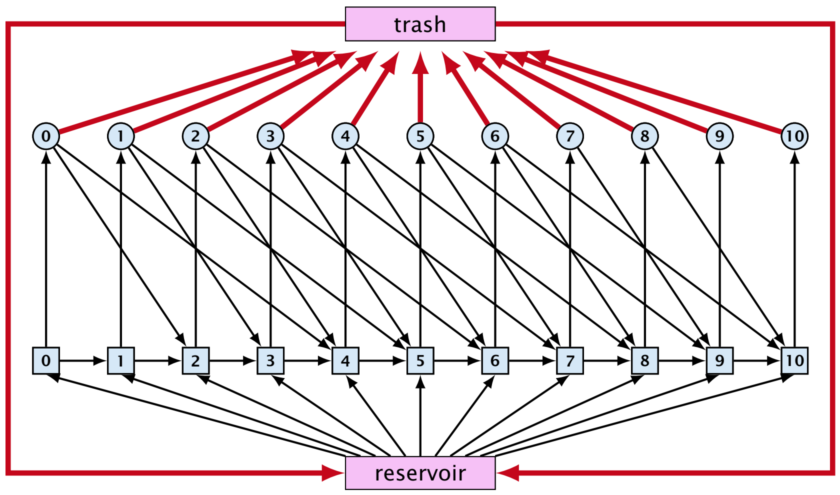
slow edges:

upper bound: $u(e_i) = \infty$;
lower bound: $\ell(e_i) = 0$;
cost: $c(e) = s$



fast edges:

upper bound: $u(e_i) = \infty$;
lower bound: $\ell(e_i) = 0$;
cost: $c(e) = f$



trash edges:

upper bound: $u(e_i) = \infty$;
lower bound: $\ell(e_i) = 0$;
cost: $c(e) = 0$

Residual Graph

The residual graph for a mincost flow is exactly defined as the residual graph for standard flows, with the only exception that one needs to define a cost for the residual edge.

For a flow of z from u to v the residual edge (v, u) has capacity z and a cost of $-c((u, v))$.

15 Mincost Flow

A circulation in a graph $G = (V, E)$ is a function $f : E \to \mathbb{R}^+$ that has an excess flow $f(v) = 0$ for every node $v \in V$.

A circulation is feasible if it fulfills capacity constraints, i.e., $f(e) \le u(e)$ for every edge of G.

Lemma 1
A given flow is a mincost-flow if and only if the corresponding residual graph G_f does not have a feasible circulation of negative cost.

⇒ Suppose that g is a feasible circulation of negative cost in the residual graph.

Then $f + g$ is a feasible flow with cost $\text{cost}(f) + \text{cost}(g) < \text{cost}(f)$. Hence, f is not minimum cost.

⇐ Let f be a non-mincost flow, and let f^* be a min-cost flow. We need to show that the residual graph has a feasible circulation with negative cost.

Clearly $f^* - f$ is a circulation of negative cost. One can also easily see that it is feasible for the residual graph. (after sending $-f$ in the residual graph (pushing all flow back) we arrive at the original graph; for this f^* is clearly feasible)

For previous slide:
$g = f^* - f$ is obtained by computing $\Delta(e) = f^*(e) - f(e)$ for every edge $e = (u, v)$. If the result is positive set $g((u,v)) = \Delta(e)$ and $g((v,u)) = 0$. Otherwise set $g((u,v)) = 0$ and $g((v,u)) = -\Delta(e)$.

15 Mincost Flow

Lemma 2

A graph (without zero-capacity edges) has a feasible circulation of negative cost if and only if it has a negative cycle w.r.t. edge-weights $c : E \to \mathbb{R}$.

Proof.

- Suppose that we have a negative cost circulation.
- Find directed path only using edges that have non-zero flow.
- If this path has negative cost you are done.
- Otherwise send flow in opposite direction along the cycle until the bottleneck edge(s) does not carry any flow.
- You still have a circulation with negative cost.
- Repeat.

15 Mincost Flow

Algorithm 51 CycleCanceling$(G = (V, E), c, u, b)$

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establish a feasible flow f in G
while G_f contains negative cycle do
    use Bellman-Ford to find a negative circuit Z
    δ ← min{u_f(e) | e ∈ Z}
    augment δ units along Z and update G_f
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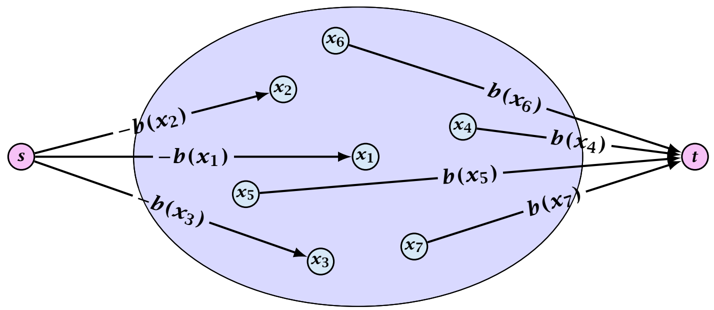
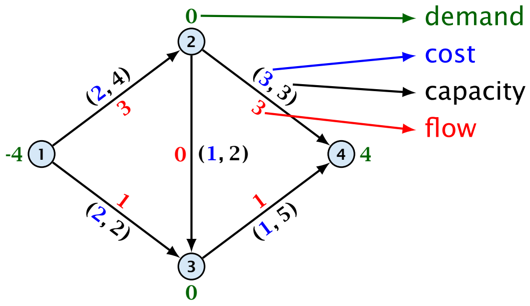
How do we find the initial feasible flow?



- Connect new node s to all nodes with negative $b(v)$-value.
- Connect nodes with positive $b(v)$-value to a new node t.
- There exist a feasible flow in the original graph iff in the resulting graph there exists an s-t flow of value

$$\sum_{v:b(v)<0} (-b(v)) = \sum_{v:b(v)>0} b(v) .$$

15 Mincost Flow



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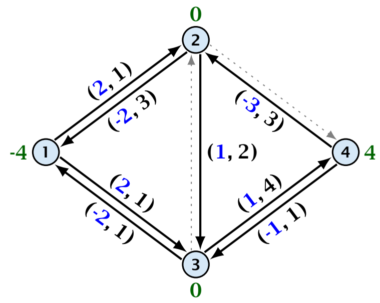

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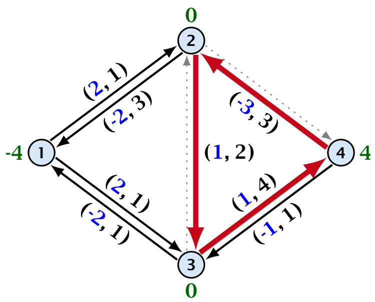

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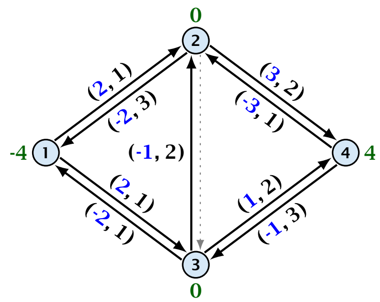

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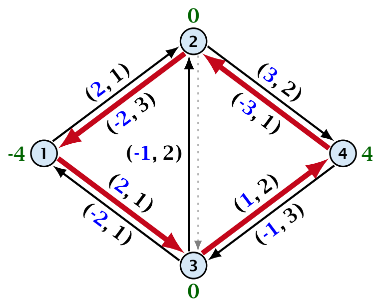

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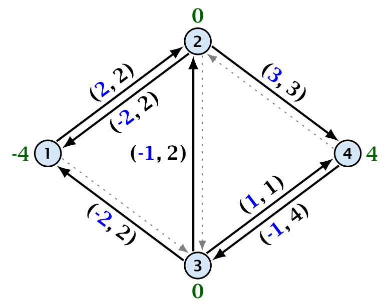

15 Mincost Flow

Lemma 3
The improving cycle algorithm runs in time $\mathcal{O}(nm^2CU)$, for integer capacities and costs, when for all edges e, $|c(e)| \le C$ and $|u(e)| \le U$.

- Running time of Bellman-Ford is $\mathcal{O}(mn)$.
- Pushing flow along the cycle can be done in time $\mathcal{O}(n)$.
- Each iteration decreases the total cost by at least 1.
- The true optimum cost must lie in the interval $[-mCU, \ldots, +mCU]$.

Note that this lemma is weak since it does not allow for edges with infinite capacity.

15 Mincost Flow

A general mincost flow problem is of the following form:

$$\begin{array}{lll} \text{min} & \sum_e c(e)f(e) & \\ \text{s.t.} & \forall e \in E: & \ell(e) \le f(e) \le u(e) \\ & \forall v \in V: & a(v) \le f(v) \le b(v) \end{array}$$

where $a: V \to \mathbb{R}$, $b: V \to \mathbb{R}$; $\ell: E \to \mathbb{R} \cup \{-\infty\}$, $u: E \to \mathbb{R} \cup \{\infty\}$ $c: E \to \mathbb{R}$;

Lemma 4 (without proof)
A general mincost flow problem can be solved in polynomial time.